# High-dimensional approximation and sparse FFT using (multiple) rank-1 lattices

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joint work with Lutz Kämmerer, Daniel Potts, and Tino Ullrich

- torus $\mathbb{T} \simeq [0,1)$, $\{\mathrm{e}^{2\pi \mathrm{i} k x}\}_{k\in\mathbb{Z}}$ orthonormal basis of $L_2(\mathbb{T})$
- function $f \in L_2(\mathbb{T})$, $f(x) = \sum_{k\in\mathbb{Z}} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i} k x}$, $\hat{f}_k = \int_{\mathbb{T}} f(x) \, \mathrm{e}^{-2\pi \mathrm{i} k x} \, \mathrm{d}x \in \mathbb{C}$
- smooth function $f \Longrightarrow$ fast decay of Fourier coefficients $\hat{f}_k$
- truncated Fourier series $S_I f(x) = \sum_{k\in I} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i} k x} \approx f(x)$
- $\hat{f}_k = \int_{\mathbb{T}} f(x) \, \mathrm{e}^{-2\pi \mathrm{i} k x} \, \mathrm{d}x \approx \tilde{\hat{f}}_k := \frac{1}{2N} \sum_{j=0}^{2N-1} f(x_j) \, \mathrm{e}^{-2\pi \mathrm{i} k x_j}$, $x_j := \frac{j}{2N}$

$\Rightarrow$ transfer to multivariate case (tensorization)




$$(\tilde{\hat{f}}_k)_{k\in I} \xleftrightarrow[\text{FFT}]{\text{1-dim.}} (f(x_j))_{j=0}^{2N-1}$$

$\mathcal{O}(N \log N)$

[Gauß 1866] [Cooley, Tukey 1965]

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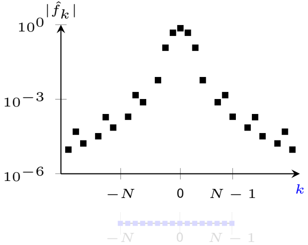


$(\tilde{\hat{f}}_k)_{k\in I}$ $\xleftrightarrow[\text{FFT}]{\text{1-dim.}}$ $(f(x_j))_{j=0}^{2N-1}$

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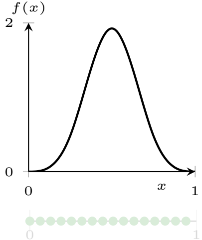

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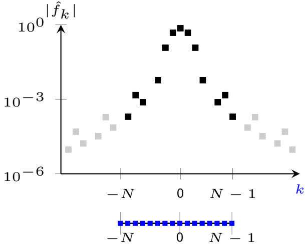


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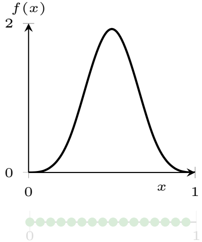

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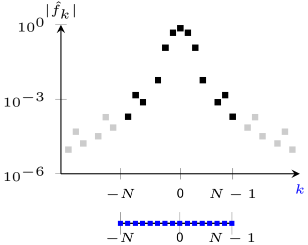


$(\hat{\tilde{f}}_k)_{k\in I}$ $\longleftrightarrow$ (1-dim. FFT) $(f(x_j))_{j=0}^{2N-1}$

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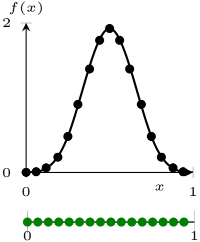

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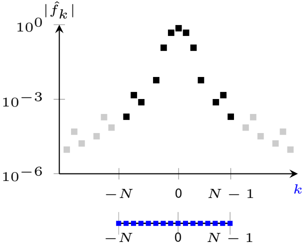


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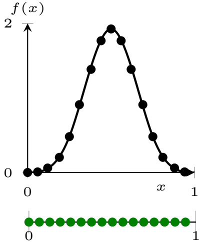
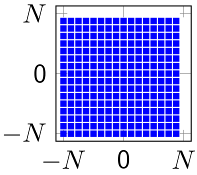


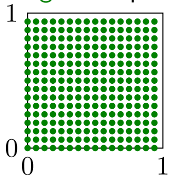
- full grid in frequency domain — equispaced full grid in spatial domain




$$(\hat{\tilde{\hat{f}}}_{\boldsymbol{k}})_{\boldsymbol{k}\in I} \xleftrightarrow[\text{FFT}]{d\text{-dim.}} (f(\boldsymbol{x}_j))_{j=0}^{|I|-1}$$

$$\mathcal{O}(N^d \log N)$$

curse of dimensionality




? high-dimensional case (e.g. spatial dimension $d = 10$)

$\Rightarrow$ assumption: sparsity or smoothness

- hyperbolic cross in frequency domain — sparse grid in spatial domain

[Smolyak '63; Temlyakov '85; Baszenski, Delvos '89; Hallatschek '92; Gradinaru '07; Griebel, Hamaekers '14]




$$(\hat{\tilde{\hat{f}}}_{\boldsymbol{k}})_{\boldsymbol{k}\in I} \xleftrightarrow[\text{HCFFT}]{} (f(\boldsymbol{x}_j))_{j=0}^{|I|-1}$$

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condition number of Fourier matrix

[Kämmerer, Kunis '11]

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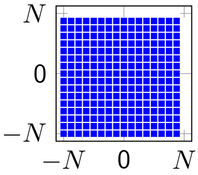


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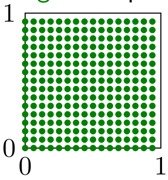


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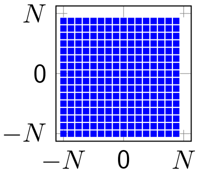


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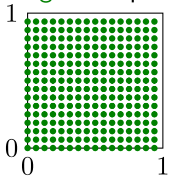


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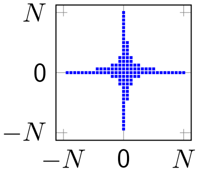


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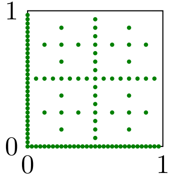
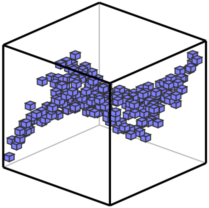


first part:

- fast reconstruction of arbitrary high-dimensional trigonometric polynomials $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ using 1-dimensional FFTs

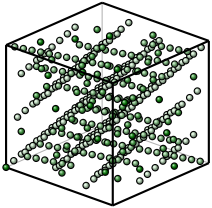
general known frequency index set $I \subset \mathbb{Z}^d$

spatial domain: multiple rank-1 lattice



$$\left(\hat{p}_{\boldsymbol{k}}\right)_{\boldsymbol{k}\in I} \xleftrightarrow[\text{FFTs}]{\text{1-dim.}} \left(f(\boldsymbol{x}_j)\right)_{j=0}^{M-1}$$

$$\mathcal{O}\left(|I|\,(d+\log|I|)\log^3|I|\right)$$



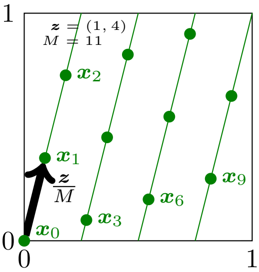
- fast approximation $f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ of functions from samples

second part:

- unknown frequency index set $I$ / weights / function space in high dimensions

$\Rightarrow$ dimension-incremental sparse FFT using multiple rank-1 lattices

- $f(\boldsymbol{x}) = p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, arbitrary freq. index set $I \subset \mathbb{Z}^d$, $|I| < \infty$
- rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, M) := \{\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1}\}_{j=0}^{M-1}$, $\boldsymbol{z} \in \mathbb{Z}^d$, $M \in \mathbb{N}$, as discretization in spatial domain




Korobov '59
Maisonneuve '72
Sloan & Kachoyan '84,'87,'90
Temlyakov '86
Lyness '89
Sloan & Joe '94
Sloan & Reztsov '01
Li & Hickernell '03
Kämmerer & Kunis & Potts '12

- numerical integration: $\hat{p}_{\boldsymbol{0}} = \int_{\mathbb{T}^d} p_I(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \overset{=}{\approx} \sum_{j=0}^{M-1} \frac{1}{M}\, p_I(\boldsymbol{x}_j)$
- reconstruction / approx.: $\hat{p}_{\boldsymbol{k}} = \left\langle p_I, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\circ}\right\rangle_{L_2(\mathbb{T}^d)} = \int_{\mathbb{T}^d} p_I(\boldsymbol{x})\,\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}$

$$\overset{=}{\approx} \sum_{j=0}^{M-1} \frac{1}{M}\, p_I(\boldsymbol{x}_j)\,\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \boldsymbol{k} \in I$$

- $f(\boldsymbol{x}) = p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, arbitrary freq. index set $I \subset \mathbb{Z}^d$, $|I| < \infty$
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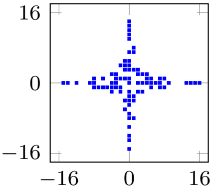
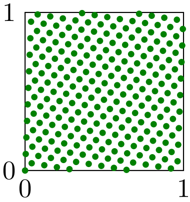
- fast reconstruction of $\hat{p}_{\boldsymbol{k}}$ using 1-dim. FFT? $\hat{p}_{\boldsymbol{k}} \overset{?}{=} \frac{1}{M} \sum_{j=0}^{M-1} p_I(\boldsymbol{x}_j) \, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_j}$




$$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I} \xleftarrow{\;?\;} (p_I(\boldsymbol{x}_j))_{j=0}^{M-1}$$

$$\mathcal{O}(M \log M + d\,|I|)$$



## Fast reconstruction of $\hat{p}_{\boldsymbol{k}}$ and approximation of $f$ using rank-1 lattices

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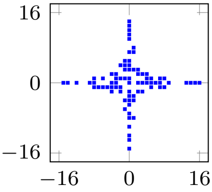
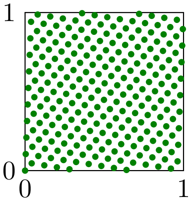
  $\Rightarrow$ reconstruction property: [Kämmerer,Kunis,Potts '12]
  $\boldsymbol{k}\cdot\boldsymbol{z} \not\equiv \boldsymbol{k}'\cdot\boldsymbol{z} \pmod{M}$ for all $\boldsymbol{k}, \boldsymbol{k}' \in I$, $\boldsymbol{k} \neq \boldsymbol{k}'$

  - $|I| \leq M \leq |I|^2$, simple CBC construction method [Kämmerer '12]




$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I} \xleftarrow[\text{FFT}]{\text{1-dim.}} (p_I(\boldsymbol{x}_j))_{j=0}^{M-1}$

$\mathcal{O}(M \log M + d\,|I|)$



- $f(\boldsymbol{x}) \approx p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, arbitrary freq. index set $I \subset \mathbb{Z}^d$, $|I| < \infty$
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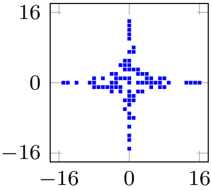
  - $|I| \le M \le |I|^2$, simple CBC construction method [Kämmerer '12]
- fast approximation of $f \in L_2(\mathbb{T}^d) \cap C(\mathbb{T}^d)$ using rank-1 lattice sampling error estimates in [Byrenheid,Kämmerer,Ullrich,V. '17] [V. '17]

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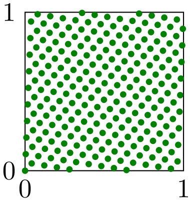
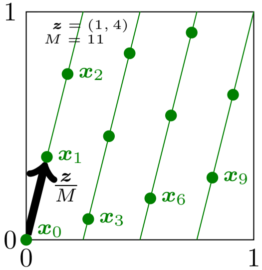


Some aspects:




- $\mathrm{R1L}(\boldsymbol{z}, M) = \{\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1}\}_{j=0}^{M-1}$, $\boldsymbol{z} \in \mathbb{Z}^d$, $M \in \mathbb{N}$, has group structure
- $\{(\boldsymbol{h}, \boldsymbol{h}') \subset \mathbb{Z}^d \times \mathbb{Z}^d : \boldsymbol{h}\cdot\boldsymbol{z} \equiv \boldsymbol{h}'\cdot\boldsymbol{z} \pmod{M}\}$ equivalence relation, (up to) $M$ equivalence classes
- $\boldsymbol{k}\cdot\boldsymbol{z} \not\equiv \boldsymbol{k}'\cdot\boldsymbol{z} \pmod{M}\ \forall \boldsymbol{k}, \boldsymbol{k}' \in I$, $\boldsymbol{k} \neq \boldsymbol{k}'$ $\Longrightarrow$ reconstruction is exact
- reconstruction/approximation: 1-dim. FFT + simple index transform

```
nodes_x = mod((0:M-1)'*z,M)/M; samples = f(nodes_x);
g_hat = fft(samples)/M;
p_hat = g_hat(mod(I*z',M)+1);
```

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$\Rightarrow$ all $\hat{p}_{\boldsymbol{k}}$ with frequency $\boldsymbol{k}$ in the same equivalence class ($\boldsymbol{k}\cdot\boldsymbol{z} \bmod M$) get identical value

Some aspects:

- $\mathrm{R1L}(\boldsymbol{z}, M) = \{\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1}\}_{j=0}^{M-1}$, $\boldsymbol{z} \in \mathbb{Z}^d$, $M \in \mathbb{N}$, has group structure
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- reconstruction/approximation: 1-dim. FFT + simple index transform

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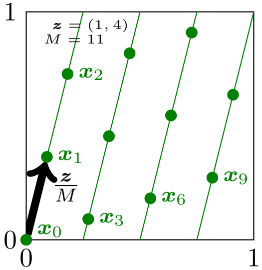

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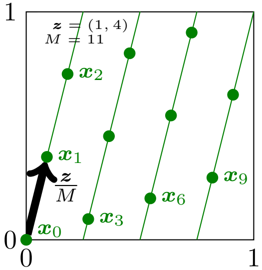


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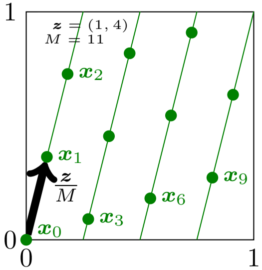


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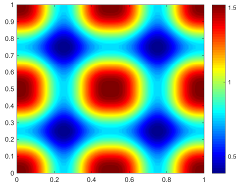
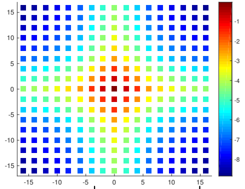
# Example – function from Sobolev space of dominating mixed smoothness

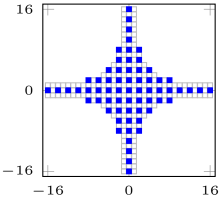
- $f(\boldsymbol{x}) := \prod_{s=1}^{d} \left(2 + \operatorname{sgn}((x_s \bmod 1) - \tfrac{1}{2})\, \sin(2\pi x_s)^3\right)$,
- hyperbolic cross $I := \{\boldsymbol{k} \in 2\,\mathbb{Z}^d \colon \prod_{s=1}^{d} \max(1, |k_s|) \le N\}$



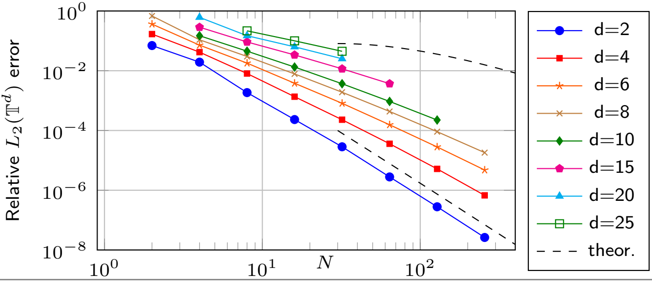

$f\left((x_1, x_2)^\top\right)$




$\log_{10} \left|\hat{f}_{(k_1,k_2)^\top}\right|$




$I$



- $\mathcal{H}^{\beta}_{\mathrm{mix}}(\mathbb{T}^d) := \left\{ f \in L_2 \colon \|f\| := \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}|^2 \prod_{s=1}^d \max(1,|k_s|)^{2\beta}} < \infty \right\}$, $\beta \geq 0$
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$$\mathrm{err} := \frac{\|f - p_I | L_2(\mathbb{T}^d)\|}{\|f | \mathcal{H}^{\beta}_{\mathrm{mix}}(\mathbb{T}^d)\|}, \ \beta > 1/2$$

| best approximation | sparse grid [Sickel, Ullrich '07] | rank-1 lattice [Byrenheid, Kämmerer, Ullrich, V. '17] |
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| $\asymp N^{-\beta}$ | $\lesssim N^{-\beta} \log^{\frac{d-1}{2}} N$ | |
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  - − number of samples $M$: $|I| \leq M \leq |I|^2$, construction: $\mathcal{O}(d\,|I|^3)$
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⇒ improvements? Use more than one rank-1 lattice! (union of several)

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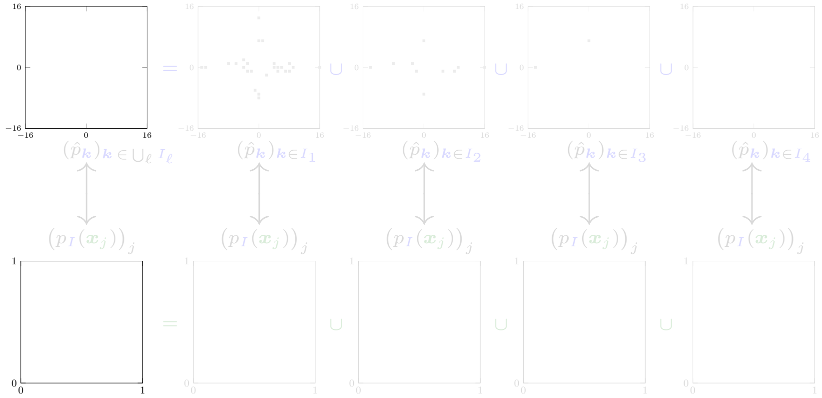
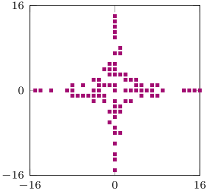
## Fast reconstruction of $\hat{p}_{\boldsymbol{k}}$ using multiple rank-1 lattices [Kämmerer '16] [Kämmerer '17]




remaining freq. from $I$:

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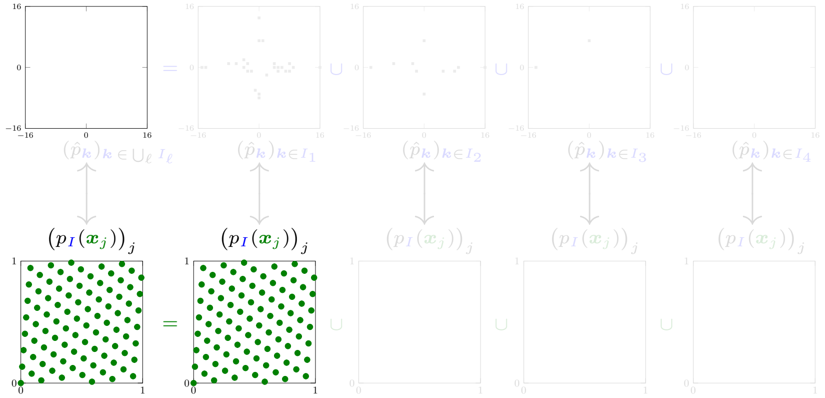


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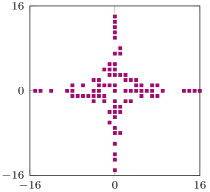

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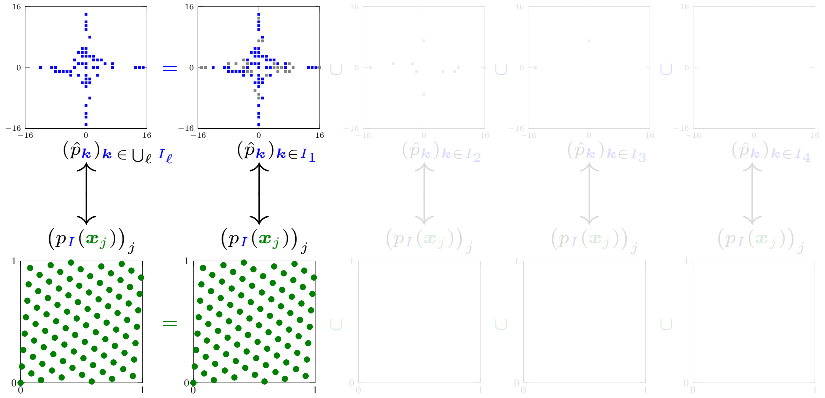


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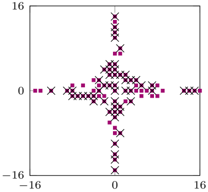

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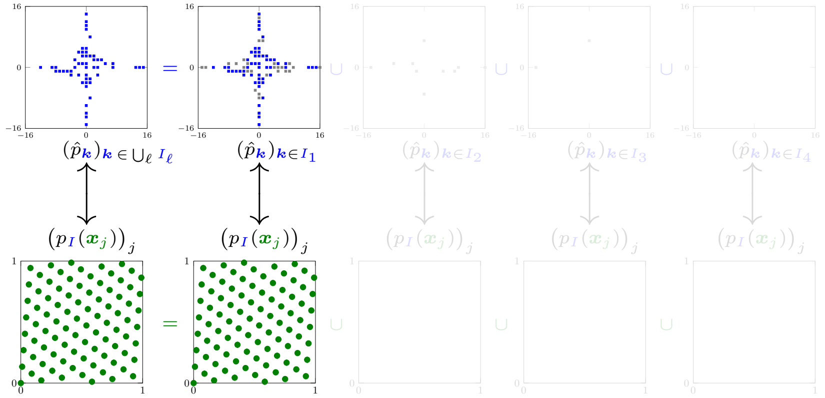


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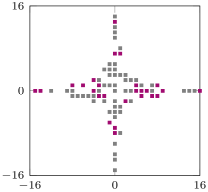


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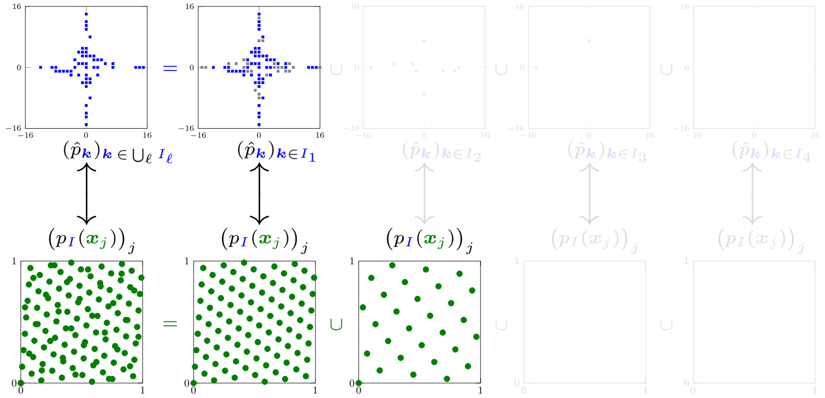


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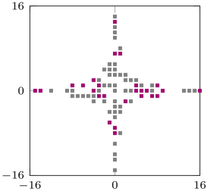

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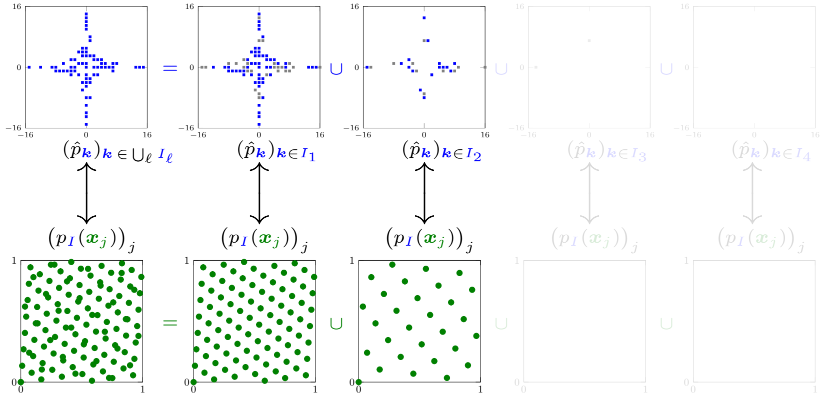


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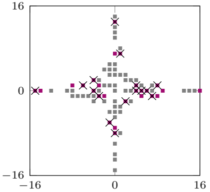


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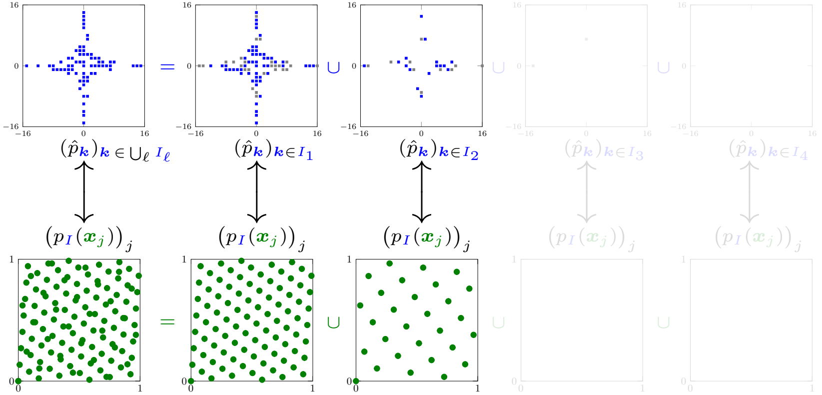


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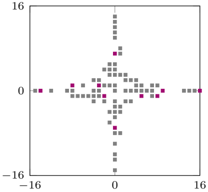


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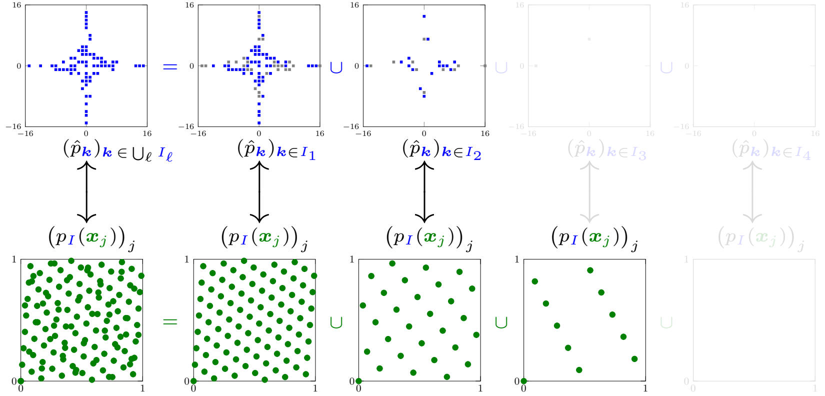
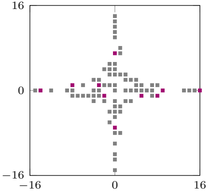



remaining freq. from $I$:



- number of samples: $\mathcal{O}\big(|I| \log^2 |I|\big)$
  (w.h.p., constants independent from $d$)
- construction complexity: $\mathcal{O}\big(|I|\,(d + \log|I|)\log^3|I|\big)$
  (w.h.p., constants independent from $d$)
- reconstruction $(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I}$: $\mathcal{O}\big(|I|\,(d + \log|I|)\log^3|I|\big)$

# Multivariate trigonometric polynomials $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \approx f(\boldsymbol{x})$

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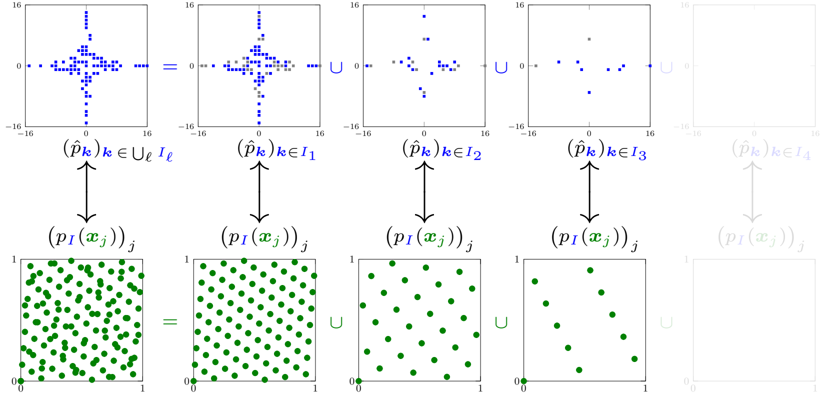
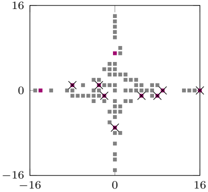



remaining freq. from $I$:



- number of samples: $\mathcal{O}\big(|I| \log^2 |I|\big)$ (w.h.p., constants independent from $d$)
- construction complexity: $\mathcal{O}\big(|I|\,(d + \log|I|)\log^3|I|\big)$ (w.h.p., constants independent from $d$)
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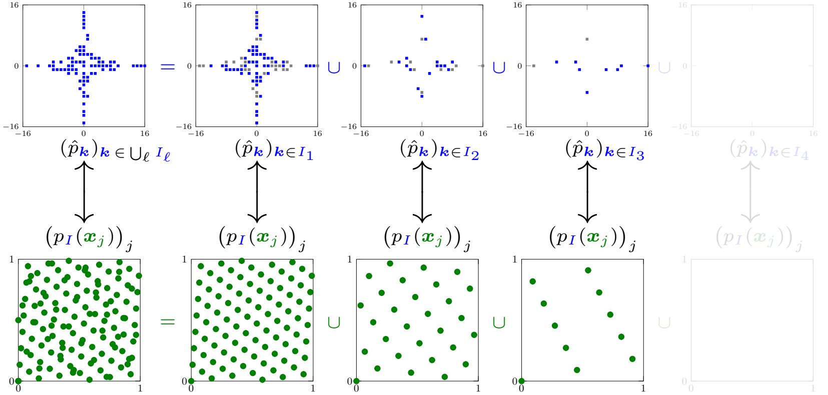
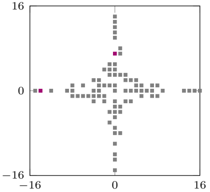



remaining freq. from $I$:



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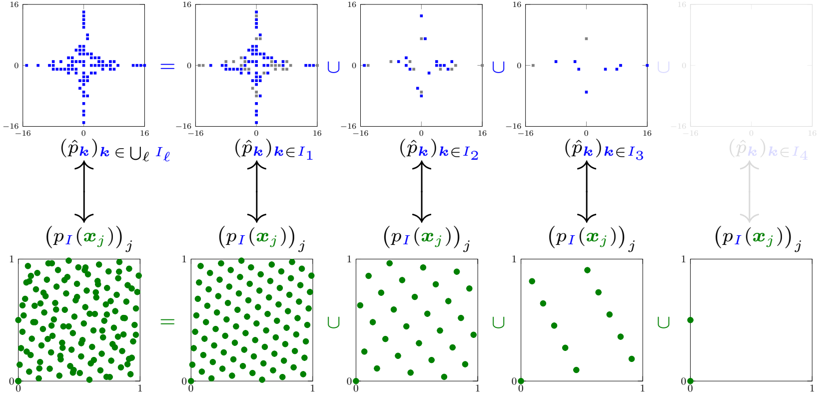


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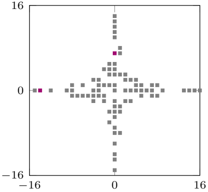


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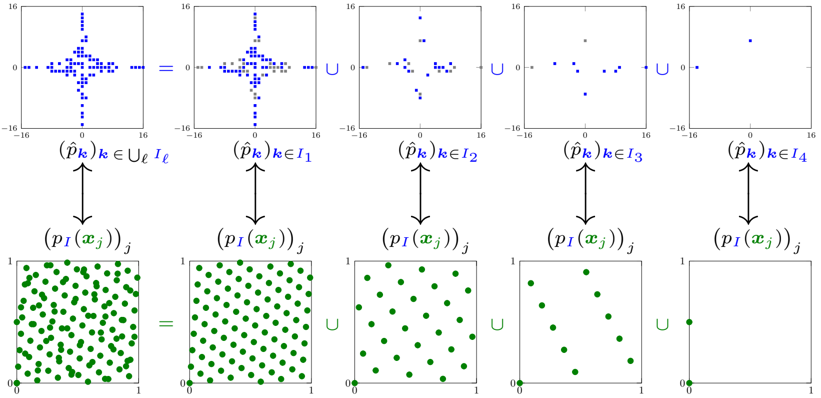


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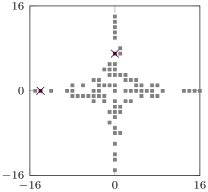

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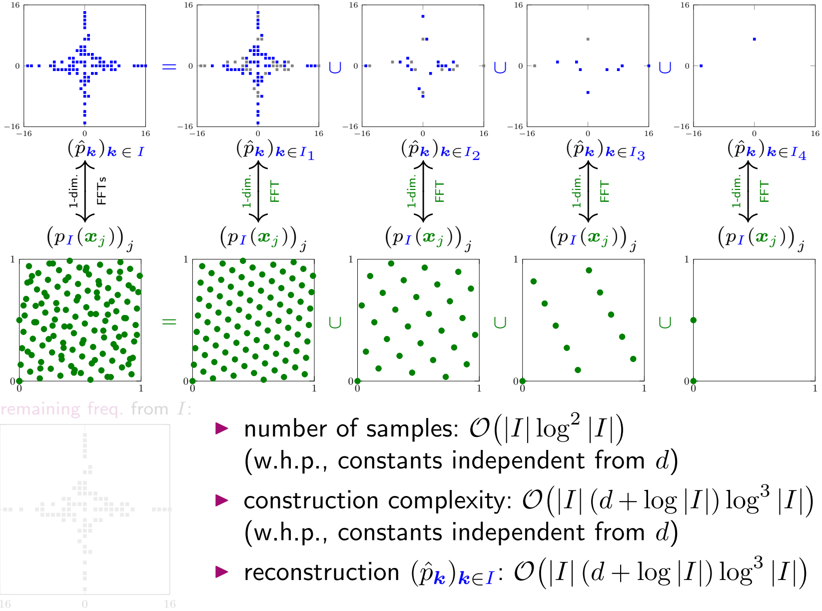


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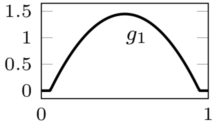
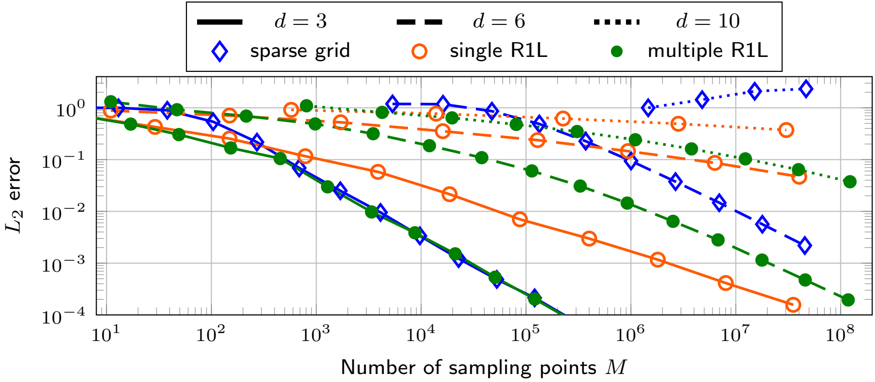
kink function $g_d \colon \mathbb{T}^d \to \mathbb{R}$,

$$g_d(\boldsymbol{x}) = \prod_{s=1}^{d} \left( \frac{5^{3/4} 15}{4\sqrt{3}} \max\left\{ \frac{1}{5} - \left(x_s - \frac{1}{2}\right)^2, 0 \right\} \right)$$




- error estimates for (multiple) rank-1 lattice sampling in [Byrenheid, Kämmerer, Ullrich, V. '17] [V. '17] [Kämmerer, V. '18]

# Sampling error when using (multiple) rank-1 lattices

$p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \approx f(\boldsymbol{x})$

| Complexities: | multiple rank-1 lattice | single rank-1 lattice |
|---|---|---|
| samples | $\leq C \, \lvert I \rvert \log^2 \lvert I \rvert$ (w.h.p.) | $\leq C \, \lvert I \rvert^2$ |
| find lattice | $\leq C \, \lvert I \rvert \, (d + \log \lvert I \rvert) \log^3 \lvert I \rvert$ | $\leq C \, d \, \lvert I \rvert^3$ |
| reconstr./approx. | $\leq C \, \lvert I \rvert \, (d + \log \lvert I \rvert) \log^3 \lvert I \rvert$ | $\leq C \, \lvert I \rvert^2 \, (d + \log \lvert I \rvert)$ |

$\mathrm{err} := \frac{\lVert f - p_I \vert L_2(\mathbb{T}^d) \rVert}{\lVert f \vert \mathcal{H}^{\beta}_{\mathrm{mix}}(\mathbb{T}^d) \rVert}, \ \beta > 1/2, \varepsilon > 0, \quad I := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \prod_{s=1}^d \max(1, \lvert k_s \rvert) \leq N\}$

| best approximation | multiple R1L [Kämmerer, V. '18] | single R1L [Byrenheid, Kämmerer, Ullrich, V. '17] |
|---|---|---|
| $\asymp N^{-\beta}$ | $\lesssim N^{-\beta + \frac{1}{2} + \varepsilon} \log N$ | $\lesssim N^{-\beta} \log^{\frac{d-1}{2}} N$ |
| $\asymp \lvert I \rvert^{-\beta} (\log \lvert I \rvert)^{(d-1)\beta}$ | $\lesssim \lvert I \rvert^{-\beta + \frac{1}{2} + \varepsilon} \cdot \log^{(\beta - \frac{1}{2} - \varepsilon)(d-1)+1} \lvert I \rvert$ | $\lesssim \lvert I \rvert^{-\beta} (\log \lvert I \rvert)^{(d-1)(\beta + \frac{1}{2})}$ |
| $\asymp M^{-\beta} (\log M)^{(d-1)\beta}$ | $\lesssim M^{-\beta + \frac{1}{2} + \varepsilon} \cdot \log^{(\beta - \frac{1}{2} - \varepsilon)d+1} M$ | $M^{-\frac{\beta}{2}} \lesssim \mathrm{err} \lesssim M^{-\frac{\beta}{2}} (\log M)^{\frac{d-2}{2}\beta + \frac{d-1}{2}}$ |

- $\mathcal{H}^{\beta}_{\mathrm{mix}}(\mathbb{T}^d) := \left\{ f \in L_2 \colon \lVert f \rVert := \sqrt{\sum_{\boldsymbol{k} \in \mathbb{Z}^d} \lvert \hat{f}_{\boldsymbol{k}} \rvert^2 \prod_{s=1}^d \max(1, \lvert k_s \rvert)^{2\beta}} < \infty \right\}, \ \beta \geq 0$

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$p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\cdot\boldsymbol{x}} \approx f(\boldsymbol{x})$

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$$\mathrm{err} := \frac{\|f-p_I|L_2(\mathbb{T}^d)\|}{\|f|\mathcal{H}^\beta_{\mathrm{mix}}(\mathbb{T}^d)\|},\ \beta > 1/2, \varepsilon > 0, \quad I := \{\boldsymbol{k}\in\mathbb{Z}^d \colon \prod_{s=1}^d \max(1,|k_s|) \leq N\}$$

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| $\asymp \lvert I\rvert^{-\beta}(\log\lvert I\rvert)^{(d-1)\beta}$ | $\lesssim \lvert I\rvert^{-\beta+\frac{1}{2}+\varepsilon} \cdot \log^{(\beta-\frac{1}{2}-\varepsilon)(d-1)+1}\lvert I\rvert$ | $\lesssim \lvert I\rvert^{-\beta}(\log\lvert I\rvert)^{(d-1)(\beta+\frac{1}{2})}$ |
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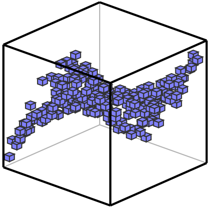
- $\mathcal{H}^{\beta}_{\mathrm{mix}}(\mathbb{T}^d) := \left\{ f \in L_2\colon \|f\| := \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}|^2 \prod_{s=1}^d \max(1,|k_s|)^{2\beta}} < \infty \right\}$, $\beta \geq 0$

first part:

- fast reconstruction of arbitrary high-dimensional trigonometric polynomials $f(\boldsymbol{x}) = p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ using 1-dimensional FFTs

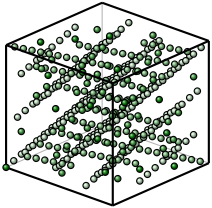
general known frequency index set $I \subset \mathbb{Z}^d$

spatial domain: multiple rank-1 lattice



$$\left(\hat{p}_{\boldsymbol{k}}\right)_{\boldsymbol{k}\in I} \xleftrightarrow[\text{FFTs}]{\text{1-dim.}} \left(f(\boldsymbol{x}_j)\right)_{j=0}^{M-1}$$

$$\mathcal{O}\left(|I|\,(d + \log|I|)\log^3|I|\right)$$



- fast approximation $f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ of functions from samples

## second part:

- unknown frequency index set $I$ / weights / function space in high dimensions

$\Rightarrow$ dimension-incremental sparse FFT using multiple rank-1 lattices

next: unknown frequency index set $I$ / weights / function space

$f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, find $\hat{p}_{\boldsymbol{k}}$ AND $I \subset \mathbb{Z}^d$ from samples of $f$

$\Rightarrow$ multi-dimensional sparse FFT

- task: determine frequency index set $I$ (out of search domain $\Gamma \subset \mathbb{Z}^d$) from samples belonging to ≈largest Fourier coefficients $\hat{f}_{\boldsymbol{k}}$ or to $\hat{f}_{\boldsymbol{k}} \neq 0$
- various existing methods, e.g., based on
  filters [Indyk, Kapralov '14] / Chinese Remainder Theorem [Cuyt, Lee '08] [Iwen '13] /
  Prony's method [Tasche, Potts '13] [Peter, Plonka, Schaback '15] [Kunis, Peter, Römer, von der Ohe '15]
- problems: non-sparsity, implementations?, stability, many frequencies

$\Rightarrow$ dimension-incremental sparse FFT based on (multiple) rank-1 lattices
[Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17]
(similar basic idea without rank-1 lattices: [Zippel '79] [Kaltofen, Lee '03] [Javadi, Monagan '10] [Potts, Tasche '13])

$\Rightarrow$ "Sparse Harmonic Transforms: A New Class of Sublinear-time Algorithms for Learning Functions of Many Variables" [Choi, Iwen, Krahmer '18]

next: unknown frequency index set $I$ / weights / function space

$$f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}},$$ find $\hat{p}_{\boldsymbol{k}}$ AND $I \subset \mathbb{Z}^d$ from samples of $f$

$\Rightarrow$ multi-dimensional sparse FFT

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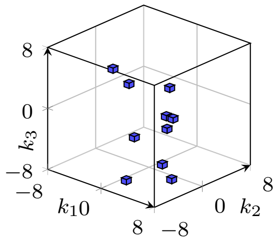



$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$

$$\hat{p}_{k_1} := \frac{1}{17} \sum_{\ell=0}^{16} p\left(\begin{pmatrix} \ell/17 \\ x_2' \\ x_3' \end{pmatrix}\right) \mathrm{e}^{-2\pi\mathrm{i}\frac{\ell k_1}{17}}$$

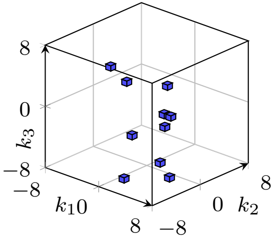
$k_1 = -8, \ldots, 8$




frequency candidates




sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$

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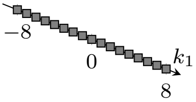


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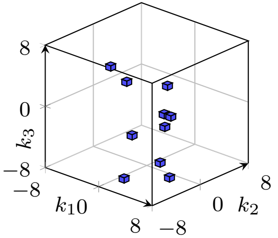


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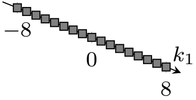
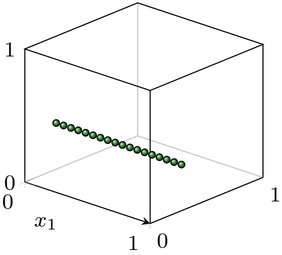
$$\hat{p}_{k_1} := \frac{1}{17} \sum_{\ell=0}^{16} p\left(\begin{pmatrix} \ell/17 \\ x_2' \\ x_3' \end{pmatrix}\right) \mathrm{e}^{-2\pi\mathrm{i}\frac{\ell k_1}{17}}$$

$k_1 = -8, \ldots, 8$

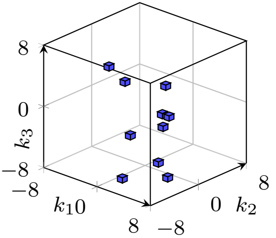



frequency candidates

construct $\longrightarrow$ sampling set




sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$

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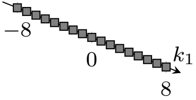
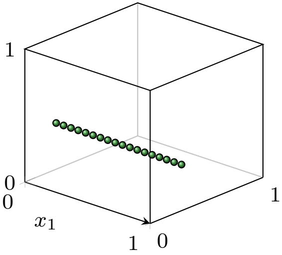
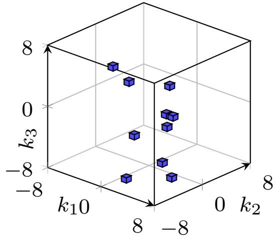



frequency candidates

1-dim. ← FFT




sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$

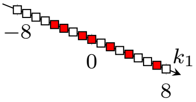



$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$

$$\hat{p}_{k_1} := \frac{1}{17}\sum_{\ell=0}^{16} p\left(\begin{pmatrix}\ell/17\\ x_2'\\ x_3'\end{pmatrix}\right) \mathrm{e}^{-2\pi\mathrm{i}\frac{\ell k_1}{17}}$$

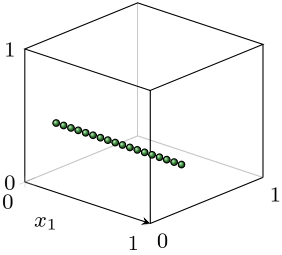
$$= \sum_{\substack{(h_2,h_3)\in\{-8,\ldots,8\}^2\\ (k_1,h_2,h_3)^\top\in\operatorname{supp}\hat{p}}} \hat{p}_{\begin{pmatrix}k_1\\ h_2\\ h_3\end{pmatrix}}\, \mathrm{e}^{2\pi\mathrm{i}(h_2x_2'+h_3x_3')},$$

$k_1 = -8, \ldots, 8$

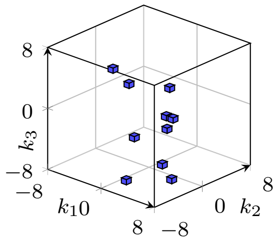



detected frequencies $I^{(1)} = \mathcal{N}_0$

1-dim.
⟵
FFT



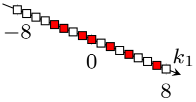

sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$




$I \subset \Gamma = \hat{G}_8^3 := \{-8,-7,\dots,8\}$

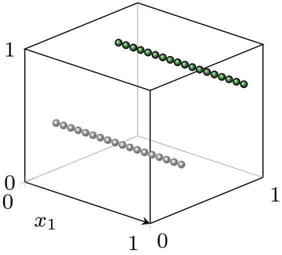
$$\hat{p}_{k_1} := \frac{1}{17}\sum_{\ell=0}^{16} p\left(\begin{pmatrix}\ell/17\\ x_2'\\ x_3'\end{pmatrix}\right) \mathrm{e}^{-2\pi\mathrm{i}\frac{\ell k_1}{17}}$$
$$= \sum_{\substack{(h_2,h_3)\in\{-8,\dots,8\}^2\\ (k_1,h_2,h_3)^\top\in\operatorname{supp}\hat{p}}} \hat{p}\begin{pmatrix}k_1\\h_2\\h_3\end{pmatrix} \mathrm{e}^{2\pi\mathrm{i}(h_2x_2'+h_3x_3')},$$

$k_1 = -8,\dots,8$




detected frequencies $I^{(1)} = \mathcal{N}_0$

1-dim.
$\longleftarrow$
FFT

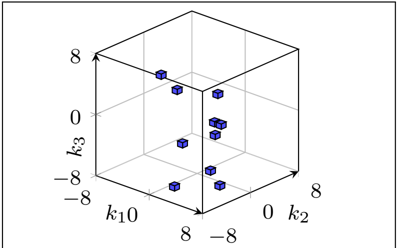



sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$

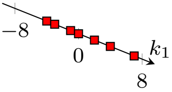
+ repeat ($r$ detection iterations)

# High-dimensional dimension-incremental sparse FFT

## Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$




$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$




detected frequencies

$I^{(1)} = \mathcal{N}_0 = \mathcal{P}_0$

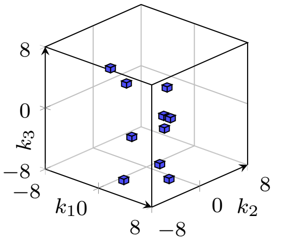


frequency candidates

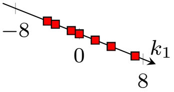


sampling nodes

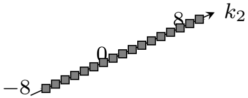
## High-dimensional dimension-incremental sparse FFT
## Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$




$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \dots, 8\}$



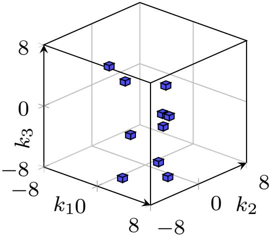

detected frequencies
$I^{(1)} = \mathcal{N}_0 = \mathcal{P}_0$



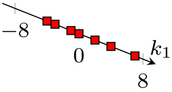

frequency candidates



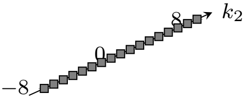

sampling nodes




$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$

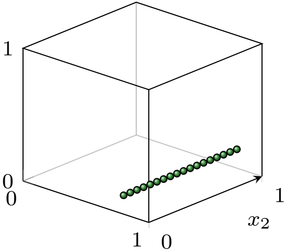



detected frequencies
$I^{(1)} = \mathcal{N}_0 = \mathcal{P}_0$




frequency candidates

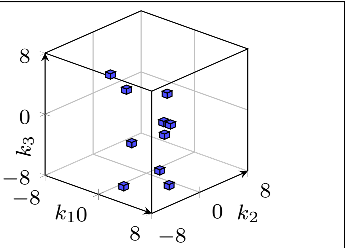
construct $\longrightarrow$ sampling set



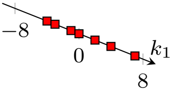

sampling nodes

# High-dimensional dimension-incremental sparse FFT

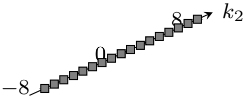
## Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



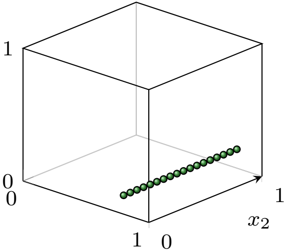

$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$



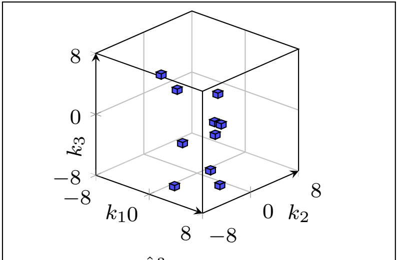

detected frequencies
$I^{(1)} = \mathcal{N}_0 = \mathcal{P}_0$



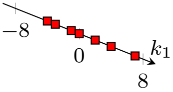

frequency candidates

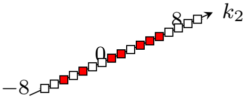

sampling nodes




$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$

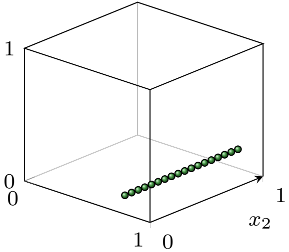



detected frequencies
$I^{(1)} = \mathcal{N}_0 = \mathcal{P}_0$

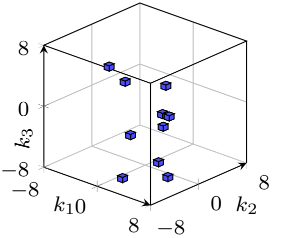



detected frequencies $I^{(2)} = \mathcal{N}_1$

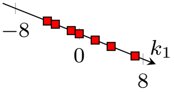
1-dim.
$\longleftarrow$
FFT



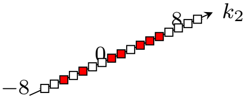

sampling nodes




$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$

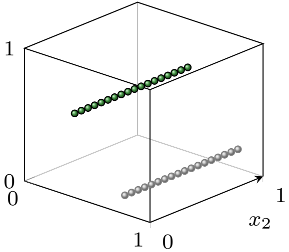



detected frequencies
$I^{(1)} = \mathcal{N}_0 = \mathcal{P}_0$



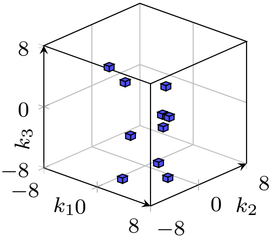

detected frequencies $I^{(2)} = \mathcal{N}_1$

1-dim. ⟵ FFT

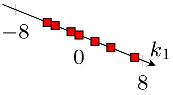



sampling nodes

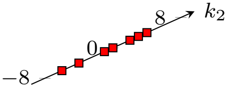
+ repeat ($r$ detection iterations)




$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$



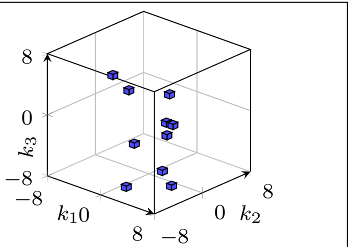

detected frequencies
$I^{(1)} = \mathcal{N}_0 = \mathcal{P}_0$



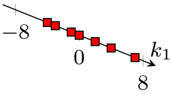

detected frequencies
$I^{(2)} = \mathcal{N}_1$



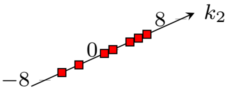

frequency candidates
$I^{(1)} \times I^{(2)} = \mathcal{P}_0 \times \mathcal{N}_1$



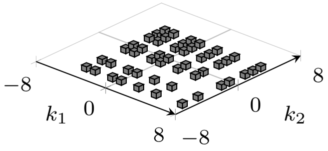

$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$



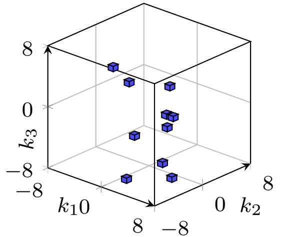

detected frequencies
$I^{(1)} = \mathcal{N}_0 = \mathcal{P}_0$



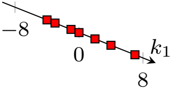

detected frequencies
$I^{(2)} = \mathcal{N}_1$



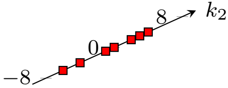

frequency candidates
$I^{(1)} \times I^{(2)} = \mathcal{P}_0 \times \mathcal{N}_1$



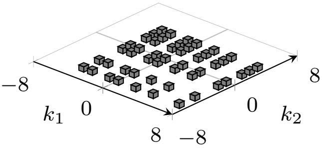

$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$




detected frequencies
$I^{(1)} = \mathcal{N}_0 = \mathcal{P}_0$

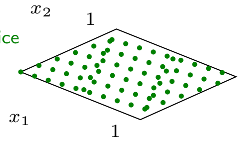



detected frequencies
$I^{(2)} = \mathcal{N}_1$




frequency candidates
$I^{(1)} \times I^{(2)} = \mathcal{P}_0 \times \mathcal{N}_1$

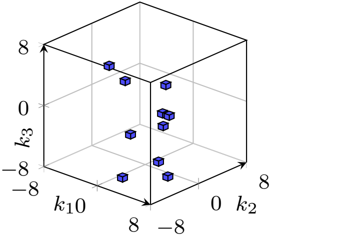
reconstructing
$\longrightarrow$
multiple rank-1 lattice



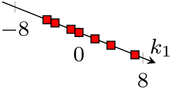

sampling nodes

# High-dimensional dimension-incremental sparse FFT

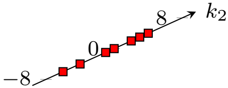
## Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



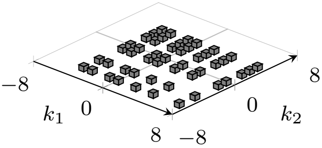

$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$




detected frequencies
$I^{(1)} = \mathcal{N}_0 = \mathcal{P}_0$

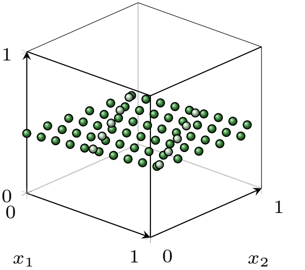



detected frequencies
$I^{(2)} = \mathcal{N}_1$

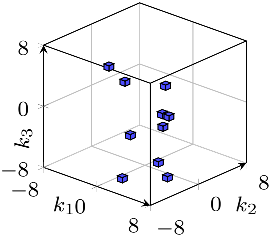



frequency candidates
$I^{(1)} \times I^{(2)} = \mathcal{P}_0 \times \mathcal{N}_1$

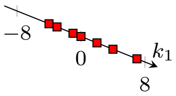
1-dim. ⟵ FFTs



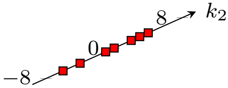

sampling nodes



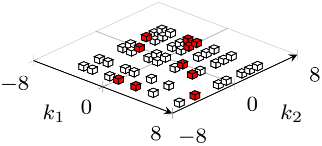

$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$




detected frequencies
$I^{(1)} = \mathcal{N}_0 = \mathcal{P}_0$

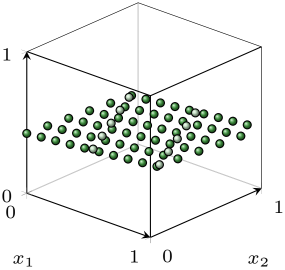



detected frequencies
$I^{(2)} = \mathcal{N}_1$



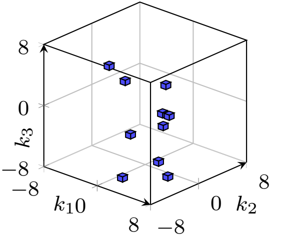

detected frequencies $I^{(1,2)} = \mathcal{P}_1$

1-dim.
⟵
FFTs

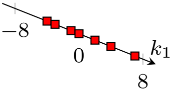



sampling nodes

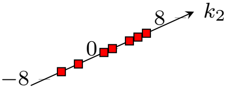
## Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



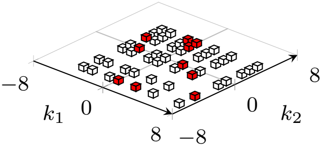

$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$




detected frequencies $I^{(1)} = \mathcal{N}_0 = \mathcal{P}_0$

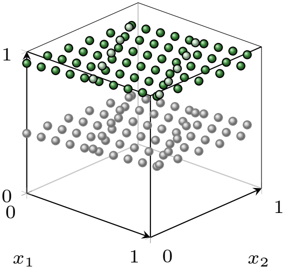



detected frequencies $I^{(2)} = \mathcal{N}_1$




detected frequencies $I^{(1,2)} = \mathcal{P}_1$

1-dim. ← FFTs

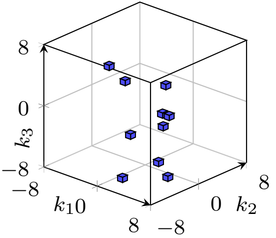



sampling nodes

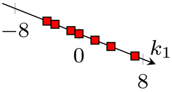
**+ repeat** ($r$ detection iterations)

# High-dimensional dimension-incremental sparse FFT

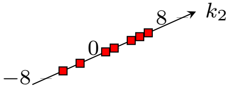
Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



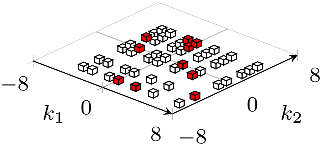

$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$




detected frequencies $I^{(1)} = \mathcal{N}_0 = \mathcal{P}_0$

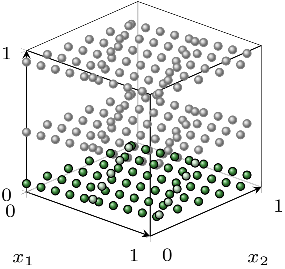



detected frequencies $I^{(2)} = \mathcal{N}_1$



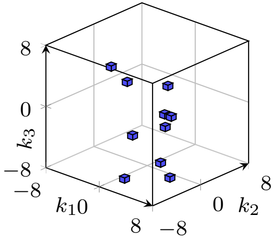

detected frequencies $I^{(1,2)} = \mathcal{P}_1$

1-dim. ← FFTs

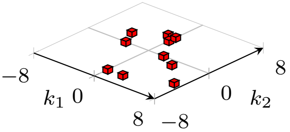



sampling nodes

**+ repeat** ($r$ detection iterations)




$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$



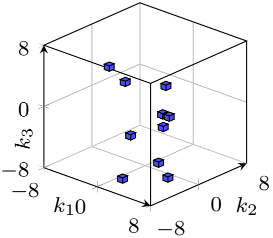

detected frequencies
$I^{(1,2)} = \mathcal{P}_1$



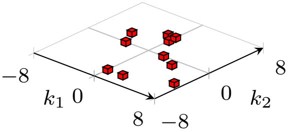

frequency candidates



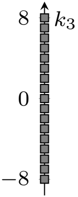

sampling nodes




$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$



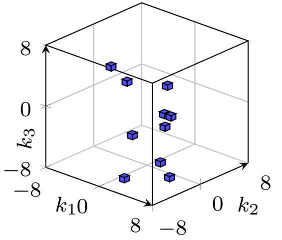

detected frequencies
$I^{(1,2)} = \mathcal{P}_1$



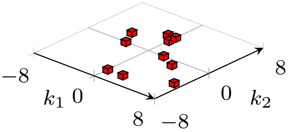

frequency candidates



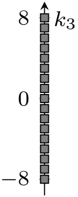

sampling nodes




$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$

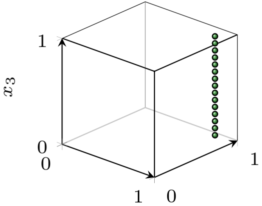



detected frequencies
$I^{(1,2)} = \mathcal{P}_1$

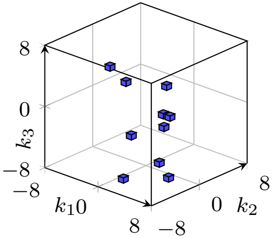



frequency candidates

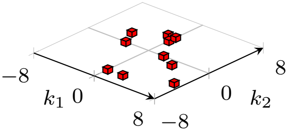
construct $\longrightarrow$ sampling set




sampling nodes

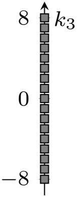



$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$



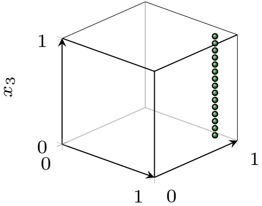

detected frequencies

$I^{(1,2)} = \mathcal{P}_1$

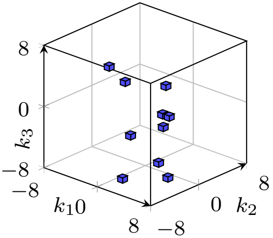



frequency candidates

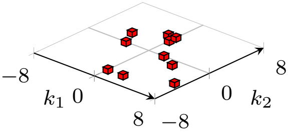
1-dim. FFT ⟵



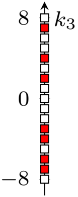

sampling nodes




$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$

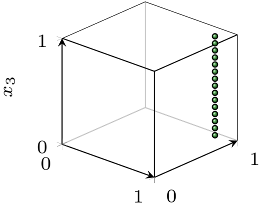



detected frequencies
$I^{(1,2)} = \mathcal{P}_1$

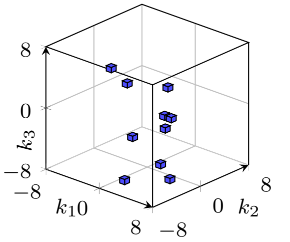



detected frequencies $I^{(3)} = \mathcal{N}_2$

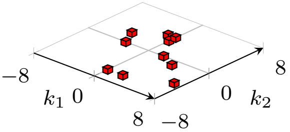
1-dim. $\longleftarrow$ FFT



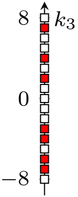

sampling nodes



$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$

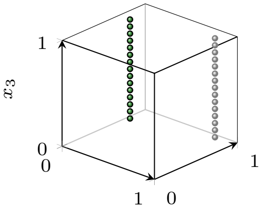


detected frequencies
$I^{(1,2)} = \mathcal{P}_1$



detected frequencies $I^{(3)} = \mathcal{N}_2$

1-dim. $\longleftarrow$ FFT

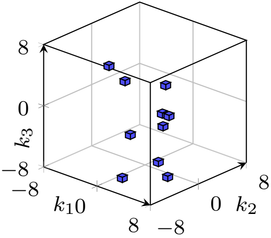


sampling nodes

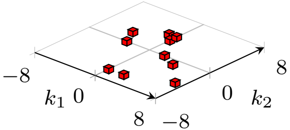
$+$ repeat ($r$ detection iterations)

# High-dimensional dimension-incremental sparse FFT

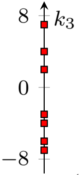
Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$




$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$




detected frequencies
$I^{(1,2)} = \mathcal{P}_1$




$I^{(3)} = \mathcal{N}_2$



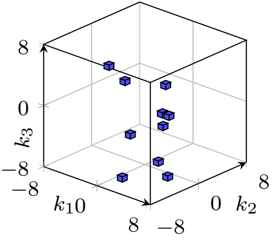

frequency candidates
$I^{(1,2)} \times I^{(3)} = \mathcal{P}_2 \times \mathcal{N}_3$



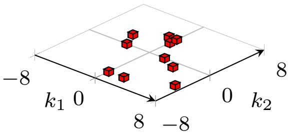

sampling nodes

# High-dimensional dimension-incremental sparse FFT

Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$

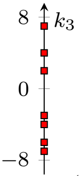



$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \dots, 8\}$

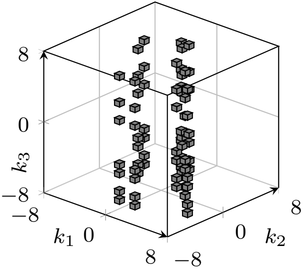



detected frequencies

$I^{(1,2)} = \mathcal{P}_1$




$I^{(3)} = \mathcal{N}_2$




frequency candidates

$I^{(1,2)} \times I^{(3)} = \mathcal{P}_2 \times \mathcal{N}_3$




sampling nodes

# High-dimensional dimension-incremental sparse FFT

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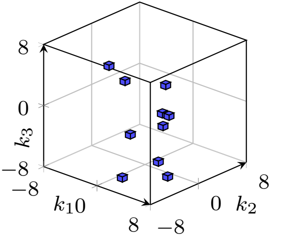


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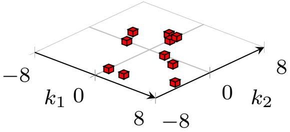
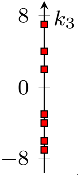
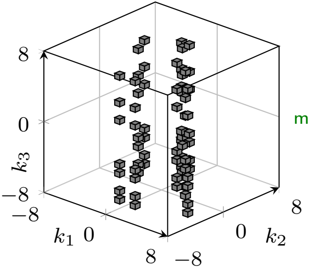



detected frequencies

$I^{(1,2)} = \mathcal{P}_1$



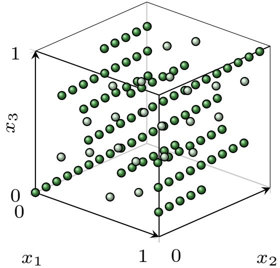

$I^{(3)} = \mathcal{N}_2$




frequency candidates

$I^{(1,2)} \times I^{(3)} = \mathcal{P}_2 \times \mathcal{N}_3$

reconstructing $\longrightarrow$ multiple rank-1 lattice




sampling nodes

# High-dimensional dimension-incremental sparse FFT

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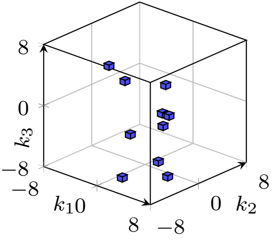


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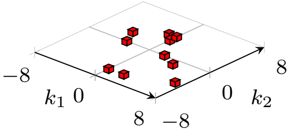
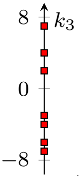
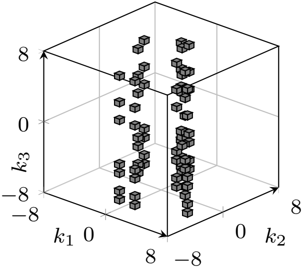



detected frequencies

$I^{(1,2)} = \mathcal{P}_1$



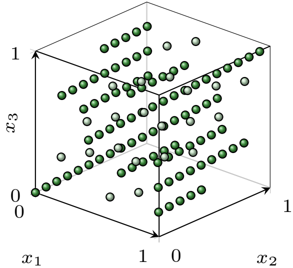

$I^{(3)} = \mathcal{N}_2$




frequency candidates

$I^{(1,2)} \times I^{(3)} = \mathcal{P}_2 \times \mathcal{N}_3$

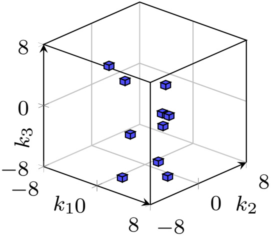
1-dim.
$\longleftarrow$
FFTs



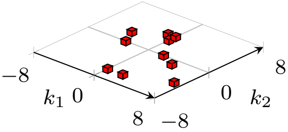

sampling nodes

# High-dimensional dimension-incremental sparse FFT

Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$

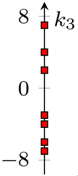



$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$

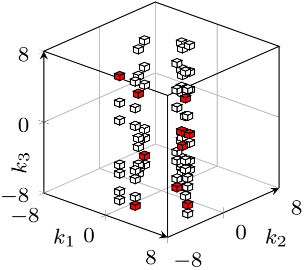



detected frequencies

$I^{(1,2)} = \mathcal{P}_1$



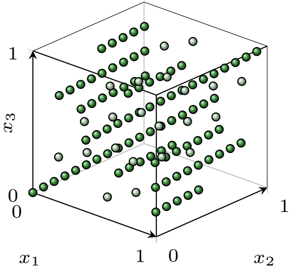

$I^{(3)} = \mathcal{N}_2$




detected frequencies

$I^{(1,2,3)} = \mathcal{P}_2 = I$

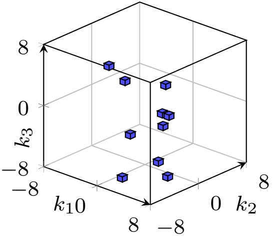
1-dim. $\longleftarrow$ FFTs



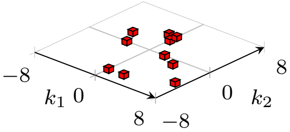

sampling nodes

# High-dimensional dimension-incremental sparse FFT

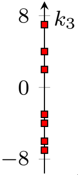
## Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



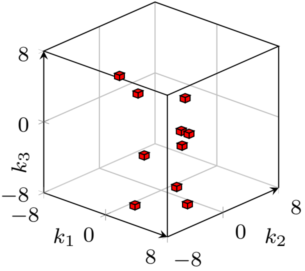

$I \subset \Gamma = \hat{G}_8^3 := \{-8,-7,\ldots,8\}$




detected frequencies
$I^{(1,2)} = \mathcal{P}_1$

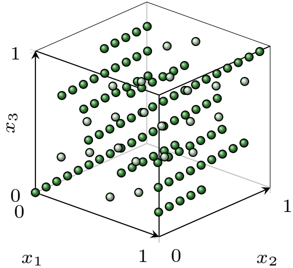



$I^{(3)} = \mathcal{N}_2$




detected frequencies
$I^{(1,2,3)} = \mathcal{P}_2 = I$

1-dim. ⟵ FFTs




sampling nodes

reconstruction of $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ with unknown $I$ using rank-1 lattices:

- sparsity $s = |I|$, search domain $\Gamma = \hat{G}_N^d := \{-N,\dots,N\}^d \supset I$, number of detection iterations $r$

| | single rank-1 lattices | multiple rank-1 lattices |
|---|---|---|
| samples | $\mathcal{O}(d\,r^2 s^2 N)$ | $\mathcal{O}(d\,r s N \log^2(rsN))$ (w.h.p.) |
| arithmetic op. | $\mathcal{O}(d\,r^3 s^3 + d\,r^2 s^2 N \log(rsN))$ | $\mathcal{O}\big(d^2 r s N \log^4(rsN)\big)$ (w.h.p.) |

- if $(\mathrm{Re}(\hat{p}_{\boldsymbol{k}})$ identical sign) AND $(\mathrm{Im}(\hat{p}_{\boldsymbol{k}})$ identical sign) $\Rightarrow r = 1$; otherwise
  - in theory: $r = 2s(\log 3 + \log d + \log s - \log \varepsilon)$ for failure probability $\varepsilon \in (0,1)$
  - $\Rightarrow$ samples: $\mathcal{O}(d\,s^3 \log^3(\dots))$ for $s \gtrsim N$
  - in practice: $r = 1$ or 2
- can be applied for approximate reconstruction of a function $f \in L_2(\mathbb{T}^d) \cap C(\mathbb{T}^d)$
- MATLAB implementation
- numerically tested for up to 30 spatial dimensions

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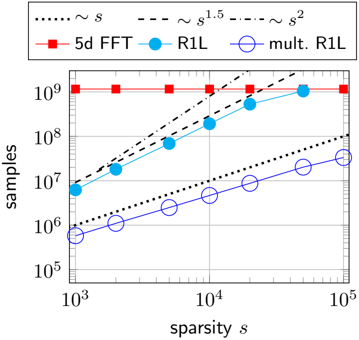
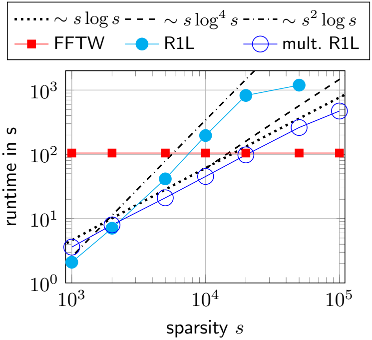
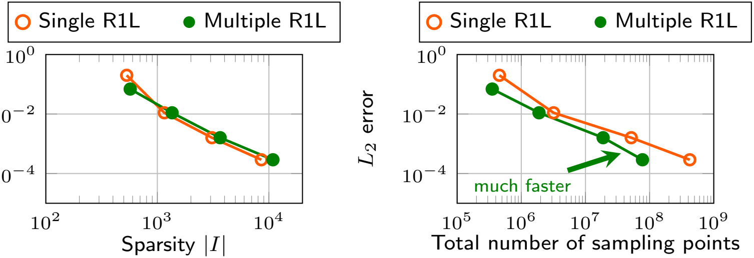
example: $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, $I \subset \Gamma = \hat{G}_{32}^5 := \{-32,\dots,32\}^5$, $|I| = s$,

$|\Gamma| \approx 1.16 \cdot 10^9$

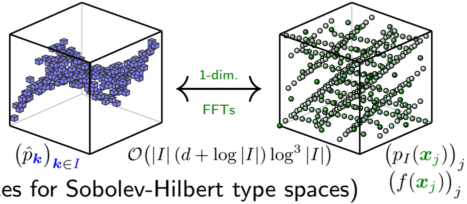


- $f(\boldsymbol{x}) := \prod_{t\in\{1,3,8\}} B_2(x_t) + \prod_{t\in\{2,5,6,10\}} B_4(x_t) + \prod_{t\in\{4,7,9\}} B_6(x_t)$,
  $B_m(x) = \sum_{k\in\mathbb{Z}} C_m \operatorname{sinc}\left(\frac{\pi}{m}k\right)^m (-1)^k \mathrm{e}^{2\pi\mathrm{i}kx}$
  univariate B-spline of order $m \in \mathbb{N}$
- dimension-incremental sparse FFT for $\Gamma = \hat{G}_{64}^{10}$ ($|\hat{G}_{64}^{10}| \approx 1.28 \cdot 10^{21}$), $r = 10$:



- known frequency index set $I \subset \mathbb{Z}^d$, multiple rank-1 lattice
  - fast reconstruction of high-dim. trigonometric polynomials $p_I$ [Kämmerer '16] [Kämmerer '17]
  - fast approximation (error estimates for Sobolev-Hilbert type spaces) [Kämmerer, Potts, V. '15] [Byrenheid, Kämmerer, Ullrich, V. '17] [V. '17] [Kämmerer, V. '18]




- **unknown** $I \subset \mathbb{Z}^d$, sampling along (multiple) rank-1 lattices
  - high-dimensional dimension-incremental sparse FFT [Potts, V. '16] [V. '17] [Kämmerer, V. '17]
  - very good numerical results for high-dimensional sparse trigonometric polynomials and for high-dimensional functions (non-sparse in frequency domain)
- can be transfered to non-periodic case (tensor product Chebyshev bases)
- see also

L. Kämmerer, D. Potts, T. V. **High-dimensional sparse FFT based on sampling along multiple rank-1 lattices**. *ArXiv e-prints* 1711.05152, Nov. 2017.

L. Kämmerer, T. V. **Approximation of multivariate periodic functions based on sampling along multiple rank-1 lattices**. *ArXiv e-prints* 1802.06639, Feb. 2018.

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- L. Kämmerer, T. V. **Approximation of multivariate periodic functions based on sampling along multiple rank-1 lattices**. *ArXiv e-prints* 1802.06639, Feb. 2018.


Software: MATLAB toolboxes (for single rank-1 lattices) `https://www.tu-chemnitz.de/~tovo`