

Multiple rank-1 lattice sampling and high-dimensional sparse FFT

Toni Volkmer



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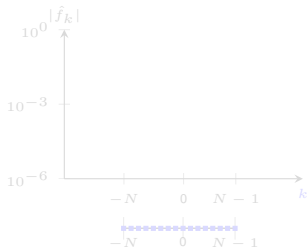
joint work with Lutz Kämmerer and Daniel Potts

supported by

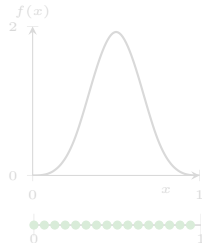


- ▶ torus $\mathbb{T} \simeq [0, 1)$, $\{e^{2\pi i k x}\}_{k \in \mathbb{Z}}$ orthonormal basis of $L_2(\mathbb{T})$
- ▶ function $f \in L_2(\mathbb{T})$, $f(x) = \sum_{k \in \mathbb{Z}} \hat{f}_k e^{2\pi i k x}$, $\hat{f}_k \in \mathbb{C}$
- ▶ smooth function $f \implies$ fast decay of Fourier coefficients \hat{f}_k
- ▶ truncated Fourier series $S_I f(x) = \sum_{k \in I} \hat{f}_k e^{2\pi i k x} \approx f(x)$
- ▶ $\hat{f}_k = \int_{\mathbb{T}} f(x) e^{-2\pi i k x} dx \approx \tilde{f}_k := \frac{1}{2N} \sum_{j=0}^{2N-1} f(x_j) e^{-2\pi i k x_j}$, $x_j := \frac{j}{2N}$

\Rightarrow transfer to multivariate case

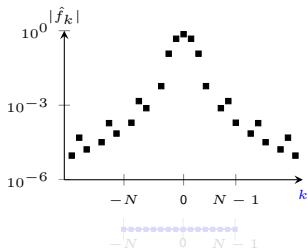


$$(\tilde{f}_k)_{k \in I} \begin{array}{c} \xleftarrow{1\text{-dim.}} \\ \xrightarrow{\text{FFT}} \end{array} (f(x_j))_{j=0}^{2N-1} \\
 \mathcal{O}(N \log N) \\
 [\text{Gau\ss} 1866] [\text{Cooley, Tukey 1965}]$$



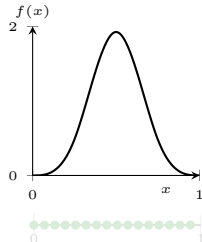
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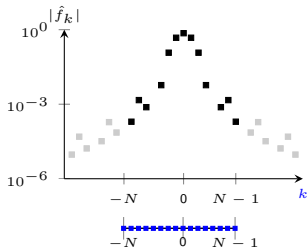
$$(\tilde{f}_k)_{k \in I} \begin{matrix} \xleftarrow{1\text{-dim.}} \\ \text{FFT} \\ \xrightarrow{\quad} \end{matrix} (f(x_j))_{j=0}^{2N-1}$$

$\mathcal{O}(N \log N)$
[Gauß 1866] [Cooley, Tukey 1965]



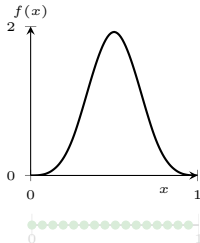
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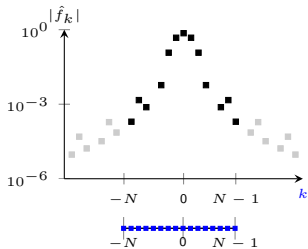
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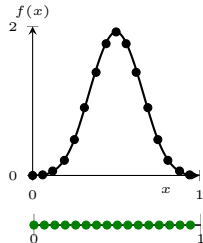
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$$(\tilde{f}_k)_{k \in I} \begin{matrix} \xleftarrow{1\text{-dim.}} \\ \xrightarrow{\text{FFT}} \end{matrix} (f(x_j))_{j=0}^{2N-1}$$

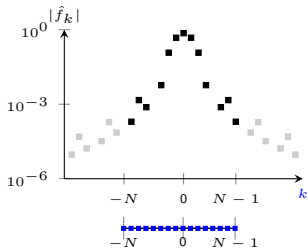
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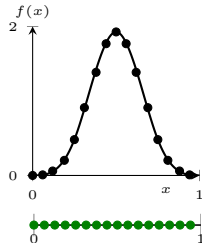
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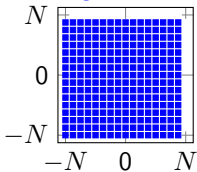
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[Gauß 1866] [Cooley, Tukey 1965]



▶ full grid in frequency domain

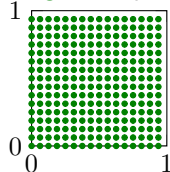


$$(\tilde{f}_{\mathbf{k}})_{\mathbf{k} \in I} \begin{array}{c} \xleftarrow{d\text{-dim.}} \\ \xrightarrow{\text{FFT}} \end{array} (f(\mathbf{x}_j))_{j=0}^{|I|-1}$$

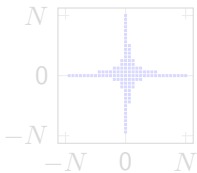
$$\mathcal{O}(N^d \log N)$$

curse of dimensionality

equispaced full grid in spatial domain


 ? high-dimensional case (e.g. $d = 10$) \implies assumption: sparsity or smoothness

▶ hyperbolic cross in frequency domain

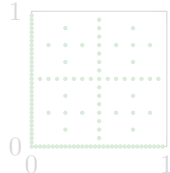


$$(\tilde{f}_{\mathbf{k}})_{\mathbf{k} \in I} \begin{array}{c} \xleftarrow{\text{HCFFT}} \\ \xrightarrow{\text{HCFFT}} \end{array} (f(\mathbf{x}_j))_{j=0}^{|I|-1}$$

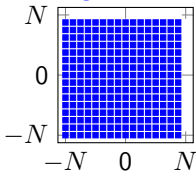
$$\mathcal{O}(N \log^d N)$$

 condition number of
 Fourier matrix
 [Kämmerer, Kunis '11]

sparse grid in spatial domain



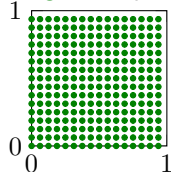
► full grid in frequency domain



$$(\tilde{f}_{\mathbf{k}})_{\mathbf{k} \in I} \begin{array}{c} \xleftarrow{\mathcal{O}\text{-dim.}} \\ \text{FFT} \\ \xrightarrow{\mathcal{O}(N^d \log N)} \end{array} (f(\mathbf{x}_j))_{j=0}^{|I|-1}$$

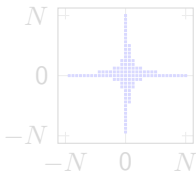
curse of dimensionality

equispaced full grid in spatial domain



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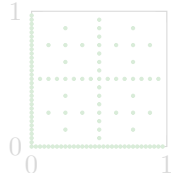
► hyperbolic cross in frequency domain



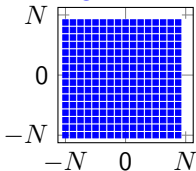
$$(\tilde{f}_{\mathbf{k}})_{\mathbf{k} \in I} \begin{array}{c} \xleftarrow{\text{HCFFT}} \\ \xrightarrow{\mathcal{O}(N \log^d N)} \end{array} (f(\mathbf{x}_j))_{j=0}^{|I|-1}$$

condition number of
 Fourier matrix
 [Kämmerer, Kunis '11]

sparse grid in spatial domain



► **full grid** in frequency domain

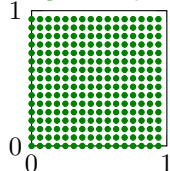


$$(\tilde{f}_{\mathbf{k}})_{\mathbf{k} \in I} \begin{array}{c} \xleftarrow{d\text{-dim.}} \\ \xrightarrow{\text{FFT}} \end{array} (f(\mathbf{x}_j))_{j=0}^{|I|-1}$$

$$\mathcal{O}(N^d \log N)$$

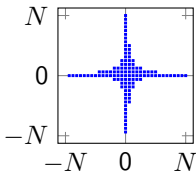
curse of dimensionality

equispaced full grid in spatial domain



? high-dimensional case (e.g. $d = 10$) \implies assumption: sparsity or smoothness

► **hyperbolic cross** in frequency domain

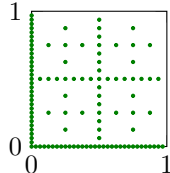


$$(\tilde{f}_{\mathbf{k}})_{\mathbf{k} \in I} \begin{array}{c} \xleftarrow{\text{HCFFT}} \\ \xrightarrow{\text{HCFFT}} \end{array} (f(\mathbf{x}_j))_{j=0}^{|I|-1}$$

$$\mathcal{O}(N \log^d N)$$

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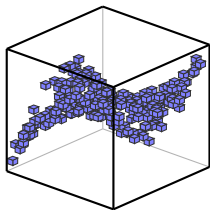
sparse grid in spatial domain



first part:

- ▶ fast reconstruction of arbitrary **high-dimensional** trigonometric polynomials using **1-dimensional FFTs**

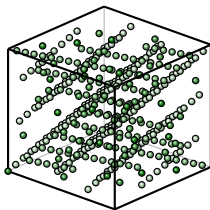
general frequency index set $I \subset \mathbb{Z}^d$



$$\left(\tilde{f}_k \right)_{k \in I} \begin{array}{c} \xleftrightarrow{\text{1-dim.}} \\ \xleftrightarrow{\text{FFTs}} \end{array} \left(f(\mathbf{x}_j) \right)_{j=0}^{M-1}$$

$$\mathcal{O}(|I| (d + \log |I|) \log^3 |I|)$$

spatial domain:
multiple rank-1 lattice

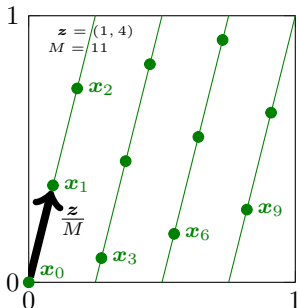


- ▶ fast approximation of functions from samples

second part:

- ▶ **unknown** frequency index set I / weights / function space in **high dimensions**
- ⇒ dimension-incremental sparse FFT using **multiple rank-1 lattices**

- ▶ $f(\mathbf{x}) = p_I(\mathbf{x}) = \sum_{\mathbf{k} \in I} \hat{p}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}$, arbitrary freq. index set $I \subset \mathbb{Z}^d$, $|I| < \infty$
- ▶ rank-1 lattice $\mathbf{R1L}(\mathbf{z}, M) := \{\mathbf{x}_j := \frac{j}{M} \mathbf{z} \bmod \mathbf{1}\}_{j=0}^{M-1}$, $\mathbf{z} \in \mathbb{Z}^d$, $M \in \mathbb{N}$, as discretization in spatial domain

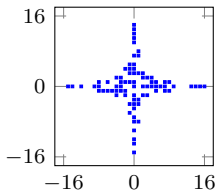


Korobov '59
 Maisonneuve '72
 Sloan & Kachoyan '84, '87, '90
 Temlyakov '86
 Lyness '89
 Sloan & Joe '94
 Sloan & Reztsov '01
 Li & Hickernell '03
 Kämmerer & Kunis & Potts '12

$$\hat{p}_0 = \int_{\mathbb{T}^d} p_I(\mathbf{x}) d\mathbf{x} \approx \sum_{j=0}^{M-1} \frac{1}{M} p_I(\mathbf{x}_j)$$

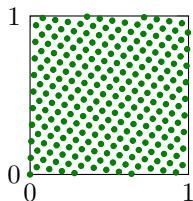
$$\hat{p}_{\mathbf{k}} = \int_{\mathbb{T}^d} p_I(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} d\mathbf{x}, \mathbf{k} \in I$$

- ▶ $f(\mathbf{x}) = p_I(\mathbf{x}) = \sum_{\mathbf{k} \in I} \hat{p}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}$, **arbitrary** freq. index set $I \subset \mathbb{Z}^d$, $|I| < \infty$
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- ▶ fast reconstruction of $\hat{p}_{\mathbf{k}}$ using 1-dim. FFT? $\hat{p}_{\mathbf{k}} \stackrel{?}{=} \frac{1}{M} \sum_{j=0}^{M-1} p_I(\mathbf{x}_j) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}_j}$

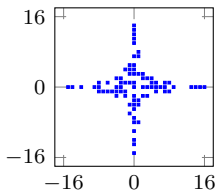


$$(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in I} \xleftarrow{?} (p_I(\mathbf{x}_j))_{j=0}^{M-1}$$

$\mathcal{O}(M \log M + d |I|)$

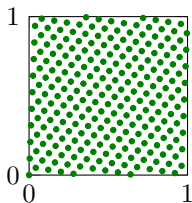


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 - ⇒ reconstruction property: [Kämmerer, Kunis, Potts '12]
 $\mathbf{k} \cdot \mathbf{z} \not\equiv \mathbf{k}' \cdot \mathbf{z} \pmod{M}$ for all $\mathbf{k}, \mathbf{k}' \in I$, $\mathbf{k} \neq \mathbf{k}'$
 - ▶ $|I| \leq M \leq |I|^2$, simple CBC construction method [Kämmerer '12]

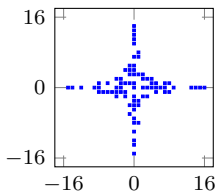


$$(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in I} \xleftarrow[\text{FFT}]{\text{1-dim.}} (p_I(\mathbf{x}_j))_{j=0}^{M-1}$$

$$\mathcal{O}(M \log M + d |I|)$$

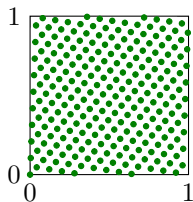


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 - ▶ $|I| \leq M \leq |I|^2$, simple CBC construction method [Kämmerer '12]
- ▶ fast approximation of $f \in L_2(\mathbb{T}^d) \cap C(\mathbb{T}^d)$ using rank-1 lattice sampling error estimates in [Byrnehed, Kämmerer, Ullrich, V. '17] [V. '17]

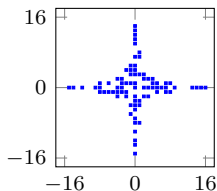


$$(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in I} \xleftarrow[\text{FFT}]{\text{1-dim.}} (f(\mathbf{x}_j))_{j=0}^{M-1}$$

$$\mathcal{O}(M \log M + d |I|)$$

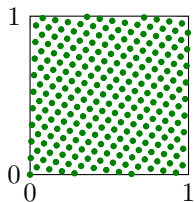


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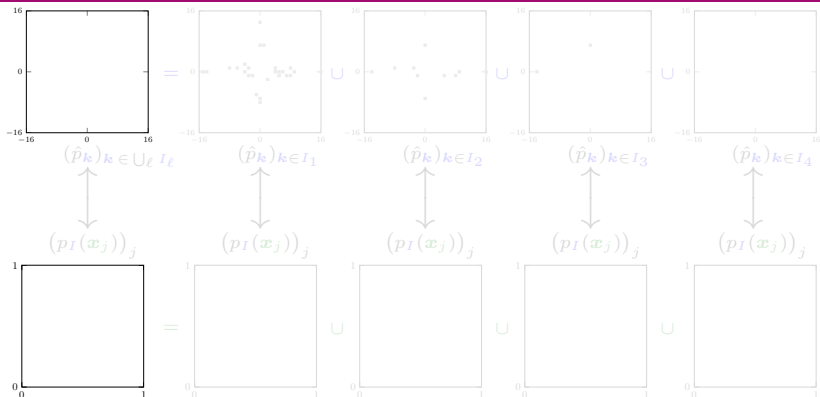
$$(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in I} \xleftarrow[\text{FFT}]{\text{1-dim.}} (f(\mathbf{x}_j))_{j=0}^{M-1}$$

$$\mathcal{O}(M \log M + d|I|)$$

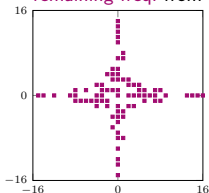


! number of samples: $M = \mathcal{O}(|I|^2)$, construction complexity: $\mathcal{O}(d|I|^3)$

$$\hat{p}_{\mathbf{k}} := \int_{\mathbb{T}^d} p_I(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} d\mathbf{x} = \frac{1}{M_\ell} \sum_{j=0}^{M_\ell-1} p_I(\mathbf{x}_j) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}_j}$$

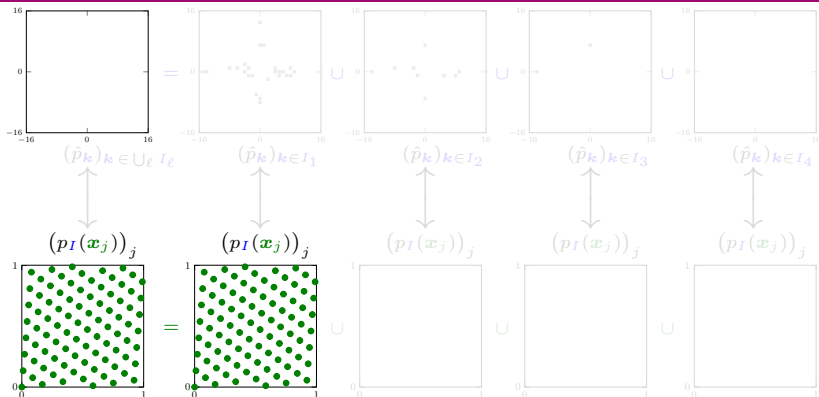
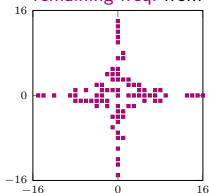


remaining freq. from I :



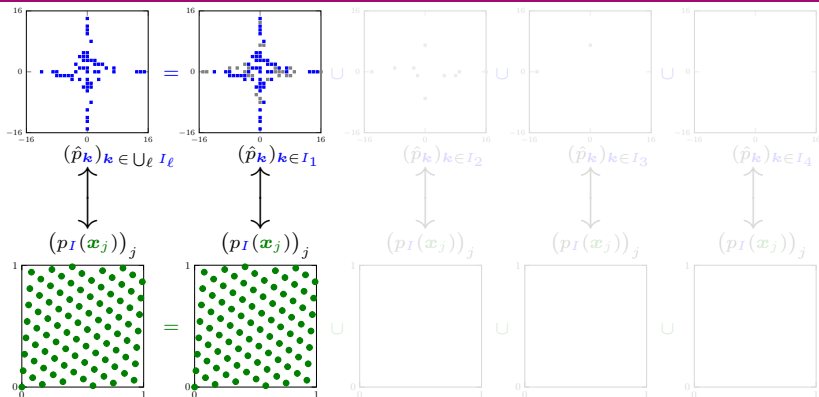
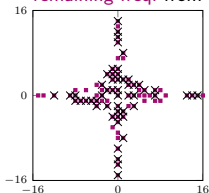
- ▶ number of samples: $\mathcal{O}(|I| \log^2 |I|)$
(w.h.p., constants independent from d)
- ▶ construction complexity: $\mathcal{O}(|I| (d + \log |I|) \log^3 |I|)$
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- ▶ reconstruction $(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in I}$: $\mathcal{O}(|I| (d + \log |I|) \log^3 |I|)$

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 remaining freq. from I :


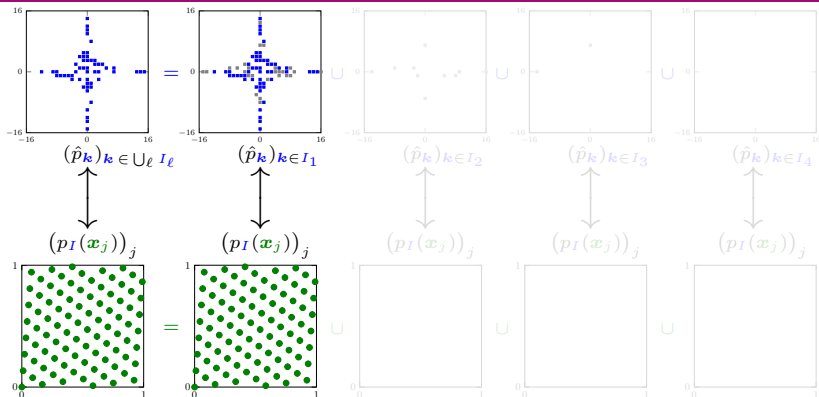
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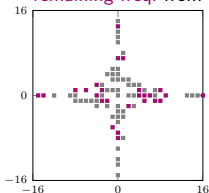

 remaining freq. from I :


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(w.h.p., constants independent from d)
- ▶ construction complexity: $\mathcal{O}(|I| (d + \log |I|) \log^3 |I|)$
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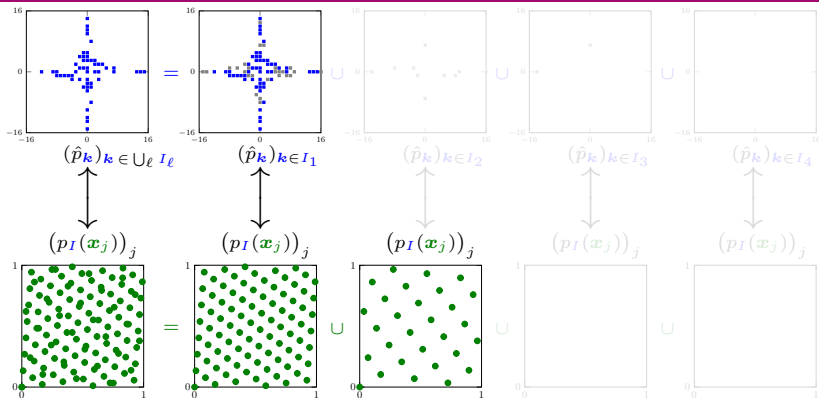
remaining freq. from I :



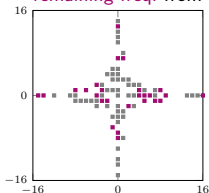
- ▶ number of samples: $\mathcal{O}(|I| \log^2 |I|)$
(w.h.p., constants independent from d)
- ▶ construction complexity: $\mathcal{O}(|I| (d + \log |I|) \log^3 |I|)$
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Reconstructing multiple rank-1 lattices [Kämmerer '16] [Kämmerer '17]

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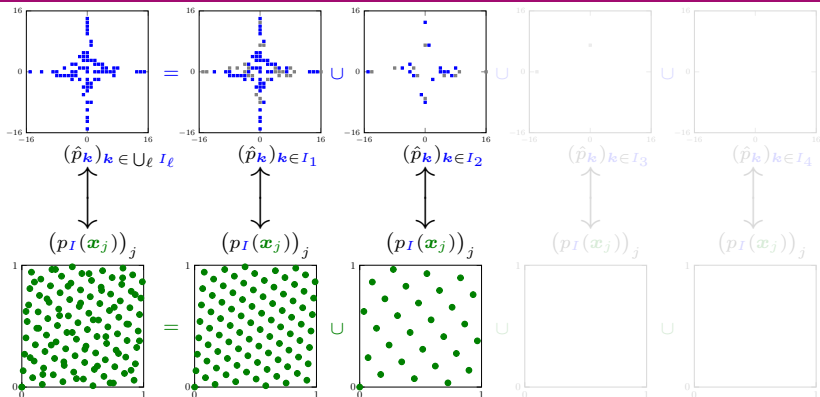
remaining freq. from I :



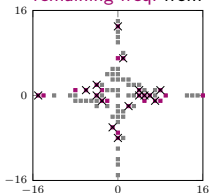
- ▶ number of samples: $\mathcal{O}(|I| \log^2 |I|)$
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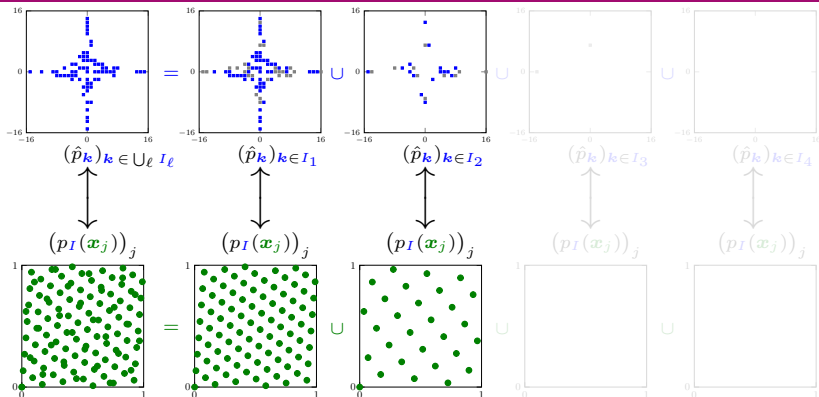


remaining freq. from I :

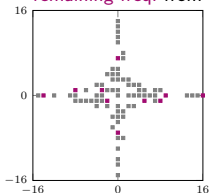


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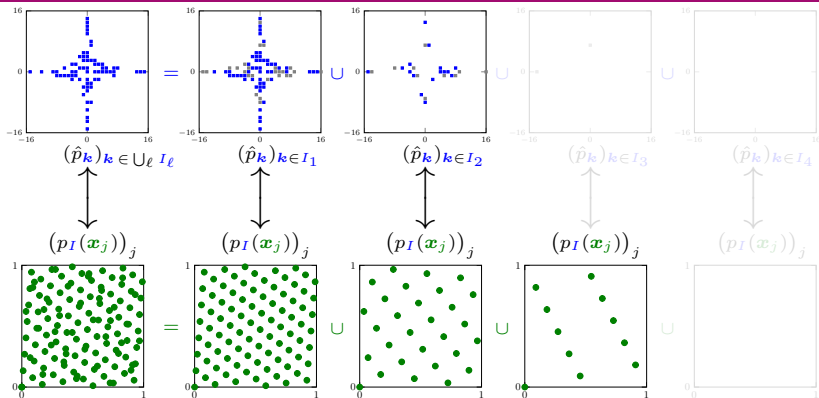
remaining freq. from I :



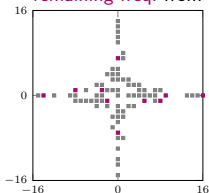
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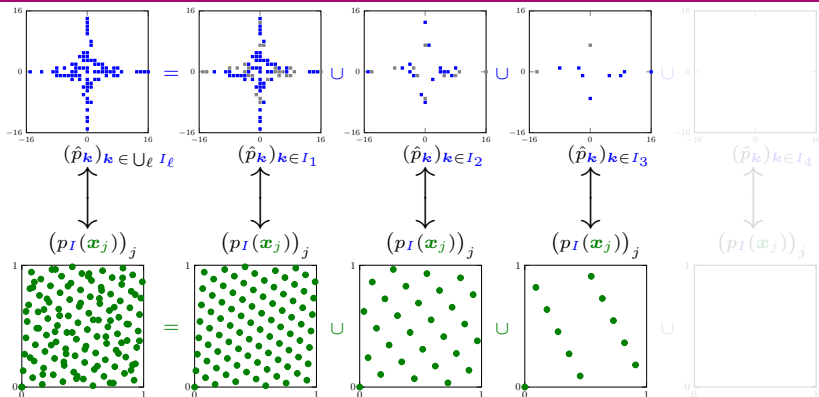


remaining freq. from I :

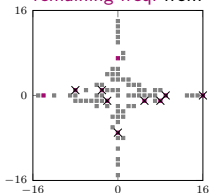


- ▶ number of samples: $\mathcal{O}(|I| \log^2 |I|)$
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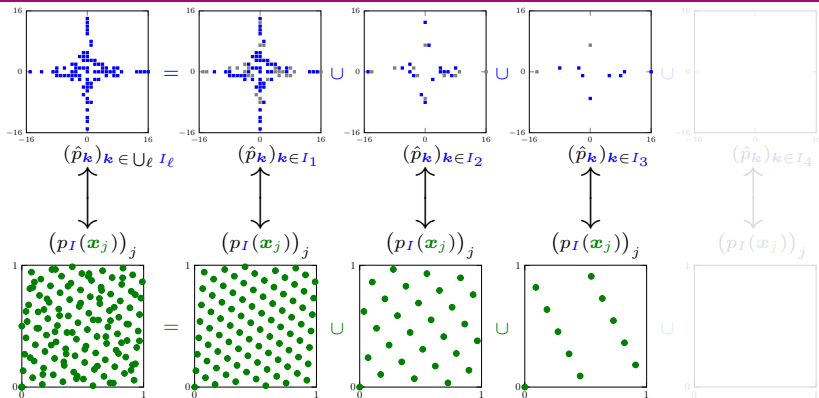
remaining freq. from I :



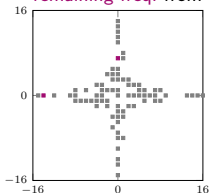
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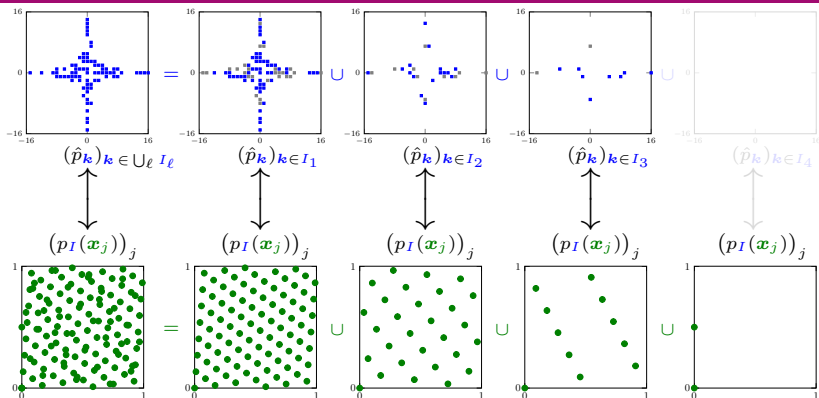


remaining freq. from I :

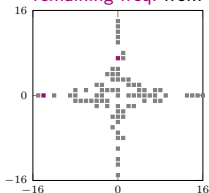


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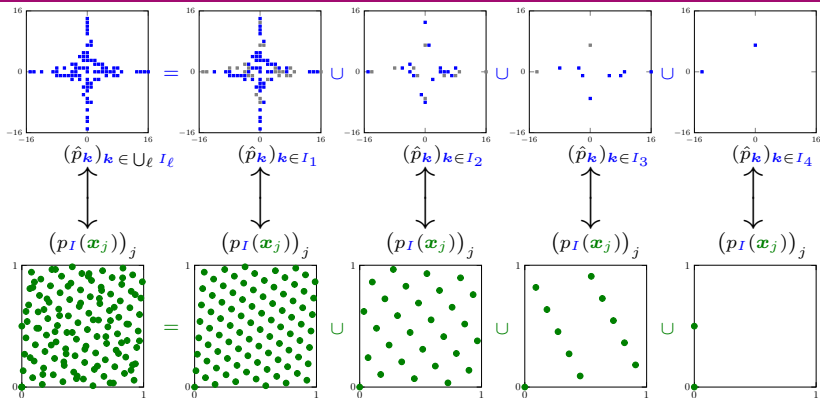
remaining freq. from I :



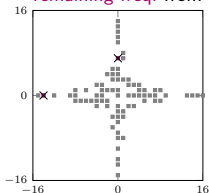
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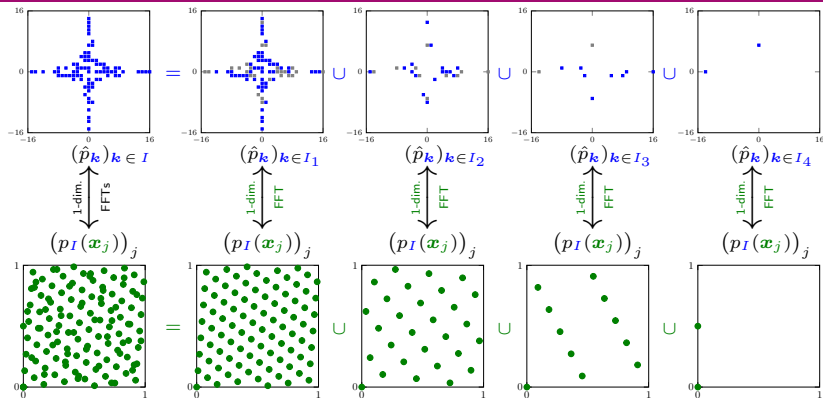
remaining freq. from I :



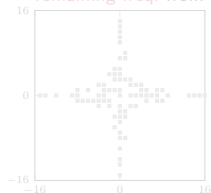
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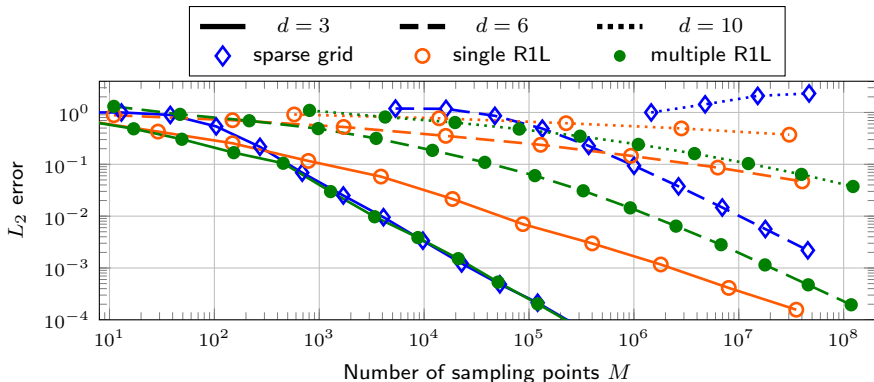
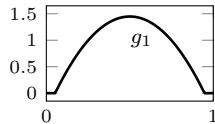
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kink function $g_d: \mathbb{T}^d \rightarrow \mathbb{R}$,

$$g_d(\mathbf{x}) = \prod_{s=1}^d \left(\frac{5^{3/4} 15}{4\sqrt{3}} \max \left\{ \frac{1}{5} - \left(x_s - \frac{1}{2}\right)^2, 0 \right\} \right)$$

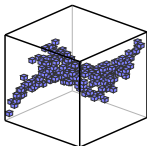


- error estimates for (multiple) rank-1 lattice sampling
in [Byrenheid, Kämmerer, Ullrich, V. '17] [V. '17] [Kämmerer, V. '18]



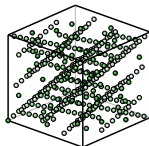
Second Part

general frequency index set $I \subset \mathbb{Z}^d$



$$\begin{array}{ccc}
 \left(\tilde{f}_{\mathbf{k}} \right)_{\mathbf{k} \in I} & \begin{array}{c} \xleftrightarrow{\text{1-dim.}} \\ \xleftarrow{\text{FFTs}} \end{array} & \left(f(\mathbf{x}_j) \right)_{j=0}^{M-1} \\
 \mathcal{O}(|I| (d + \log |I|) \log^3 |I|) & &
 \end{array}$$

multiple rank-1 lattice



until now: known frequency index set I

$$f(\mathbf{x}) \approx \sum_{\mathbf{k} \in I} \tilde{f}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}, \text{ find } \tilde{f}_{\mathbf{k}} \text{ from samples of } f$$

next: unknown frequency index set I / weights / function space

$$f(\mathbf{x}) \approx \sum_{\mathbf{k} \in I} \tilde{f}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}, \text{ find } \tilde{f}_{\mathbf{k}} \text{ AND } I \subset \mathbb{Z}^d \text{ from samples of } f$$

⇒ multi-dimensional sparse FFT

- ▶ task: determine frequency index set I (out of search domain $\Gamma \subset \mathbb{Z}^d$) from samples belonging to \approx largest Fourier coefficients $\hat{f}_{\mathbf{k}}$ or to $\hat{f}_{\mathbf{k}} \neq 0$
- ▶ various existing methods, e.g., based on filters [Indyk, Kapralov '14] / Chinese Remainder Theorem [Cuyt, Lee '08] [Iwen '13] / Prony's method [Tasche, Potts '13] [Peter, Plonka, Schaback '15] [Kunis, Peter, Römer, von der Ohe '15]
- ▶ **problems:** non-sparsity, implementations?, stability, many frequencies

⇒ dimension-incremental sparse FFT based on (multiple) rank-1 lattices

[Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17]

(similar basic idea without rank-1 lattices: [Zippel '79] [Kaltofen, Lee '03] [Javadi Monagan '10] [Potts, Tasche '13])

next: unknown frequency index set I / weights / function space

$$f(\mathbf{x}) \approx \sum_{\mathbf{k} \in I} \tilde{f}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}, \text{ find } \tilde{f}_{\mathbf{k}} \text{ AND } I \subset \mathbb{Z}^d \text{ from samples of } f$$

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⇒ multi-dimensional sparse FFT

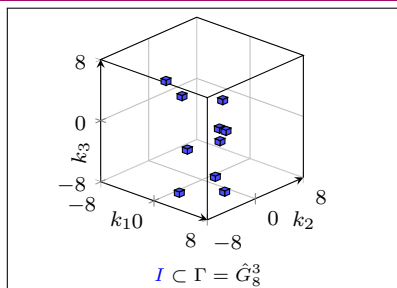
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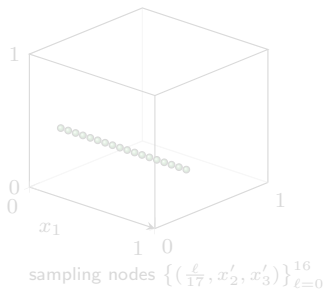
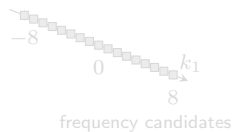
[Potts, V. '15] [V. '17] [Potts, Kämmerer, V. '17]

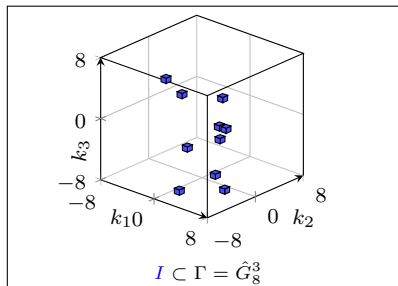
(similar basic idea without rank-1 lattices: [Zippel '79] [Kaltofen, Lee '03] [Javadi Monagan '10] [Potts, Tasche '13])



$$\hat{p}_{k_1} := \frac{1}{17} \sum_{\ell=0}^{16} p \left(\begin{pmatrix} \ell/17 \\ x_2' \\ x_3' \end{pmatrix} \right) e^{-2\pi i \frac{\ell k_1}{17}}$$

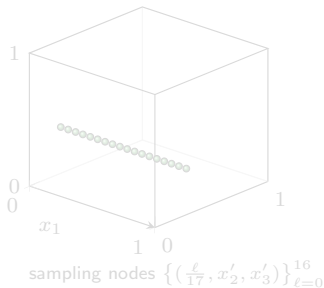
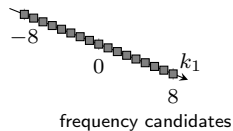
$$k_1 = -8, \dots, 8$$

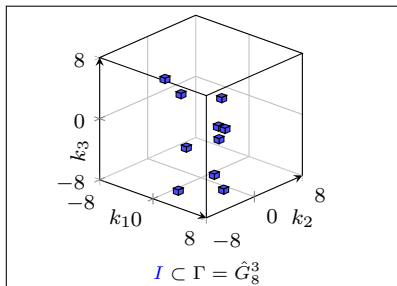




$$\hat{p}_{k_1} := \frac{1}{17} \sum_{\ell=0}^{16} p \left(\begin{pmatrix} \ell/17 \\ x_2' \\ x_3' \end{pmatrix} \right) e^{-2\pi i \frac{\ell k_1}{17}}$$

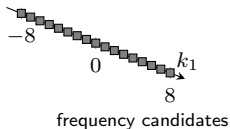
$$k_1 = -8, \dots, 8$$



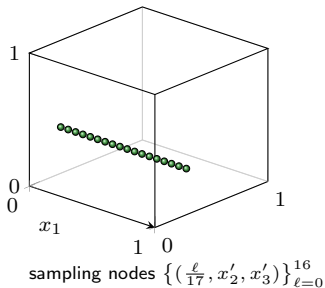


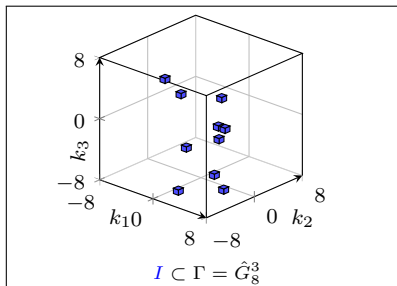
$$\hat{p}_{k_1} := \frac{1}{17} \sum_{\ell=0}^{16} p \left(\begin{pmatrix} \ell/17 \\ x'_2 \\ x'_3 \end{pmatrix} \right) e^{-2\pi i \frac{\ell k_1}{17}}$$

$$k_1 = -8, \dots, 8$$



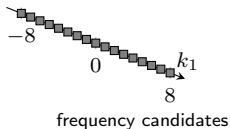
construct
→
sampling set



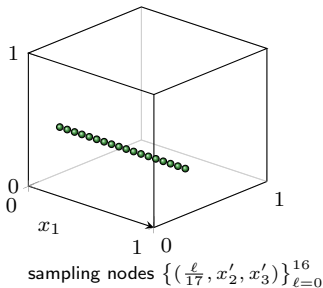


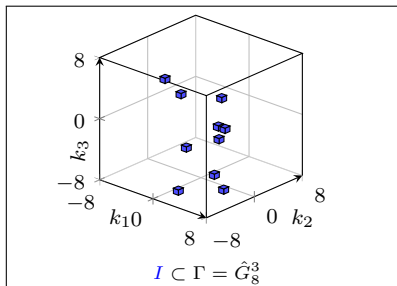
$$\hat{p}_{k_1} := \frac{1}{17} \sum_{\ell=0}^{16} p \left(\begin{pmatrix} \ell/17 \\ x_2' \\ x_3' \end{pmatrix} \right) e^{-2\pi i \frac{\ell k_1}{17}}$$

$$k_1 = -8, \dots, 8$$



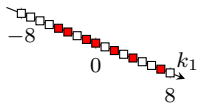
1-dim.
←
FFT





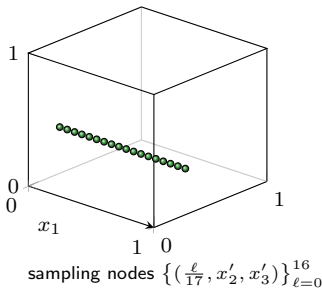
$$\begin{aligned} \hat{p}_{k_1} &:= \frac{1}{17} \sum_{\ell=0}^{16} p \left(\begin{pmatrix} \ell/17 \\ x'_2 \\ x'_3 \end{pmatrix} \right) e^{-2\pi i \frac{\ell k_1}{17}} \\ &= \sum_{\substack{(h_2, h_3) \in \{-8, \dots, 8\}^2 \\ (k_1, h_2, h_3)^\top \in \text{supp } \hat{p}}} \hat{p} \begin{pmatrix} k_1 \\ h_2 \\ h_3 \end{pmatrix} e^{2\pi i (h_2 x'_2 + h_3 x'_3)}, \end{aligned}$$

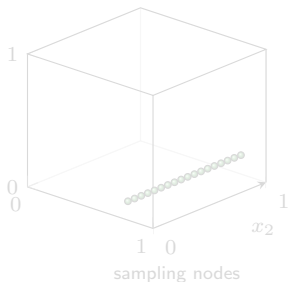
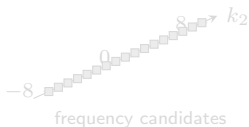
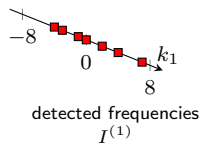
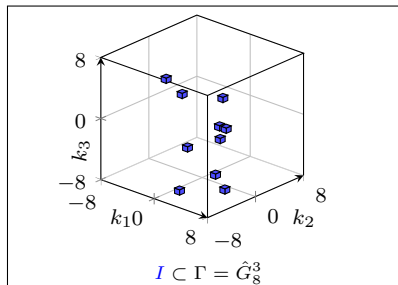
$$k_1 = -8, \dots, 8$$

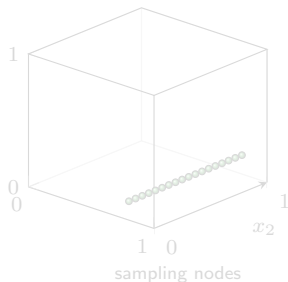
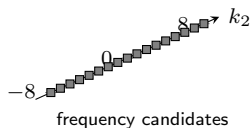
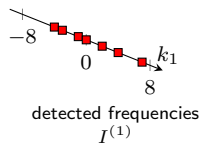
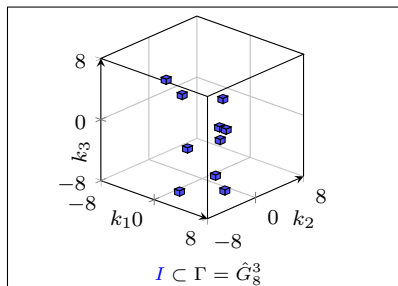


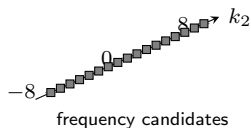
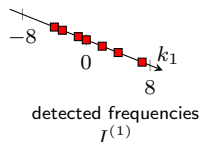
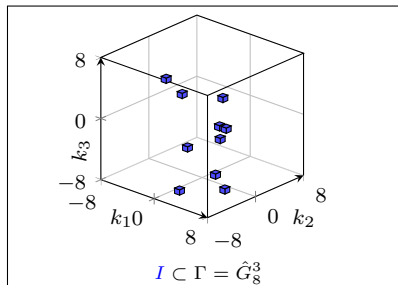
detected frequencies $I^{(1)}$

1-dim.
←
FFT

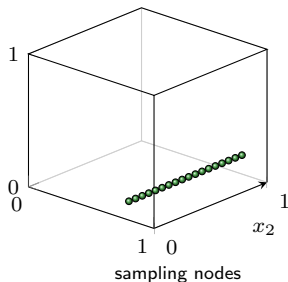


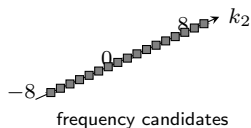
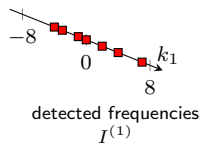
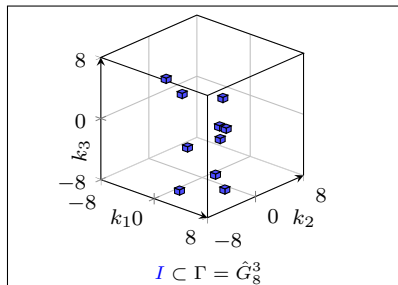




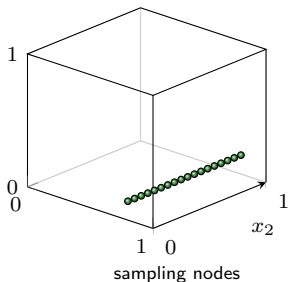


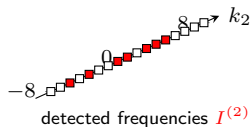
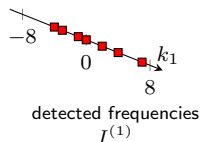
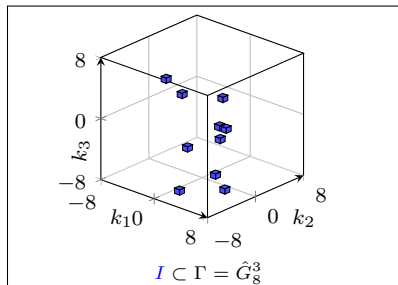
construct
→
sampling set



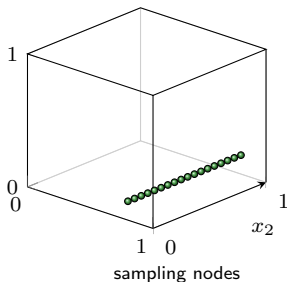


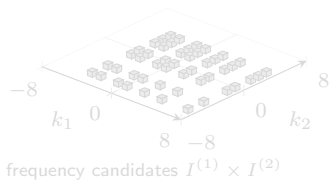
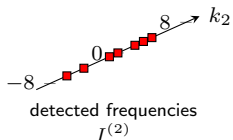
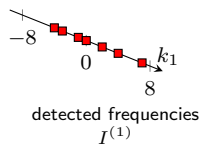
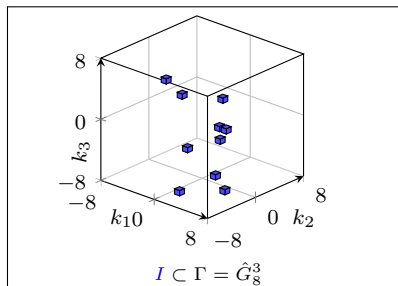
1-dim.
←
FFT

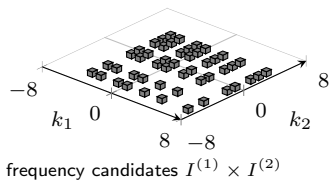
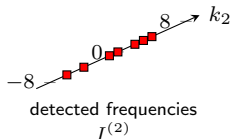
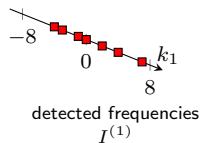
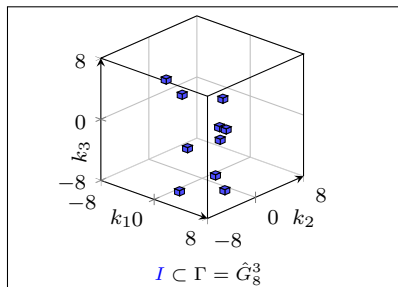


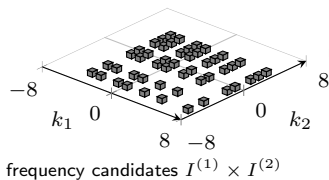
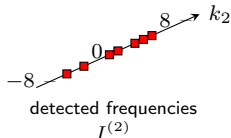
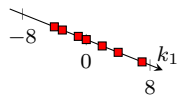
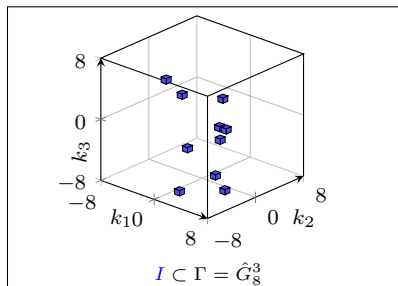


1-dim.
←
FFT

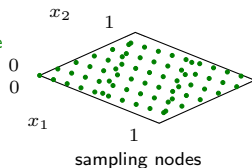


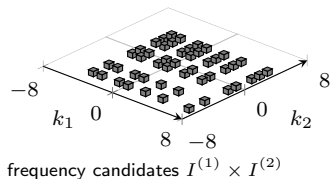
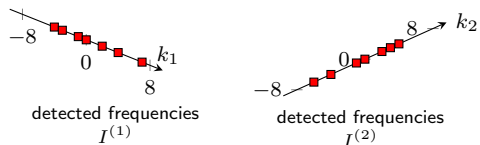
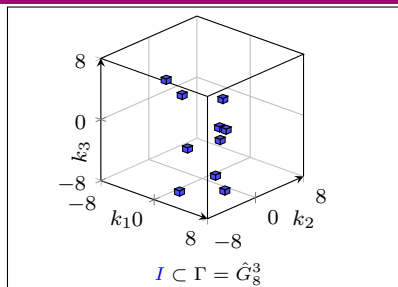




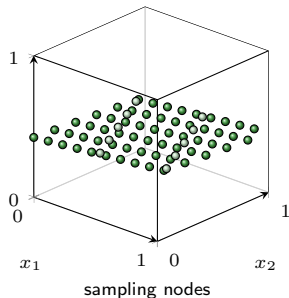


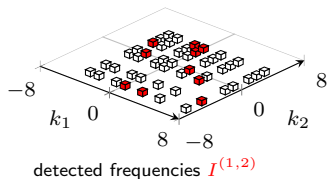
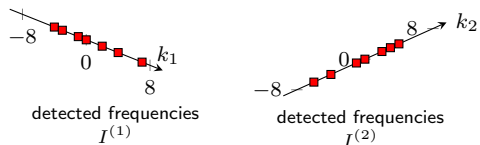
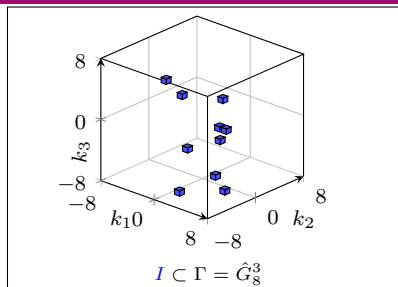
reconstructing
→
multiple rank-1 lattice



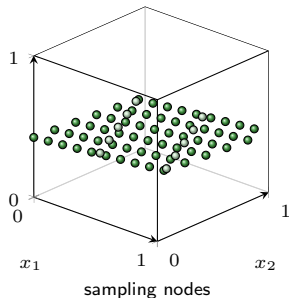


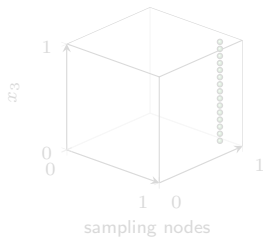
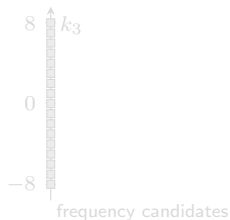
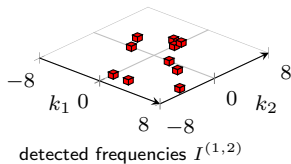
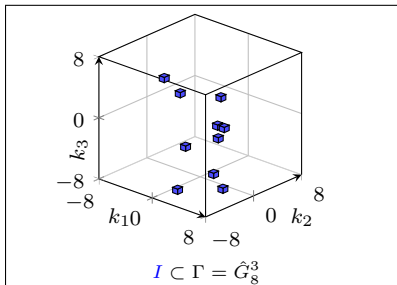
1-dim.
←
FFTs

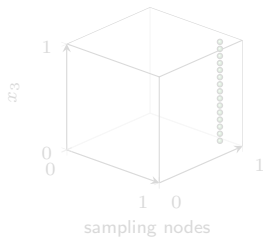
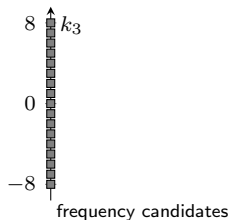
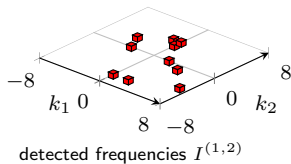
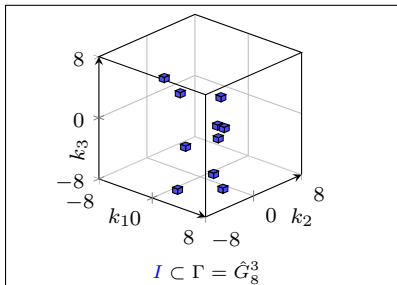


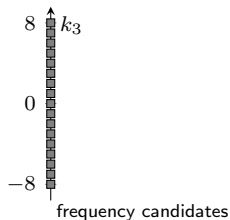
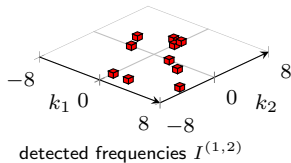
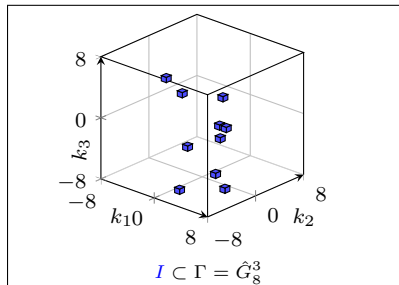


1-dim.
←
FFTs

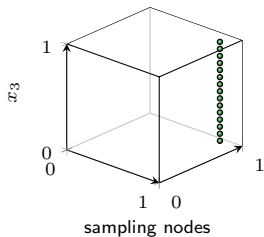


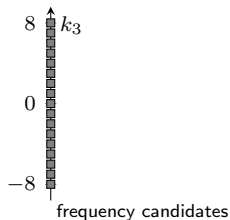
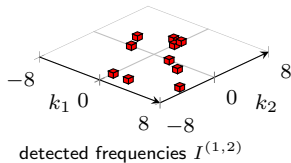
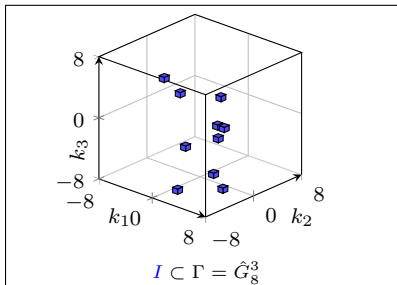




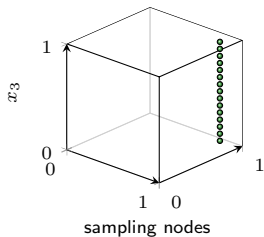


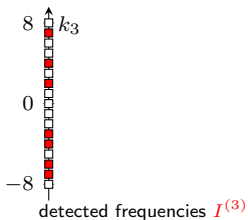
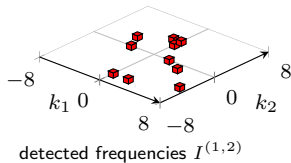
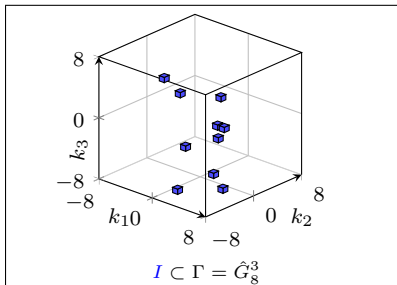
construct
→
sampling set



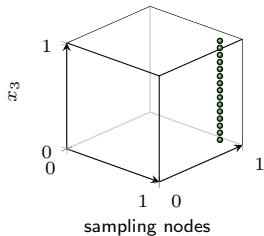


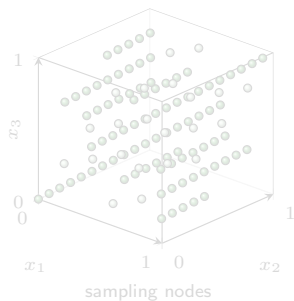
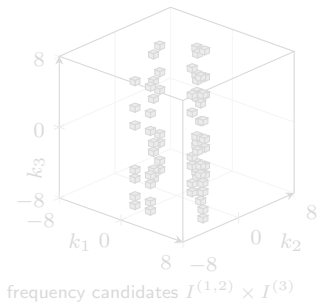
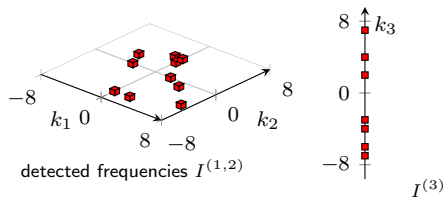
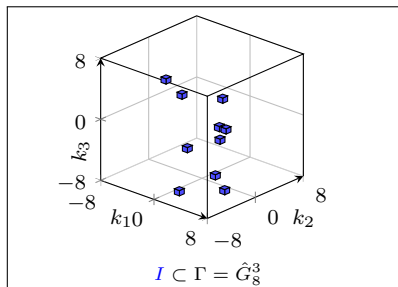
1-dim.
←
FFT

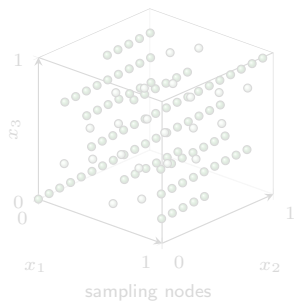
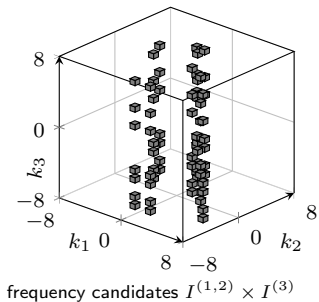
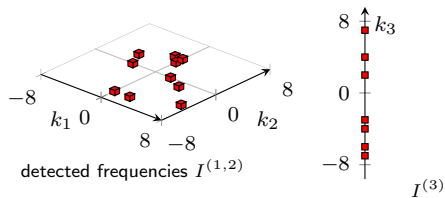
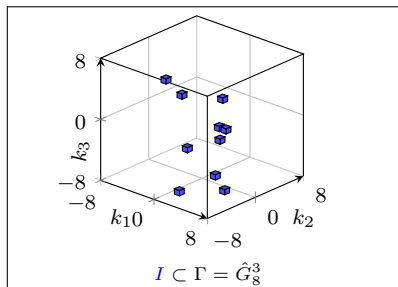


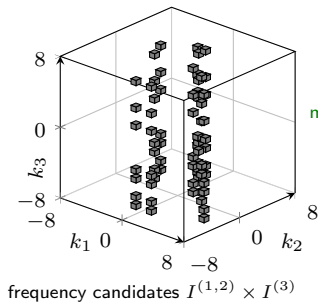
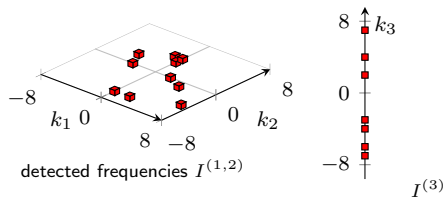
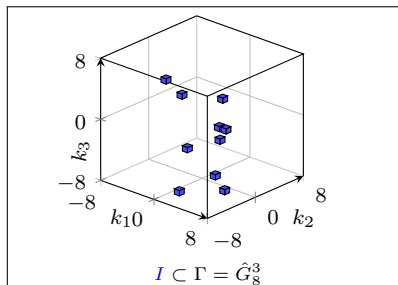


1-dim.
←
FFT

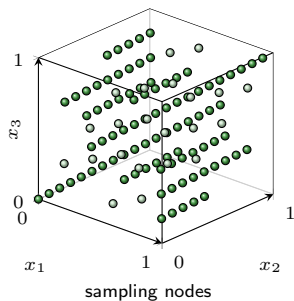


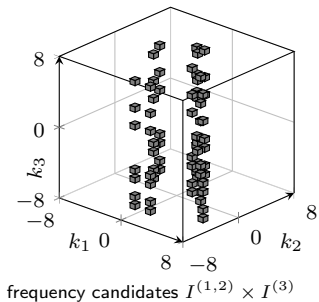
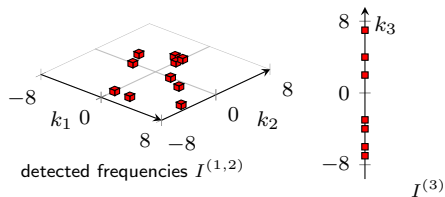
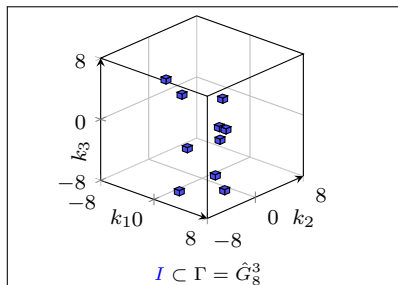




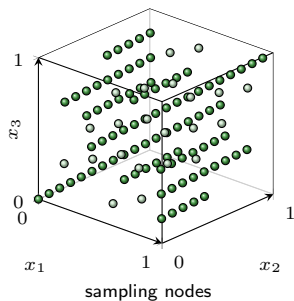


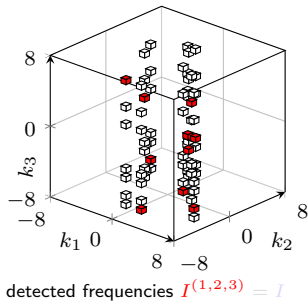
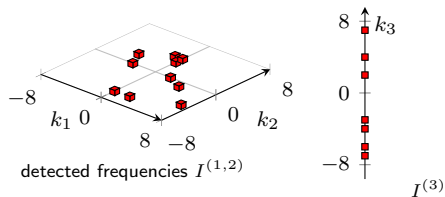
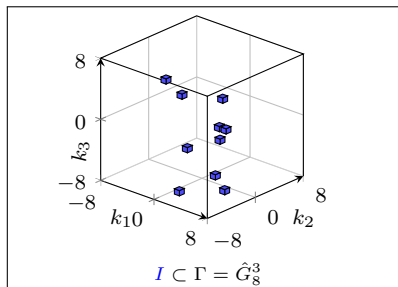
reconstructing
→
multiple rank-1 lattice



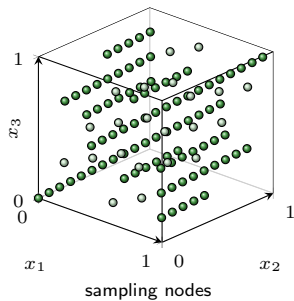


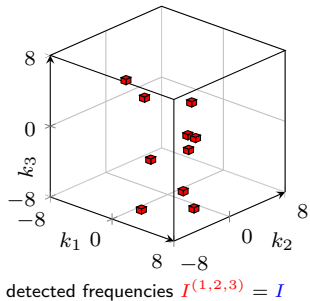
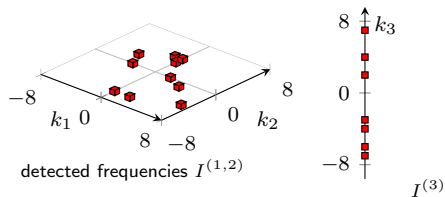
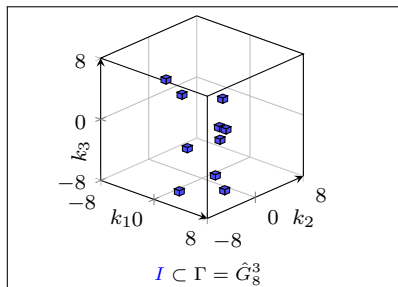
1-dim.
←
FFT_s



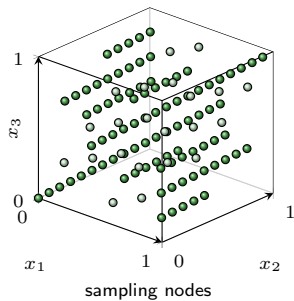


1-dim.
←
FFT_s





1-dim.
←
FFTs



reconstruction of $p_I(\mathbf{x}) = \sum_{\mathbf{k} \in I} \hat{p}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}$ with unknown I using **rank-1 lattices**:

- ▶ sparsity $s = |I|$, search domain $\Gamma = \hat{G}_N^d := \{-N, \dots, N\}^d \supset I$,
 number of detection iterations r

	single rank-1 lattices	multiple rank-1 lattices
samples	$\mathcal{O}(dr^2 s^2 N)$	$\mathcal{O}(drsN \log^2(rsN))$ (w.h.p.)
arithmetic op.	$\mathcal{O}(dr^3 s^3 + dr^2 s^2 N \log(rsN))$	$\mathcal{O}(d^2 rsN \log^4(rsN))$ (w.h.p.)

- ▶ if ($\text{Re}(\hat{p}_{\mathbf{k}})$ identical sign) AND ($\text{Im}(\hat{p}_{\mathbf{k}})$ identical sign) $\Rightarrow r = 1$;
 otherwise
 - ▶ in theory: $r = 2s(\log 3 + \log d + \log s - \log \varepsilon)$ for failure probability $\varepsilon \in (0, 1)$
 - ▶ in practice: $r = 1$ or 2
- ▶ can be applied for approximate reconstruction of function
 $f \in L_2(\mathbb{T}^d) \cap C(\mathbb{T}^d)$
- ▶ MATLAB implementation
- ▶ numerically tested for up to 30 spatial dimensions

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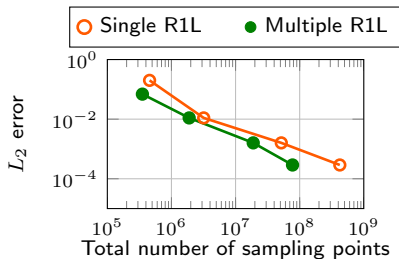
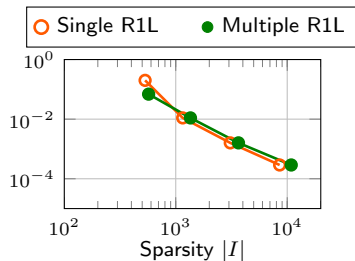
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- ▶ $f(\mathbf{x}) := \prod_{t \in \{1,3,8\}} B_2(x_t) + \prod_{t \in \{2,5,6,10\}} B_4(x_t) + \prod_{t \in \{4,7,9\}} B_6(x_t)$,
 B_m univariate B-spline of order $m \in \mathbb{N}$
- ▶ dimension-incremental sparse FFT for $\Gamma = \hat{G}_{64}^{10}$ ($|\hat{G}_{64}^{10}| \approx 1.28 \cdot 10^{21}$), $r = 10$:

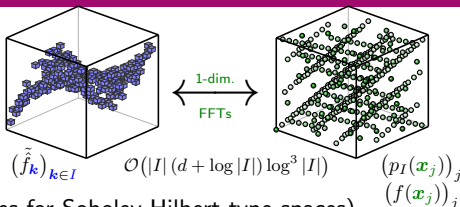


- ▶ known frequency index set $I \subset \mathbb{Z}^d$, multiple rank-1 lattices

- ▶ fast reconstruction of high-dim. trigonometric polynomials p_I
[Kämmerer '16] [Kämmerer '17]

- ▶ fast approximation (error estimates for Sobolev-Hilbert type spaces)

[Kämmerer, Potts, V. '15] [Byrenheid, Kämmerer, Ullrich, V. '17] [V. '17] [Kämmerer, V. '18]



- ▶ **unknown** $I \subset \mathbb{Z}^d$, sampling along (multiple) rank-1 lattices

- ▶ high-dimensional dimension-incremental sparse FFT

[Potts, V. '16] [V. '17] [Kämmerer, V. '17]

- ▶ very good numerical results

for high-dimensional sparse trigonometric polynomials and

for high-dimensional functions (non-sparse in frequency domain)











- ▶ see also



L. Kämmerer, D. Potts, T. V. **High-dimensional sparse FFT based on sampling along multiple rank-1 lattices.** *ArXiv e-prints* 1711.05152, Nov. 2017.



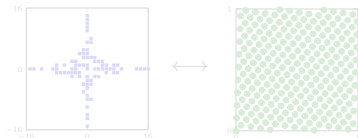
L. Kämmerer, T. V. **Approximation of multivariate periodic functions based on sampling along multiple rank-1 lattices.** *ArXiv e-prints* 1802.06639, Feb. 2018.

-  L. Kämmerer. **High Dimensional Fast Fourier Transform Based on Rank-1 Lattice Sampling.** *Dissertation (PhD thesis), Faculty of Mathematics, Chemnitz University of Technology*, 2014.
-  L. Kämmerer, D. Potts and T. V. **Approximation of multivariate periodic functions by trigonometric polynomials based on rank-1 lattice sampling.** *J. Complexity*, 31:543–576, 2015.
-  D. Potts and T. V. **Sparse high-dimensional FFT based on rank-1 lattice sampling.** *Appl. Comput. Harmon. Anal.*, 41:713–748, 2016.
-  L. Kämmerer. **Multiple Rank-1 Lattices as Sampling Schemes for Multivariate Trigonometric Polynomials.** *J. Fourier Anal. Appl.*, 2016.
-  G. Byrenheid, L. Kämmerer, T. Ullrich and T. V. **Tight error bounds for rank-1 lattice sampling in spaces of hybrid mixed smoothness.** *Numer. Math.*, 136:993–1034, 2017.
-  L. Kämmerer. **Constructing spatial discretizations for sparse multivariate trigonometric polynomials that allow for a fast discrete Fourier transform.** *Appl. Comput. Harmon. Anal.*, 2017.
-  T. V. **Multivariate Approximation and High-Dimensional Sparse FFT Based on Rank-1 Lattice Sampling.** *Dissertation (PhD thesis), Faculty of Mathematics, Chemnitz University of Technology*, 2017.
-  L. Kämmerer, D. Potts, T. V. **High-dimensional sparse FFT based on sampling along multiple rank-1 lattices.** *ArXiv e-prints* 1711.05152, Nov. 2017.
-  L. Kämmerer, T. V. **Approximation of multivariate periodic functions based on sampling along multiple rank-1 lattices.** *ArXiv e-prints* 1802.06639, Feb. 2018.
-  Software: MATLAB toolboxes (for single rank-1 lattices) <https://www.tu-chemnitz.de/~tovo>

- ▶ $\mathcal{H}_{\text{mix}}^\beta(\mathbb{T}^d) := \left\{ f \in L_2 : \|f\| := \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d} |\hat{f}_{\mathbf{k}}|^2 \prod_{s=1}^d \max(1, |k_s|)^{2\beta}} < \infty \right\}, \beta \geq 0$
- ▶ hyperbolic cross $I := \{\mathbf{k} \in \mathbb{Z}^d : \prod_{s=1}^d \max(1, |k_s|) \leq N\}$
- ▶ sparse grids: $\|f - S_I^{\text{SG}} f\|_{L_2(\mathbb{T}^d)} \lesssim N^{-\beta} \log^{\frac{d-1}{2}} N \|f\|_{\mathcal{H}_{\text{mix}}^\beta(\mathbb{T}^d)}$ [Sickel, Ullrich '07]
- ▶ rank-1 lattice: $\|f - S_I^{\text{R1L}} f\|_{L_2(\mathbb{T}^d)} \lesssim N^{-\beta} \log^{\frac{d-1}{2}} N \|f\|_{\mathcal{H}_{\text{mix}}^\beta(\mathbb{T}^d)}$
 special case of [Byrenheid, Kämmerer, Ullrich, V. '17] [V. '17]

! reconstructing $\text{R1L}(z, M, I)$ has size M
 with $|I| \leq M \leq |I|^2$

- ▶ number of samples: $\mathcal{O}(|I|^2)$
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⇒ use multiple rank-1 lattice [Kämmerer '16] [Kämmerer '17]

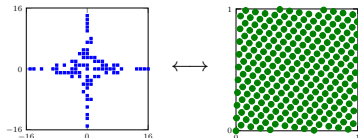
$$\|f - S_I^{\text{MR1L}} f\|_{L_2(\mathbb{T}^d)} \lesssim M^{-\beta + \frac{1}{2} + \epsilon} \log^{(\beta - \frac{1}{2} - \epsilon)d + 1} M \|f\|_{\mathcal{H}_{\text{mix}}^\beta(\mathbb{T}^d)}, \epsilon > 0$$

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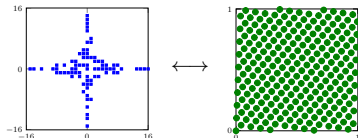
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