High-dimensional approximation and sparse FFT using (multiple) rank-1 lattices

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joint work with Lutz Kämmerer, Daniel Potts, and Tino Ullrich

- torus $\mathbb{T} \simeq [0,1)$, $\{\mathrm{e}^{2\pi\mathrm{i}kx}\}_{k\in\mathbb{Z}}$ orthonormal basis of $L_2(\mathbb{T})$
- function $f \in L_2(\mathbb{T})$, $f(x) = \sum_{k\in\mathbb{Z}} \hat{f}_k\, \mathrm{e}^{2\pi\mathrm{i}kx}$, $\hat{f}_k = \int_{\mathbb{T}} f(x)\, \mathrm{e}^{-2\pi\mathrm{i}kx}\, \mathrm{d}x \in \mathbb{C}$
- smooth function $f \implies$ fast decay of Fourier coefficients $\hat{f}_k$
- truncated Fourier series $S_I f(x) = \sum_{k\in I} \hat{f}_k\, \mathrm{e}^{2\pi\mathrm{i}kx} \approx f(x)$
- $\hat{f}_k = \int_{\mathbb{T}} f(x)\, \mathrm{e}^{-2\pi\mathrm{i}kx}\, \mathrm{d}x \approx \tilde{\hat{f}}_k := \frac{1}{2N} \sum_{j=0}^{2N-1} f(x_j)\, \mathrm{e}^{-2\pi\mathrm{i}kx_j}$, $x_j := \frac{j}{2N}$

$\Rightarrow$ transfer to multivariate case (tensorization)

$(\tilde{\hat{f}}_k)_{k\in I}$ $\xleftrightarrow[\text{FFT}]{\text{1-dim.}}$ $(f(x_j))_{j=0}^{2N-1}$

$\mathcal{O}(N \log N)$

[Gauß 1866] [Cooley, Tukey 1965]

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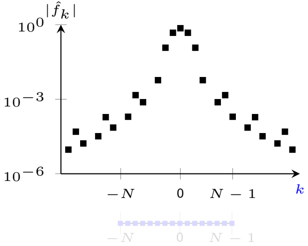

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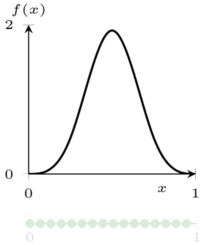

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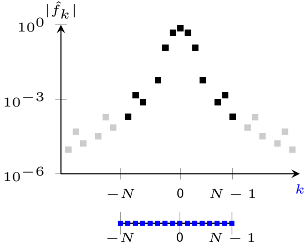

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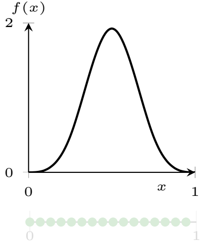

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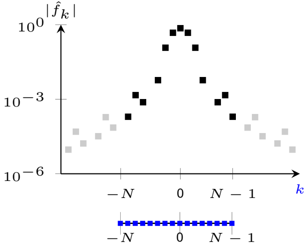

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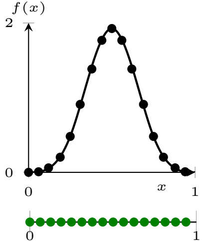

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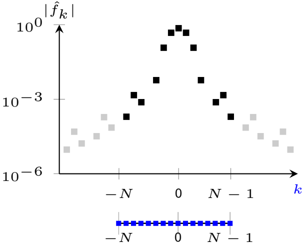

$(\tilde{\hat{f}}_k)_{k\in I}$ ⟷ (1-dim. FFT) $(f(x_j))_{j=0}^{2N-1}$

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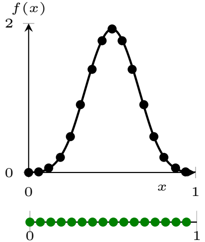
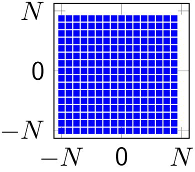


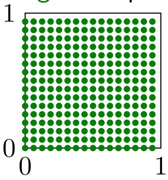
- full grid in frequency domain — equispaced full grid in spatial domain



$$(\tilde{\hat{f}}_{\boldsymbol{k}})_{\boldsymbol{k}\in I} \xleftrightarrow[\text{FFT}]{d\text{-dim.}} (f(\boldsymbol{x}_j))_{j=0}^{|I|-1}$$

$$\mathcal{O}(N^d \log N)$$

curse of dimensionality



? high-dimensional case (e.g. spatial dimension $d = 10$)

$\Rightarrow$ assumption: sparsity or smoothness

- hyperbolic cross in frequency domain — sparse grid in spatial domain

[Smolyak '63; Temlyakov '85; Baszenski, Delvos '89; Hallatschek '92; Gradinaru '07; Griebel, Hamaekers '14]

$$(\tilde{\hat{f}}_{\boldsymbol{k}})_{\boldsymbol{k}\in I} \xleftrightarrow[\text{HCFFT}]{} (f(\boldsymbol{x}_j))_{j=0}^{|I|-1}$$

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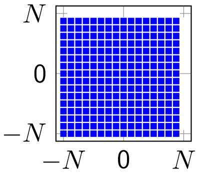
condition number of Fourier matrix
[Kämmerer, Kunis '11]

- full grid in frequency domain — equispaced full grid in spatial domain



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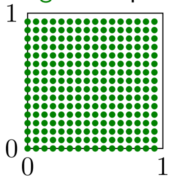

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condition number of Fourier matrix [Kämmerer, Kunis '11]

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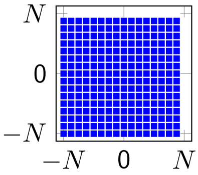

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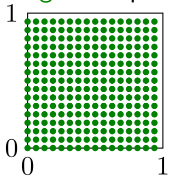

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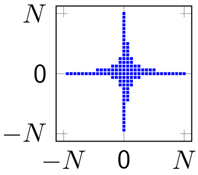

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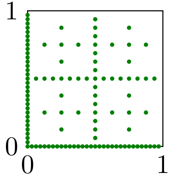
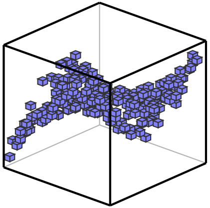


first part:

- fast reconstruction of arbitrary high-dimensional trigonometric polynomials $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\cdot\boldsymbol{x}}$ using 1-dimensional FFTs

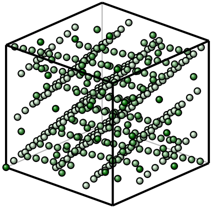
general known frequency index set $I \subset \mathbb{Z}^d$

spatial domain: multiple rank-1 lattice



$$\left(\hat{p}_{\boldsymbol{k}}\right)_{\boldsymbol{k}\in I} \xleftrightarrow[\text{FFTs}]{\text{1-dim.}} \left(f(\boldsymbol{x}_j)\right)_{j=0}^{M-1}$$

$$\mathcal{O}\left(|I|\,(d + \log |I|) \log^3 |I|\right)$$



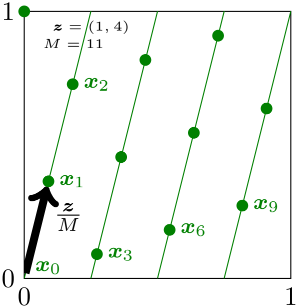
- fast approximation $f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\cdot\boldsymbol{x}}$ of functions from samples

second part:

- unknown frequency index set I / weights / function space in high dimensions

$\Rightarrow$ dimension-incremental sparse FFT using multiple rank-1 lattices

- $f(\boldsymbol{x}) = p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$,
  arbitrary freq. index set $I \subset \mathbb{Z}^d$, $|I| < \infty$
- rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, M) := \{\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1}\}_{j=0}^{M-1}$,
  $\boldsymbol{z} \in \mathbb{Z}^d$, $M \in \mathbb{N}$,
  as discretization in spatial domain



Korobov '59
Maisonneuve '72
Sloan & Kachoyan '84,'87,'90
Temlyakov '86
Lyness '89
Sloan & Joe '94
Sloan & Reztsov '01
Li & Hickernell '03
Kämmerer & Kunis & Potts '12

- $f(\boldsymbol{x}) = p_I(\boldsymbol{x}) = \sum\limits_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$,

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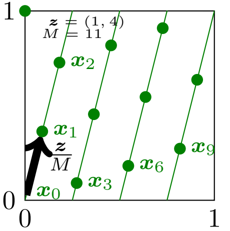
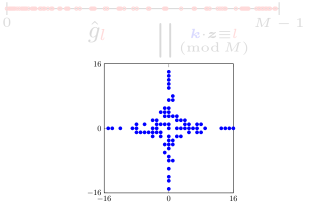
- reformulation

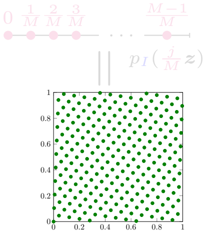


$$p_I(\boldsymbol{x}_j) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\frac{j\boldsymbol{k}\cdot\boldsymbol{z}}{M}} = \sum_{l=0}^{M-1} \underbrace{\left(\sum_{\substack{\boldsymbol{k}\in I \\ \boldsymbol{k}\cdot\boldsymbol{z}\equiv l \pmod M}} \hat{p}_{\boldsymbol{k}} \right)}_{\hat{g}_l} \mathrm{e}^{2\pi\mathrm{i}\frac{j\boldsymbol{k}\cdot\boldsymbol{z}}{M}} = \sum_{l=0}^{M-1} \hat{g}_l\, \mathrm{e}^{2\pi\mathrm{i}\, l \frac{j}{M}}$$



$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I}$

$\longrightarrow$



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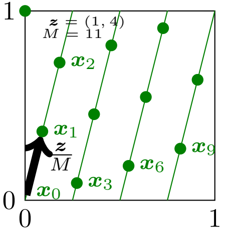
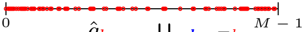
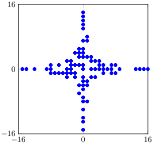


$$p_I(\boldsymbol{x}_j) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\frac{j\boldsymbol{k}\cdot\boldsymbol{z}}{M}} = \sum_{l=0}^{M-1} \underbrace{\left(\sum_{\substack{\boldsymbol{k}\in I \\ \boldsymbol{k}\cdot\boldsymbol{z}\equiv l \pmod{M}}} \hat{p}_{\boldsymbol{k}} \right)}_{\hat{g}_l} \mathrm{e}^{2\pi\mathrm{i}\frac{j\boldsymbol{k}\cdot\boldsymbol{z}}{M}} = \sum_{l=0}^{M-1} \hat{g}_l\, \mathrm{e}^{2\pi\mathrm{i}\, l\frac{j}{M}}$$



$\hat{g}_l$ $\Big|\Big|$ $\boldsymbol{k}\cdot\boldsymbol{z}\equiv l \pmod{M}$

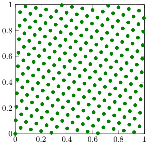


$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I}$

1-dim $\longrightarrow$ FFT

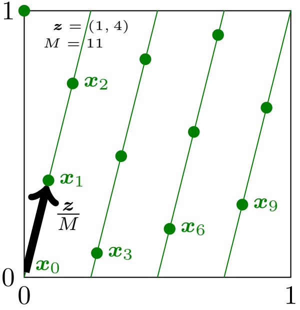
$\mathcal{O}(M \log M + d|I|)$

$\Big|\Big|$ $p_I(\frac{j}{M}\boldsymbol{z})$



$(p(\boldsymbol{x}_j))_{j=0}^{M-1}$

- $f(\boldsymbol{x}) = p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, arbitrary freq. index set $I \subset \mathbb{Z}^d$, $|I| < \infty$
- rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, M) := \{\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1}\}_{j=0}^{M-1}$, $\boldsymbol{z} \in \mathbb{Z}^d$, $M \in \mathbb{N}$, as discretization in spatial domain



$$\hat{p}_{\boldsymbol{0}} = \int_{\mathbb{T}^d} p_I(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \sum_{j=0}^{M-1} \frac{1}{M}\, p_I(\boldsymbol{x}_j)$$

$$\hat{p}_{\boldsymbol{k}} = \int_{\mathbb{T}^d} p_I(\boldsymbol{x})\,\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x},\ \boldsymbol{k} \in I$$

- $f(\boldsymbol{x}) = p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, arbitrary freq. index set $I \subset \mathbb{Z}^d$, $|I| < \infty$
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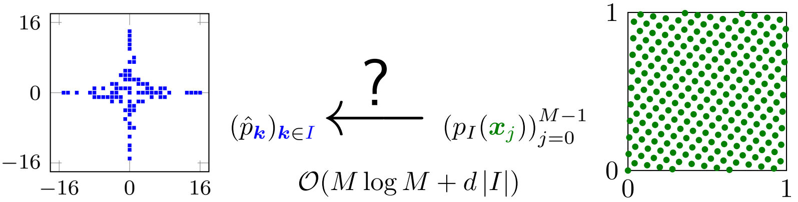
- fast reconstruction of $\hat{p}_{\boldsymbol{k}}$ using 1-dim. FFT? $\hat{p}_{\boldsymbol{k}} \stackrel{?}{=} \frac{1}{M} \sum_{j=0}^{M-1} p_I(\boldsymbol{x}_j)\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_j}$



- $f(\boldsymbol{x}) = p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, arbitrary freq. index set $I \subset \mathbb{Z}^d$, $|I| < \infty$
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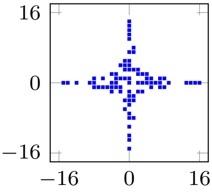
- fast reconstruction of $\hat{p}_{\boldsymbol{k}}$ using 1-dim. FFT? $\hat{p}_{\boldsymbol{k}} \overset{?}{=} \frac{1}{M} \sum_{j=0}^{M-1} p_I(\boldsymbol{x}_j)\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_j}$

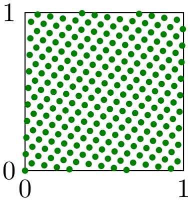
  $\Rightarrow$ reconstruction property: [Kämmerer,Kunis,Potts '12]
  $\boldsymbol{k}\cdot\boldsymbol{z} \not\equiv \boldsymbol{k}'\cdot\boldsymbol{z} \pmod{M}$ for all $\boldsymbol{k}, \boldsymbol{k}' \in I$, $\boldsymbol{k} \neq \boldsymbol{k}'$

  - $|I| \le M \le |I|^2$, simple CBC construction method [Kämmerer '12]



$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I} \xleftarrow[\text{FFT}]{\text{1-dim.}} (p_I(\boldsymbol{x}_j))_{j=0}^{M-1}$

$\mathcal{O}(M \log M + d\,|I|)$



- $f(\boldsymbol{x}) \approx p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, arbitrary freq. index set $I \subset \mathbb{Z}^d$, $|I| < \infty$
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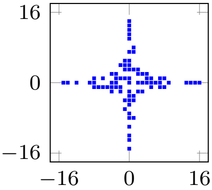
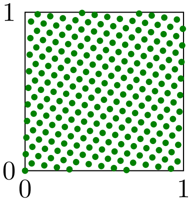
  - $|I| \leq M \leq |I|^2$, simple CBC construction method [Kämmerer '12]
- fast approximation of $f \in L_2(\mathbb{T}^d) \cap C(\mathbb{T}^d)$ using rank-1 lattice sampling error estimates in [Byrenheid,Kämmerer,Ullrich,V. '17] [V. '17]



$$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I} \xleftarrow[\text{FFT}]{\text{1-dim.}} (f(\boldsymbol{x}_j))_{j=0}^{M-1}$$

$$\mathcal{O}(M \log M + d\,|I|)$$



- $f(\boldsymbol{x}) := \prod_{s=1}^{d} \left(2 + \mathrm{sgn}((x_s \bmod 1) - \frac{1}{2})\, \sin(2\pi x_s)^3\right),\ f \in \mathcal{H}^3_{\mathrm{mix}}(\mathbb{T}^d)$
- $\mathcal{H}^\beta_{\mathrm{mix}}(\mathbb{T}^d) := \left\{ f \in L_2 \colon \sqrt{\sum_{\boldsymbol{m}\in\mathbb{N}_0^d, \|\boldsymbol{m}\|_\infty \le \beta} \|D^{\boldsymbol{m}} f | L_2(\mathbb{T}^d)\|^2} < \infty \right\},\ \beta \in \mathbb{N}$
- hyperbolic cross $I := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \prod_{s=1}^{d} \max(1, |k_s|) \le N\}$
- sparse grids: $\|f - S_I^{\mathrm{SG}} f | L_2(\mathbb{T}^d)\| \lesssim N^{-\beta} \log^{\frac{d-1}{2}} N \, \|f | \mathcal{H}^\beta_{\mathrm{mix}}(\mathbb{T}^d)\|$ [Sickel, Ullrich '07]
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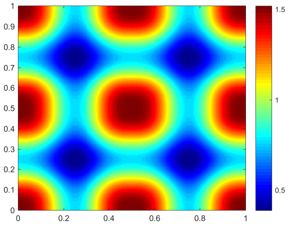
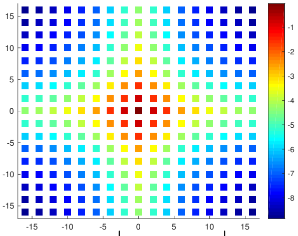
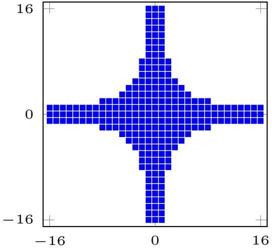


$f\left((x_1, x_2)^\top\right)$



$\log_{10} \left| \hat{f}_{(k_1,k_2)^\top} \right|$



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- $f(\boldsymbol{x}) := \prod_{s=1}^{d} \left(2 + \operatorname{sgn}((x_s \bmod 1) - \tfrac{1}{2})\, \sin(2\pi x_s)^3\right), \quad f \in \mathcal{H}_{\mathrm{mix}}^{\frac{7}{2}-\epsilon}(\mathbb{T}^d),\ \epsilon > 0$
- $\mathcal{H}_{\mathrm{mix}}^{\beta}(\mathbb{T}^d) := \left\{ f \in L_2 \colon \|f\| := \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}|^2 \prod_{s=1}^{d} \max(1,|k_s|)^{2\beta}} < \infty \right\},\ \beta > \tfrac{1}{2}$
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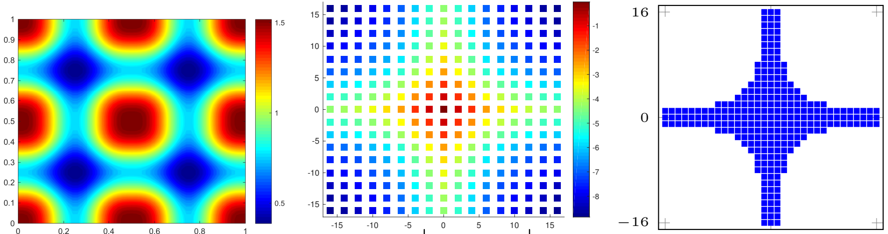

$f\left((x_1,x_2)^\top\right)$ $\quad \log_{10}\left|\hat{f}_{(k_1,k_2)^\top}\right|$ $\quad I$

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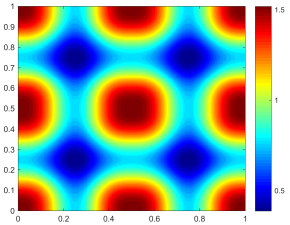
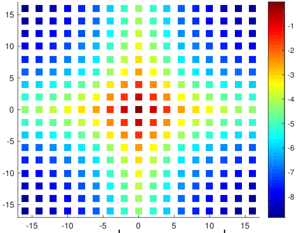
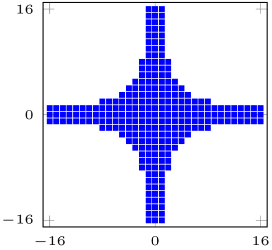


$f\left((x_1, x_2)^\top\right)$



$\log_{10} \left|\hat{f}_{(k_1,k_2)^\top}\right|$



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Example – function from Sobolev space of dominating mixed smoothness

- $f(\boldsymbol{x}) := \prod_{s=1}^{d} \left(2 + \operatorname{sgn}((x_s \bmod 1) - \tfrac{1}{2})\, \sin(2\pi x_s)^3\right), \quad f \in \mathcal{H}^{\frac{7}{2}-\epsilon}_{\mathrm{mix}}(\mathbb{T}^d),\ \epsilon > 0$
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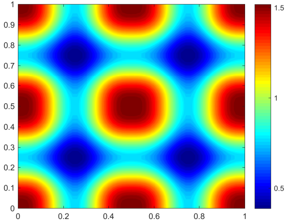
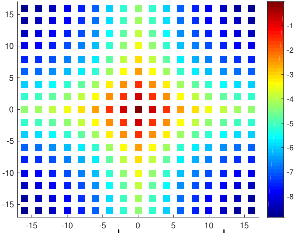
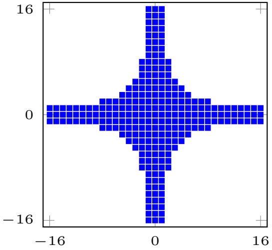


$f\left((x_1, x_2)^\top\right)$



$\log_{10} \left| \hat{f}_{(k_1,k_2)^\top} \right|$



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Multivariate trigonometric polynomials $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \approx f(\boldsymbol{x})$
Example – function from Sobolev space of dominating mixed smoothness

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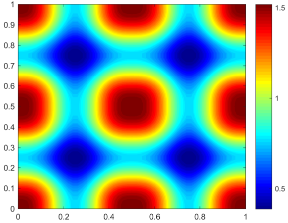

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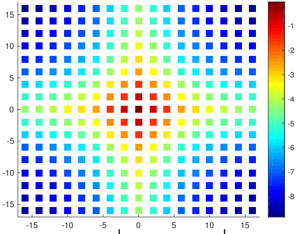

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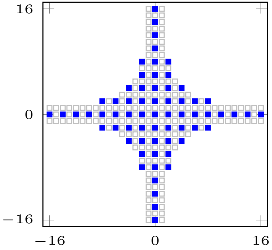

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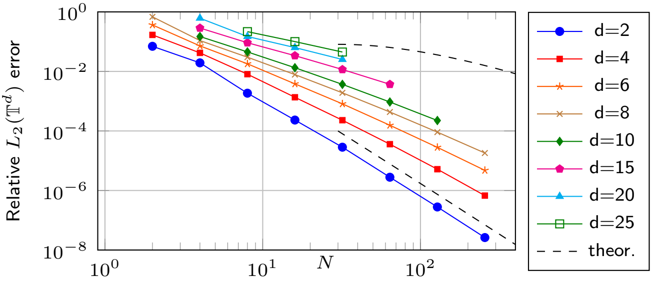

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best approximation	sparse grid	rank-1 lattice
$\asymp N^{-\beta}$	$\lesssim N^{-\beta} \log^{\frac{d-1}{2}} N$	
$\asymp \lvert I\rvert^{-\beta} (\log \lvert I\rvert)^{(d-1)\beta}$	$\lesssim \lvert I\rvert^{-\beta} (\log \lvert I\rvert)^{(d-1)(\beta+\frac{1}{2})}$	
$\asymp M^{-\beta} (\log M)^{(d-1)\beta}$	$\lesssim M^{-\beta} (\log M)^{(d-1)(\beta+\frac{1}{2})}$	$M^{-\frac{\beta}{2}} \lesssim \mathrm{err} \lesssim M^{-\frac{\beta}{2}} (\log M)^{\frac{d-2}{2}\beta + \frac{d-1}{2}}$

- (reconstructing) rank-1 lattice:
  - number of samples M: $|I| \le M \le |I|^2$, construction: $\mathcal{O}(d\,|I|^3)$
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$\Rightarrow$ improvements? Use more than one rank-1 lattice! (union of several)

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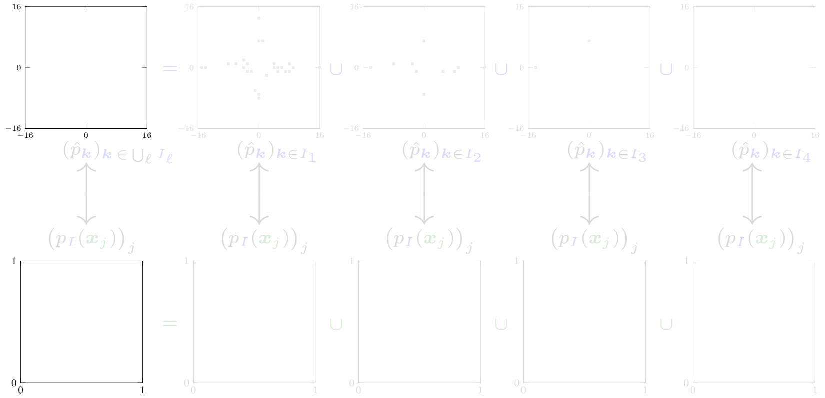
$\asymp M^{-\beta}(\log M)^{(d-1)\beta}$	$\lesssim M^{-\beta} (\log M)^{(d-1)(\beta+\frac{1}{2})}$	$M^{-\frac{\beta}{2}} \lesssim \mathrm{err} \lesssim M^{-\frac{\beta}{2}} (\log M)^{\frac{d-2}{2}\beta + \frac{d-1}{2}}$

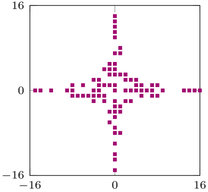
- (reconstructing) rank-1 lattice:
  - − number of samples M: $|I| \leq M \leq |I|^2$, construction: $\mathcal{O}(d\,|I|^3)$
  - + no additional dependence on spatial dimension d in M
  - + very easy and fast computation of Fourier coefficients (single 1-dim. FFT)

$\Rightarrow$ improvements? Use more than one rank-1 lattice! (union of several)

Fast reconstruction of $\hat{p}_{\boldsymbol{k}}$ using multiple rank-1 lattices [Kämmerer '16] [Kämmerer '17]



remaining freq. from I:



- number of samples: $\mathcal{O}\big(|I| \log^2 |I|\big)$ (w.h.p., constants independent from d)
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Multivariate trigonometric polynomials $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \approx f(\boldsymbol{x})$

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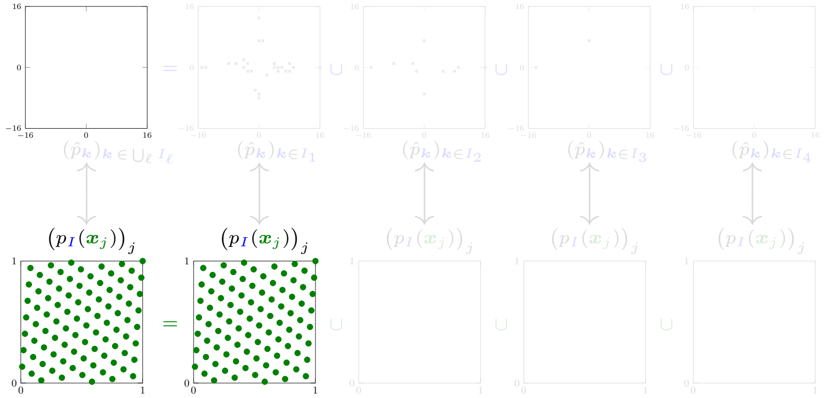

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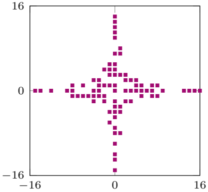

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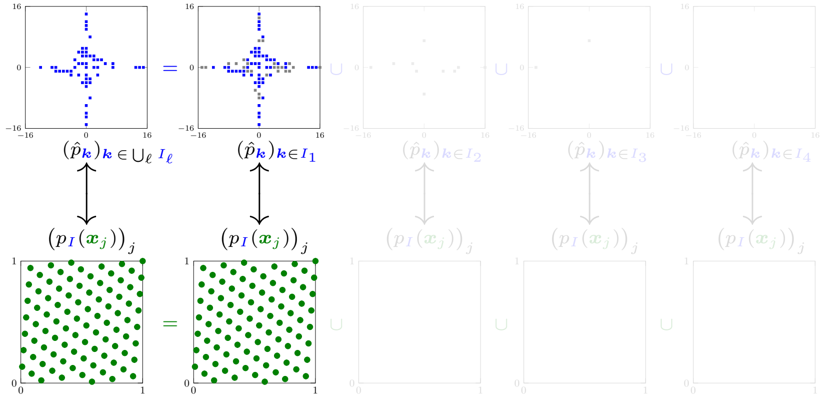

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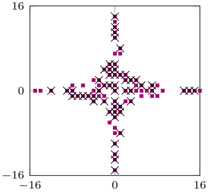

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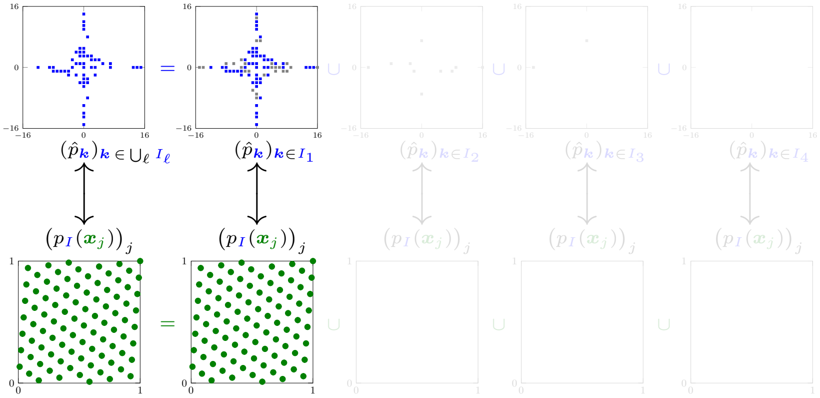

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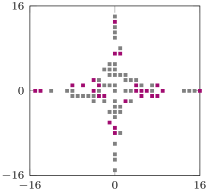

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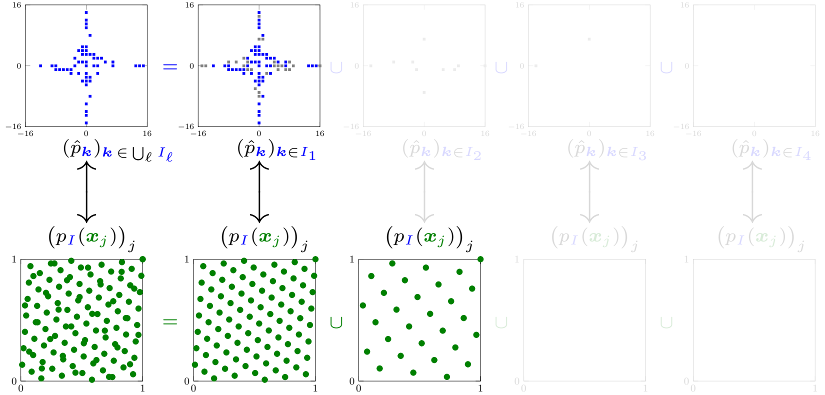

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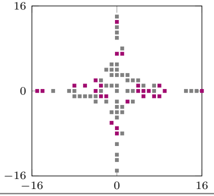

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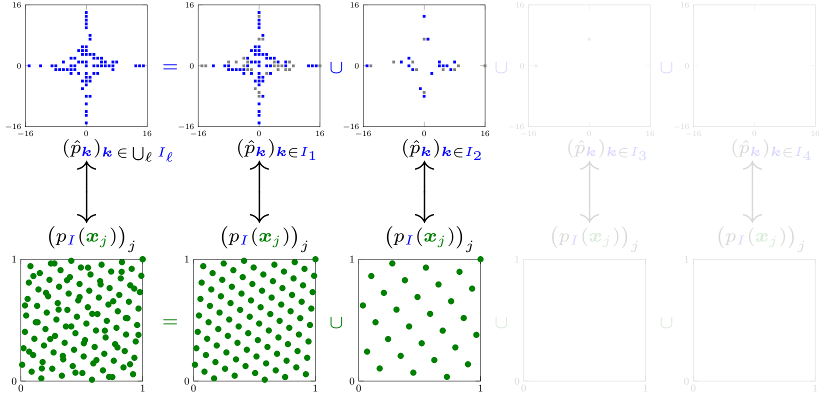

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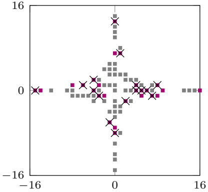

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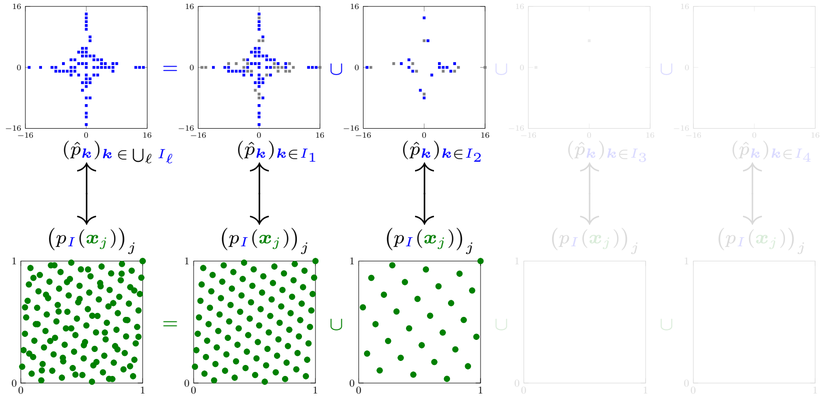
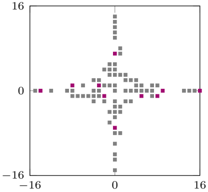


remaining freq. from I:



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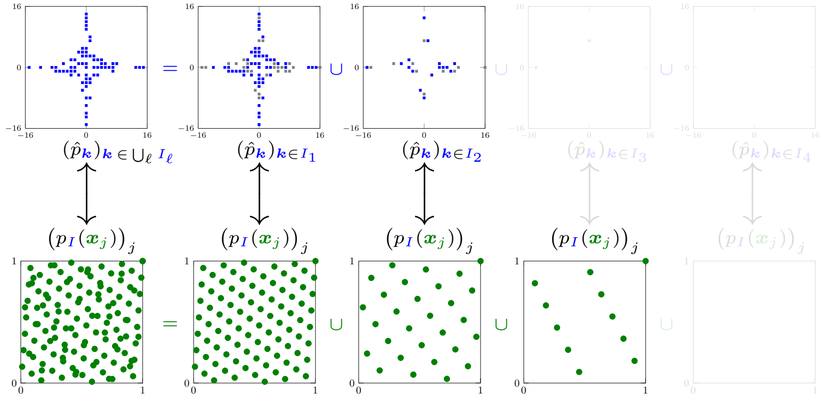

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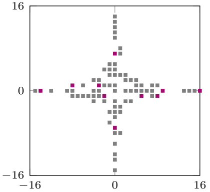

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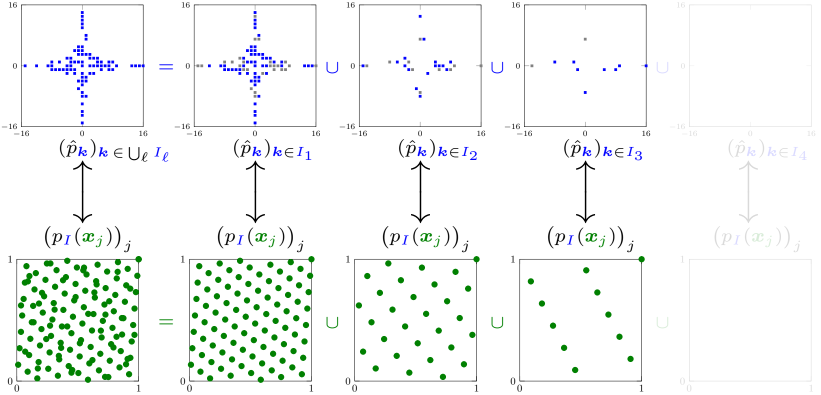

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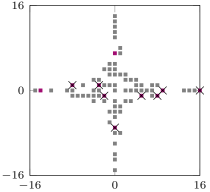

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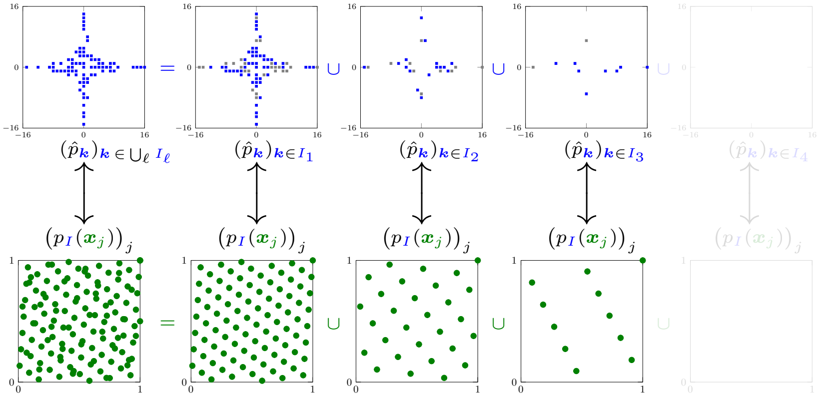

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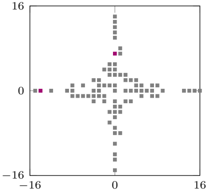

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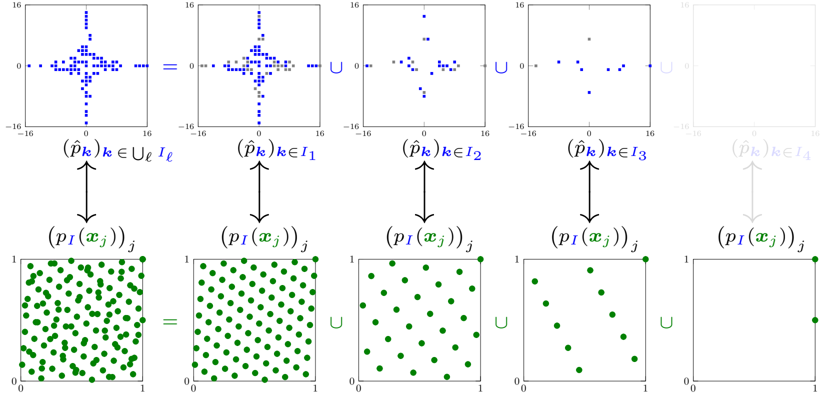

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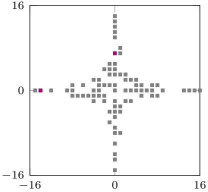

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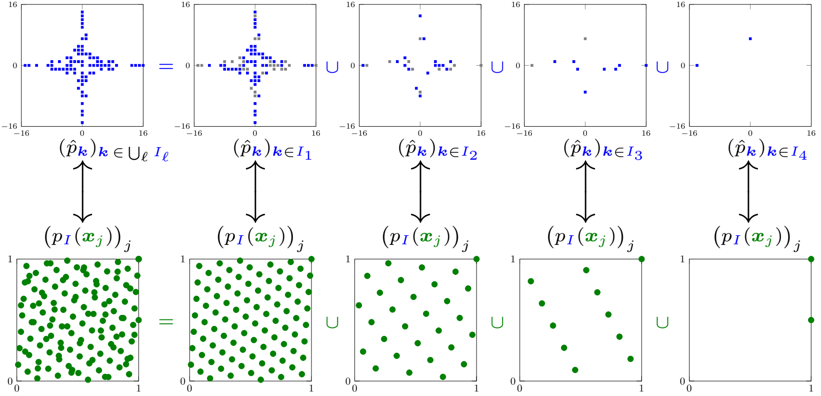

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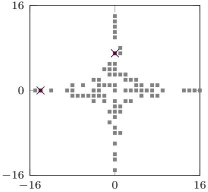

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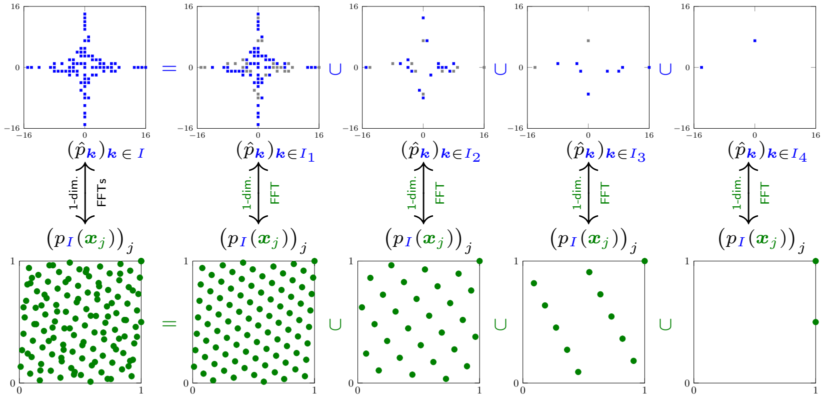

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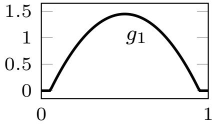
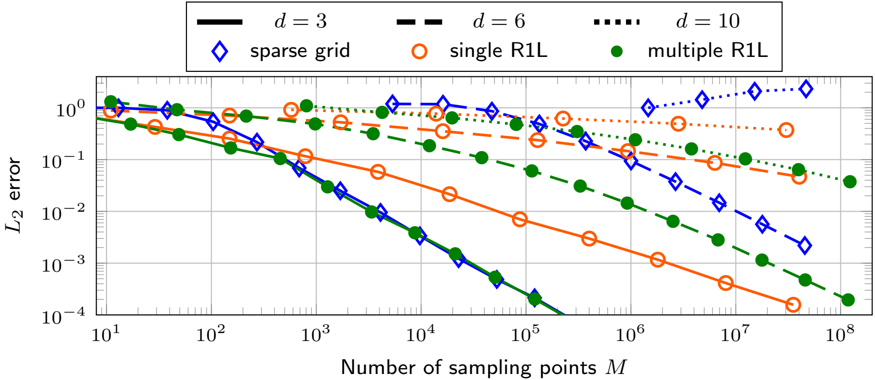
kink function $g_d\colon \mathbb{T}^d \to \mathbb{R}$,

$$g_d(\boldsymbol{x}) = \prod_{s=1}^{d} \left(\frac{5^{3/4}15}{4\sqrt{3}} \max\left\{ \frac{1}{5} - \left(x_s - \frac{1}{2}\right)^2, 0 \right\} \right)$$





- error estimates for (multiple) rank-1 lattice sampling in [Byrenheid, Kämmerer, Ullrich, V. '17] [V. '17] [Kämmerer, V. '18]

- $\mathcal{H}^{\beta}_{\mathrm{mix}}(\mathbb{T}^d) := \left\{ f \in L_2 \colon \|f\| := \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}|^2 \prod_{s=1}^{d} \max(1,|k_s|)^{2\beta}} < \infty \right\}$, $\beta > \frac{1}{2}$
- hyperbolic cross $I := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \prod_{s=1}^{d} \max(1,|k_s|) \leq N\}$
- single rank-1 lattice: special case of [Byrenheid, Kämmerer, Ullrich, V. '17] [V. '17]

$$M^{-\beta/2} \lesssim \frac{\|f - S_I^{\mathrm{R1L}} f | L_2(\mathbb{T}^d)\|}{\|f|\mathcal{H}^{\beta}_{\mathrm{mix}}(\mathbb{T}^d)\|} \lesssim M^{-\beta/2} \log^{\frac{\beta}{2}(d-2)+\frac{d-1}{2}} M$$

$\Rightarrow$ use multiple rank-1 lattice [Kämmerer '16] [Kämmerer '17]

- number of samples: $\mathcal{O}\big(|I| \log^2 |I|\big)$ (w.h.p., constants independent from d)
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- sampling error: [Kämmerer, V. '18]

$$\|f - S_I^{\mathrm{MR1L}} f | L_2(\mathbb{T}^d)\| \lesssim M^{-\beta+\frac{1}{2}+\varepsilon} \log^{(\beta-\frac{1}{2}-\varepsilon)d+1} M \; \|f|\mathcal{H}^{\beta}_{\mathrm{mix}}(\mathbb{T}^d)\|,\ \varepsilon > 0$$

$$\|f - S_I^{\mathrm{MR1L}} f | L_\infty(\mathbb{T}^d)\| \lesssim M^{-\beta+\frac{1}{2}+\varepsilon} (\log M)^{(d-1)\beta+\beta+1-d(\frac{1}{2}+\varepsilon)} \; \|f|\mathcal{H}^{\beta}_{\mathrm{mix}}(\mathbb{T}^d)\|$$

- $\mathcal{H}^{\beta}_{\mathrm{mix}}(\mathbb{T}^d) := \left\{ f \in L_2 \colon \|f\| := \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}|^2 \prod_{s=1}^{d} \max(1,|k_s|)^{2\beta}} < \infty \right\}$, $\beta > \frac{1}{2}$
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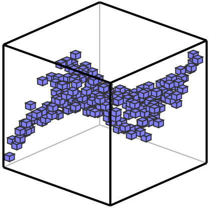
first part:

- fast reconstruction of arbitrary high-dimensional trigonometric polynomials $f(\boldsymbol{x}) = p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}$ using 1-dimensional FFTs

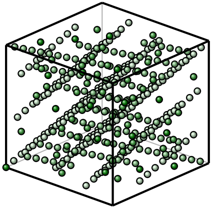
general known frequency index set $I \subset \mathbb{Z}^d$

spatial domain: multiple rank-1 lattice



$$\left(\hat{p}_{\boldsymbol{k}}\right)_{\boldsymbol{k} \in I} \xleftrightarrow[\text{FFTs}]{\text{1-dim.}} \left(f(\boldsymbol{x}_j)\right)_{j=0}^{M-1}$$

$$\mathcal{O}\left(|I| \, (d + \log |I|) \log^3 |I|\right)$$



- fast approximation $f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}$ of functions from samples

second part:

- unknown frequency index set I / weights / function space in high dimensions

$\Rightarrow$ dimension-incremental sparse FFT using multiple rank-1 lattices

next: unknown frequency index set I / weights / function space

$f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, find $\hat{p}_{\boldsymbol{k}}$ AND $I \subset \mathbb{Z}^d$ from samples of f

$\Rightarrow$ multi-dimensional sparse FFT

- task: determine frequency index set I (out of search domain $\Gamma \subset \mathbb{Z}^d$) from samples belonging to ≈largest Fourier coefficients $\hat{f}_{\boldsymbol{k}}$ or to $\hat{f}_{\boldsymbol{k}} \neq 0$
- various existing methods, e.g., based on
  filters [Indyk, Kapralov '14] / Chinese Remainder Theorem [Cuyt, Lee '08] [Iwen '13] /
  Prony's method [Tasche, Potts '13] [Peter, Plonka, Schaback '15] [Kunis, Peter, Römer, von der Ohe '15]
- problems: non-sparsity, implementations?, stability, many frequencies

$\Rightarrow$ dimension-incremental sparse FFT based on (multiple) rank-1 lattices
[Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17]
(similar basic idea without rank-1 lattices: [Zippel '79] [Kaltofen, Lee '03] [Javadi Monagan '10] [Potts, Tasche '13])

$\Rightarrow$ "Sparse Harmonic Transforms: A New Class of Sublinear-time Algorithms for Learning Functions of Many Variables" [Choi, Iwen, Krahmer '18]

next: unknown frequency index set I / weights / function space

$f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, find $\hat{p}_{\boldsymbol{k}}$ AND $I \subset \mathbb{Z}^d$ from samples of f

$\Rightarrow$ multi-dimensional sparse FFT

- task: determine frequency index set I (out of search domain $\Gamma \subset \mathbb{Z}^d$) from samples belonging to $\approx$largest Fourier coefficients $\hat{f}_{\boldsymbol{k}}$ or to $\hat{f}_{\boldsymbol{k}} \neq 0$
- various existing methods, e.g., based on
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- problems: non-sparsity, implementations?, stability, many frequencies

$\Rightarrow$ dimension-incremental sparse FFT based on (multiple) rank-1 lattices
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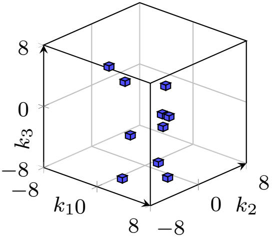


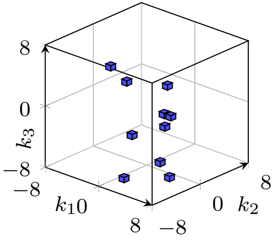
$I \subset \Gamma = \hat{G}_8^3 := \{-8,-7,\ldots,8\}$

$$\hat{p}_{k_1} := \frac{1}{17}\sum_{\ell=0}^{16} p\left(\begin{pmatrix}\ell/17\\ x_2'\\ x_3'\end{pmatrix}\right) \mathrm{e}^{-2\pi\mathrm{i}\frac{\ell k_1}{17}}$$

$k_1 = -8,\ldots,8$

frequency candidates

sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$



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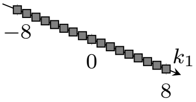

frequency candidates

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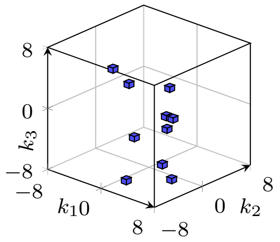

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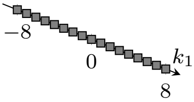
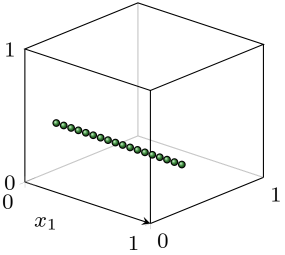
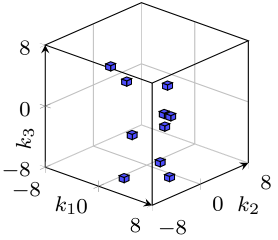


frequency candidates

construct $\longrightarrow$ sampling set



sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$



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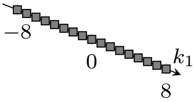
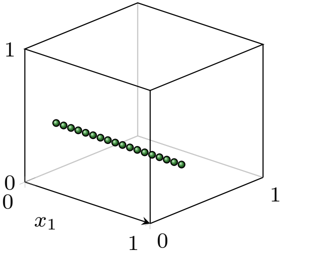
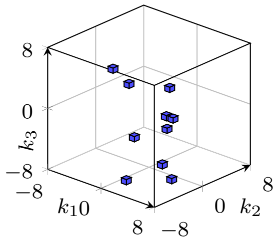


frequency candidates

1-dim.
$\longleftarrow$
FFT



sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$



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$$= \sum_{\substack{(h_2,h_3)\in\{-8,\ldots,8\}^2\\ (k_1,h_2,h_3)^\top\in\operatorname{supp}\hat{p}}} \hat{p}_{\begin{pmatrix}k_1\\h_2\\h_3\end{pmatrix}}\,\mathrm{e}^{2\pi\mathrm{i}(h_2x_2'+h_3x_3')},$$

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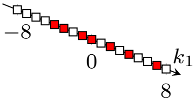
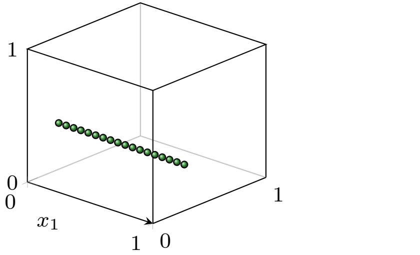
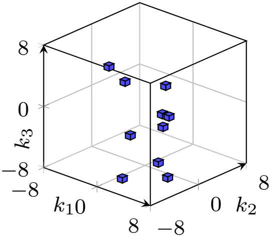


detected frequencies $I^{(1)}$

1-dim. FFT ←



sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$



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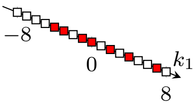
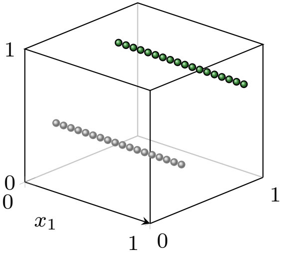


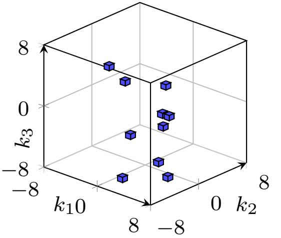
detected frequencies $I^{(1)}$

1-dim. ⟵ FFT



sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$

+ repeat (r detection iterations)



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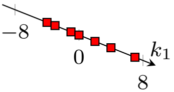
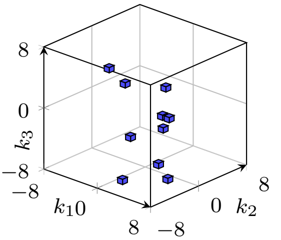


detected frequencies
$I^{(1)}$

frequency candidates

sampling nodes



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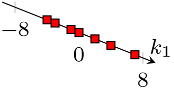

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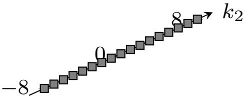
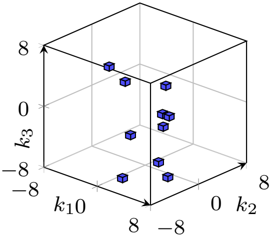


frequency candidates

sampling nodes



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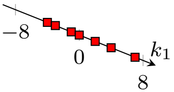
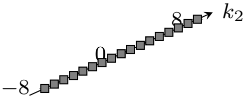
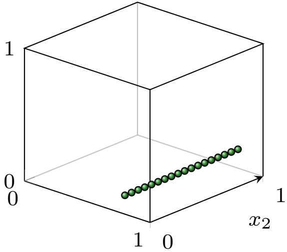


detected frequencies
$I^{(1)}$

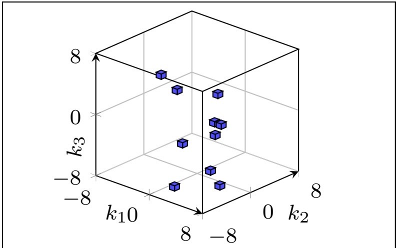


frequency candidates

construct
$\longrightarrow$
sampling set



sampling nodes



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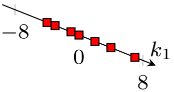

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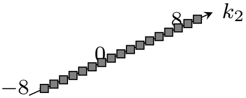
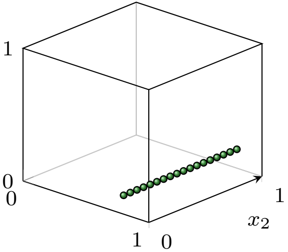
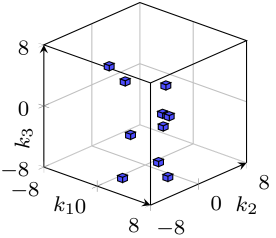


frequency candidates

1-dim. ⟵ FFT



sampling nodes



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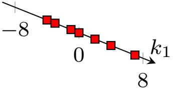
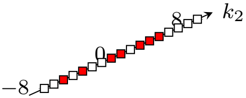
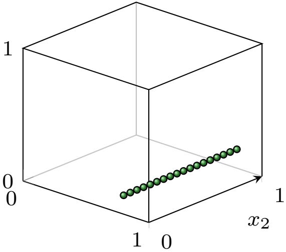


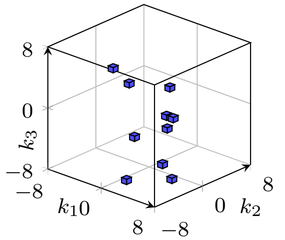
detected frequencies $I^{(1)}$



detected frequencies $I^{(2)}$



sampling nodes



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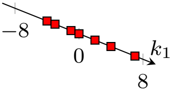
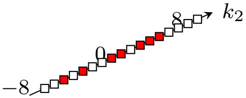
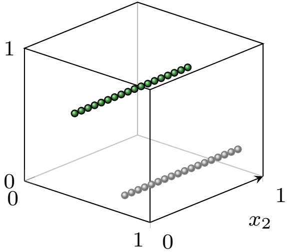


detected frequencies $I^{(1)}$



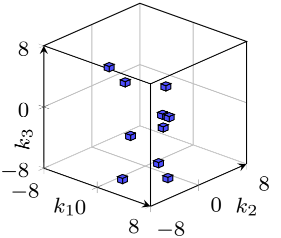
detected frequencies $I^{(2)}$

1-dim. ⟵ FFT



sampling nodes

+ repeat (r detection iterations)



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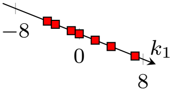
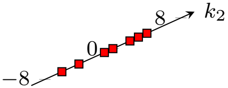
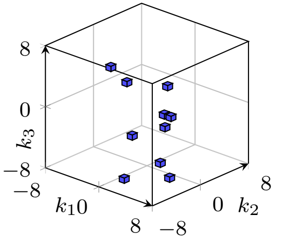


detected frequencies $I^{(1)}$



detected frequencies $I^{(2)}$

frequency candidates $I^{(1)} \times I^{(2)}$



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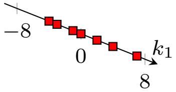
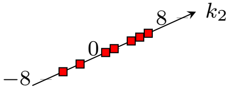
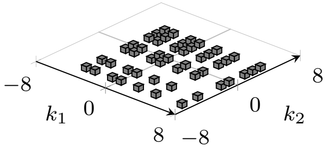


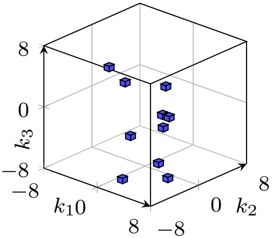
detected frequencies $I^{(1)}$



detected frequencies $I^{(2)}$



frequency candidates $I^{(1)} \times I^{(2)}$



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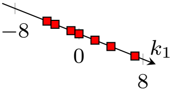
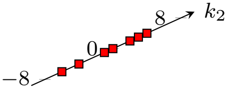


detected frequencies $I^{(1)}$



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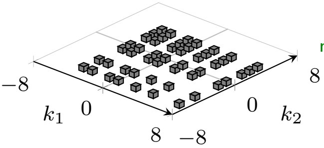
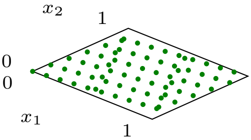


frequency candidates $I^{(1)} \times I^{(2)}$

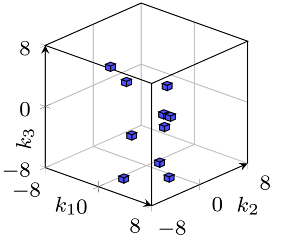
reconstructing $\longrightarrow$ multiple rank-1 lattice



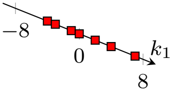
sampling nodes

High-dimensional dimension-incremental sparse FFT

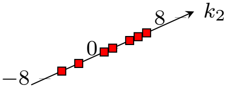
Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



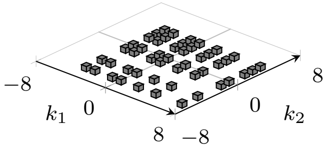
$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$



detected frequencies $I^{(1)}$

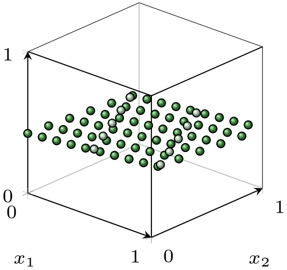


detected frequencies $I^{(2)}$

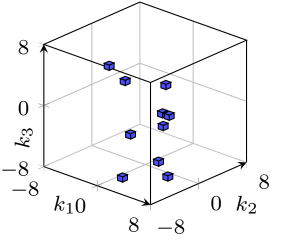


frequency candidates $I^{(1)} \times I^{(2)}$

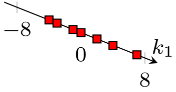
1-dim. $\longleftarrow$ FFTs



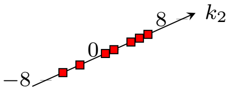
sampling nodes



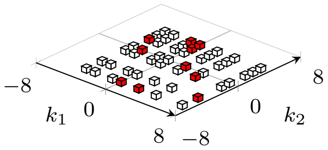
$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$



detected frequencies $I^{(1)}$

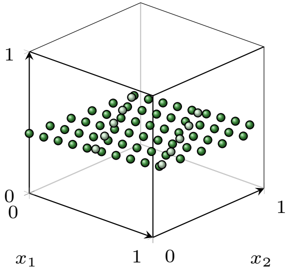


detected frequencies $I^{(2)}$



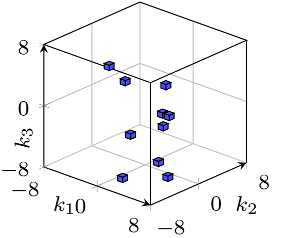
detected frequencies $I^{(1,2)}$

1-dim. $\longleftarrow$ FFTs

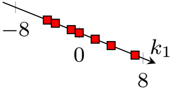


sampling nodes

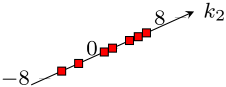
Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



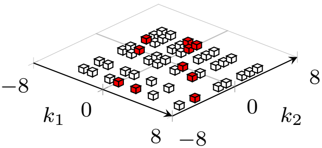
$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$



detected frequencies $I^{(1)}$

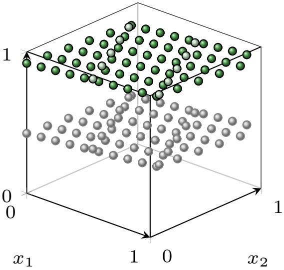


detected frequencies $I^{(2)}$



detected frequencies $I^{(1,2)}$

1-dim. ⟵ FFTs

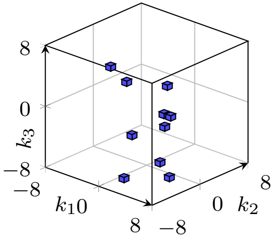


sampling nodes

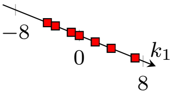
+ repeat (r detection iterations)

High-dimensional dimension-incremental sparse FFT

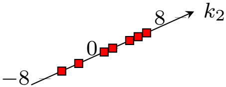
Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



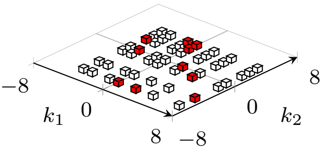
$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$



detected frequencies $I^{(1)}$

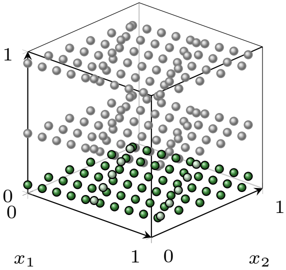


detected frequencies $I^{(2)}$



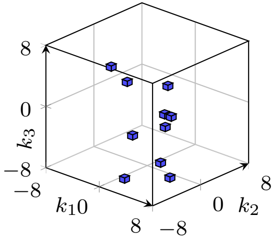
detected frequencies $I^{(1,2)}$

1-dim. ⟵ FFTs

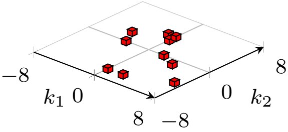


sampling nodes

\+ repeat (r detection iterations)



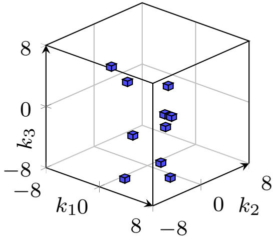
$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$



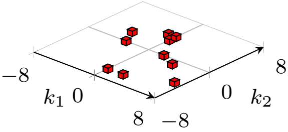
detected frequencies $I^{(1,2)}$

frequency candidates

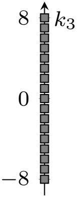
sampling nodes



$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$

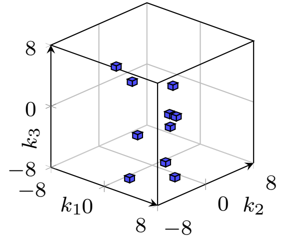


detected frequencies $I^{(1,2)}$

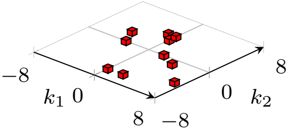


frequency candidates

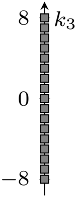
sampling nodes



$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$

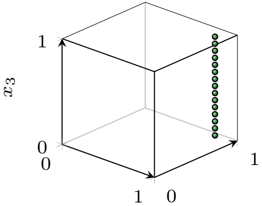


detected frequencies $I^{(1,2)}$

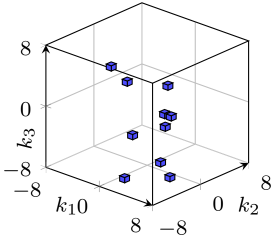


frequency candidates

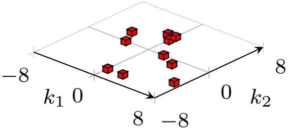
construct $\longrightarrow$ sampling set



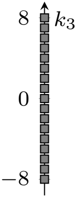
sampling nodes



$I \subset \Gamma = \hat{G}_8^3 := \{-8,-7,\ldots,8\}$

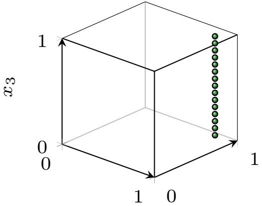


detected frequencies $I^{(1,2)}$

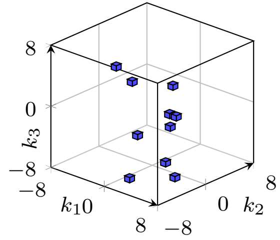


frequency candidates

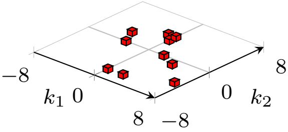
1-dim. FFT



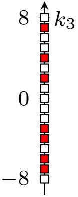
sampling nodes



$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$

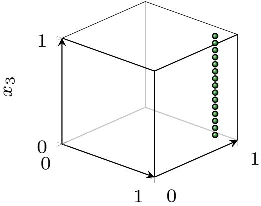


detected frequencies $I^{(1,2)}$



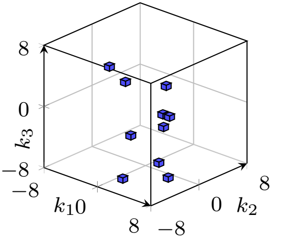
detected frequencies $I^{(3)}$

1-dim. ⟵ FFT

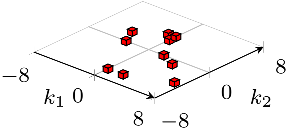


sampling nodes

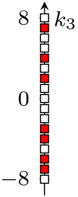
High-dimensional dimension-incremental sparse FFT
Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$

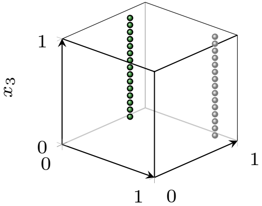


detected frequencies $I^{(1,2)}$



detected frequencies $I^{(3)}$

1-dim. ⟵ FFT

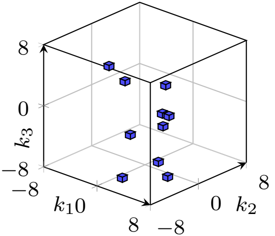


sampling nodes

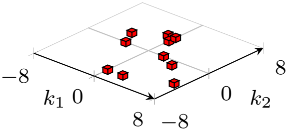
+ repeat (r detection iterations)

High-dimensional dimension-incremental sparse FFT

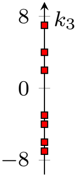
Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$



detected frequencies $I^{(1,2)}$



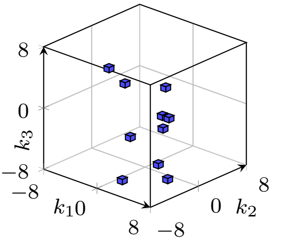
$I^{(3)}$

frequency candidates $I^{(1,2)} \times I^{(3)}$

sampling nodes

High-dimensional dimension-incremental sparse FFT

Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



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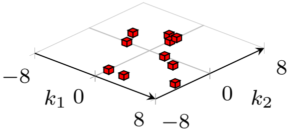
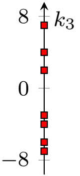
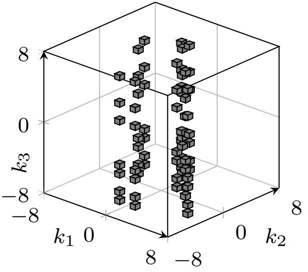


detected frequencies $I^{(1,2)}$

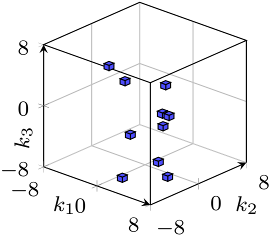


$I^{(3)}$

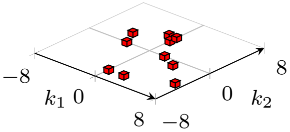


frequency candidates $I^{(1,2)} \times I^{(3)}$

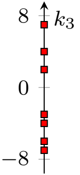
sampling nodes



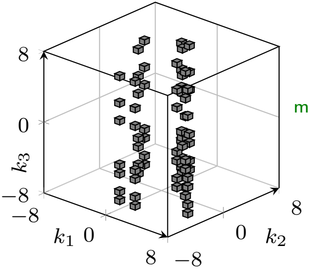
$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$



detected frequencies $I^{(1,2)}$

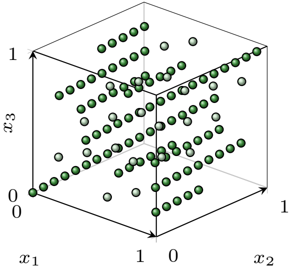


$I^{(3)}$



frequency candidates $I^{(1,2)} \times I^{(3)}$

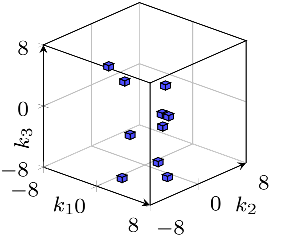
reconstructing
$\longrightarrow$
multiple rank-1 lattice



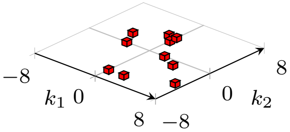
sampling nodes

High-dimensional dimension-incremental sparse FFT

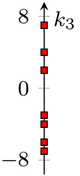
Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



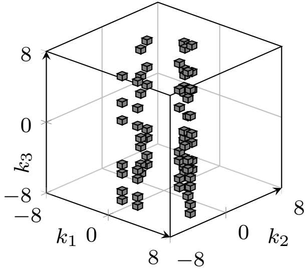
$\color{blue}{I} \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$



detected frequencies $I^{(1,2)}$

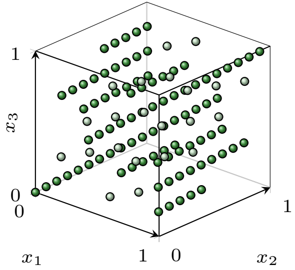


$I^{(3)}$

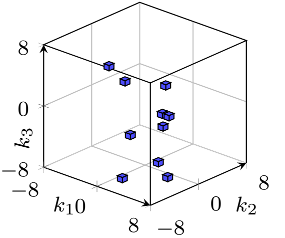


frequency candidates $I^{(1,2)} \times I^{(3)}$

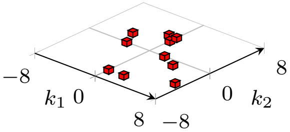
1-dim. ⟵ FFTs



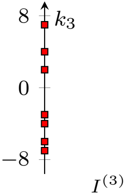
sampling nodes

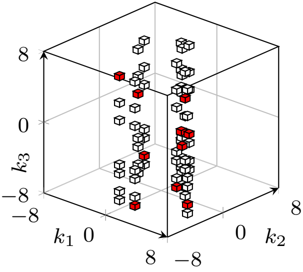


$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}$



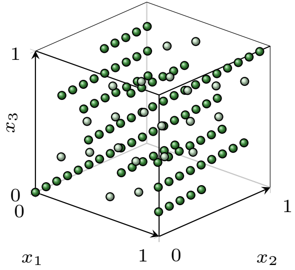
detected frequencies $I^{(1,2)}$





detected frequencies $I^{(1,2,3)} = I$

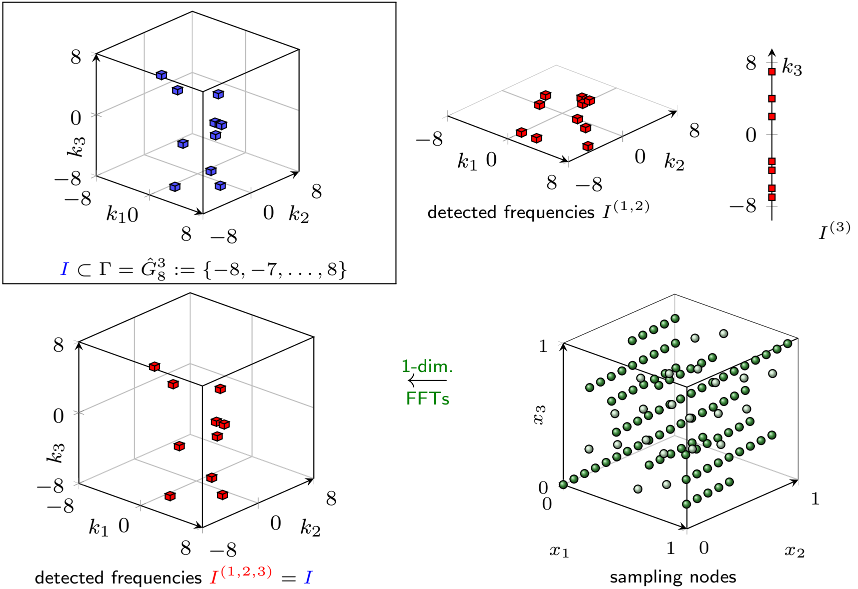
1-dim. FFTs



sampling nodes

High-dimensional dimension-incremental sparse FFT

Method [Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17], $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



reconstruction of $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ with unknown I using rank-1 lattices:

- sparsity $s = |I|$, search domain $\Gamma = \hat{G}_N^d := \{-N, \ldots, N\}^d \supset I$, number of detection iterations r

	single rank-1 lattices	multiple rank-1 lattices
samples	$\mathcal{O}(d\,r^2 s^2 N)$	$\mathcal{O}(d\,r s N \log^2(rsN))$ (w.h.p.)
arithmetic op.	$\mathcal{O}(d\,r^3 s^3 + d\,r^2 s^2 N \log(rsN))$	$\mathcal{O}\big(d^2 r s N \log^4(rsN)\big)$ (w.h.p.)

- if $(\mathrm{Re}(\hat{p}_{\boldsymbol{k}})$ identical sign) AND $(\mathrm{Im}(\hat{p}_{\boldsymbol{k}})$ identical sign) $\Rightarrow r = 1$; otherwise
  - in theory: $r = 2s(\log 3 + \log d + \log s - \log \varepsilon)$ for failure probability $\varepsilon \in (0,1)$
  - in practice: $r = 1$ or 2
- can be applied for approximate reconstruction of function $f \in L_2(\mathbb{T}^d) \cap C(\mathbb{T}^d)$
- MATLAB implementation
- numerically tested for up to 30 spatial dimensions

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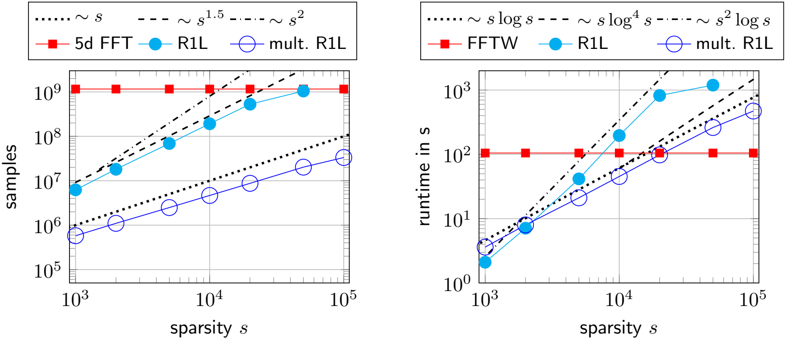
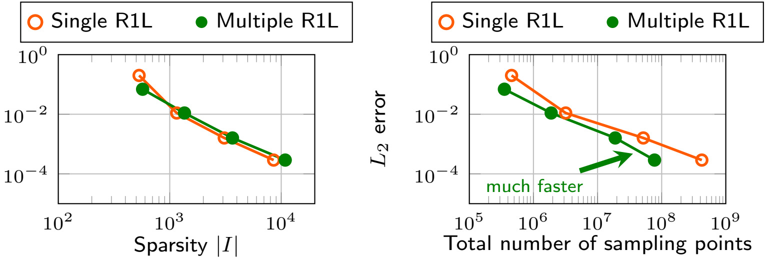
example: $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, $I \subset \Gamma = \hat{G}_{32}^5 := \{-32, \ldots, 32\}^5$, $|I| = s$,

$|\Gamma| \approx 1.16 \cdot 10^9$

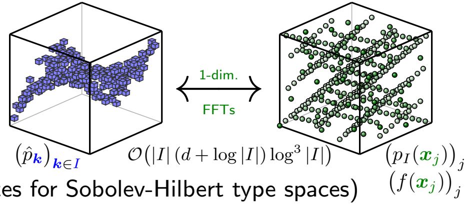


- $f(\boldsymbol{x}) := \prod_{t\in\{1,3,8\}} B_2(x_t) + \prod_{t\in\{2,5,6,10\}} B_4(x_t) + \prod_{t\in\{4,7,9\}} B_6(x_t)$,
  $B_m(x) = \sum_{k\in\mathbb{Z}} C_m \operatorname{sinc}\left(\frac{\pi}{m}k\right)^m (-1)^k \mathrm{e}^{2\pi\mathrm{i}kx}$
  univariate B-spline of order $m \in \mathbb{N}$
- dimension-incremental sparse FFT for $\Gamma = \hat{G}_{64}^{10}$ ($|\hat{G}_{64}^{10}| \approx 1.28 \cdot 10^{21}$), $r = 10$:



Conclusion

- known frequency index set $I \subset \mathbb{Z}^d$, multiple rank-1 lattice
  - fast reconstruction of high-dim. trigonometric polynomials p_I [Kämmerer '16] [Kämmerer '17]
  - fast approximation (error estimates for Sobolev-Hilbert type spaces) [Kämmerer, Potts, V. '15] [Byrenheid, Kämmerer, Ullrich, V. '17] [V. '17] [Kämmerer, V. '18]



- **unknown** $I \subset \mathbb{Z}^d$, sampling along (multiple) rank-1 lattices
  - high-dimensional dimension-incremental sparse FFT [Potts, V. '16] [V. '17] [Kämmerer, V. '17]
  - very good numerical results for high-dimensional sparse trigonometric polynomials and for high-dimensional functions (non-sparse in frequency domain)
- can be transfered to non-periodic case (tensor product Chebyshev bases)
- see also

L. Kämmerer, D. Potts, T. V. **High-dimensional sparse FFT based on sampling along multiple rank-1 lattices**. *ArXiv e-prints* 1711.05152, Nov. 2017.

L. Kämmerer, T. V. **Approximation of multivariate periodic functions based on sampling along multiple rank-1 lattices**. *ArXiv e-prints* 1802.06639, Feb. 2018.

L. Kämmerer. **High Dimensional Fast Fourier Transform Based on Rank-1 Lattice Sampling**. *Dissertation (PhD thesis), Faculty of Mathematics, Chemnitz University of Technology*, 2014.

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T. V. **Multivariate Approximation and High-Dimensional Sparse FFT Based on Rank-1 Lattice Sampling**. *Dissertation (PhD thesis), Faculty of Mathematics, Chemnitz University of Technology*, 2017.

L. Kämmerer, D. Potts, T. V. **High-dimensional sparse FFT based on sampling along multiple rank-1 lattices**. *ArXiv e-prints* 1711.05152, Nov. 2017.

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Software: MATLAB toolboxes (for single rank-1 lattices) `https://www.tu-chemnitz.de/~tovo`