High-dimensional approximation and sparse FFT

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joint work with Lutz Kämmerer and Daniel Potts

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- torus $\mathbb{T} \simeq [0,1)$, $\{\mathrm{e}^{2\pi\mathrm{i}kx}\}_{k\in\mathbb{Z}}$ orthonormal basis of $L_2(\mathbb{T})$
- function $f \in L_2(\mathbb{T})$, $f(x) = \sum_{k\in\mathbb{Z}} \hat{f}_k \, \mathrm{e}^{2\pi\mathrm{i}kx}$, $\quad \hat{f}_k \in \mathbb{C}$
- smooth function $f \Longrightarrow$ fast decay of Fourier coefficients $\hat{f}_k$
- truncated Fourier series $S_I f(x) = \sum_{k\in I} \hat{f}_k \, \mathrm{e}^{2\pi\mathrm{i}kx} \approx f(x)$
- $\hat{f}_k = \int_{\mathbb{T}} f(x) \, \mathrm{e}^{-2\pi\mathrm{i}kx} \, \mathrm{d}x \approx \tilde{\hat{f}}_k := \frac{1}{2N} \sum_{j=0}^{2N-1} f(x_j) \, \mathrm{e}^{-2\pi\mathrm{i}kx}$, $x_j := \frac{j}{2N}$

$\Rightarrow$ transfer to multivariate case

$(\tilde{\hat{f}}_k)_{k\in I} \xleftrightarrow[\text{FFT}]{\text{1-dim.}} (f(x_j))_{j=0}^{2N-1}$

$\mathcal{O}(N \log N)$

[Gauß 1866] [Cooley, Tukey 1965]

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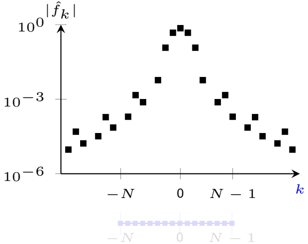

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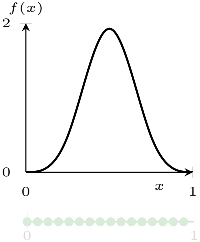

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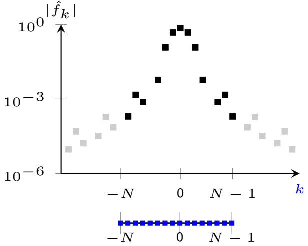

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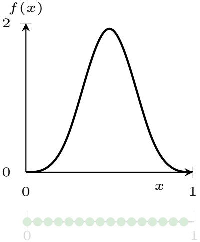

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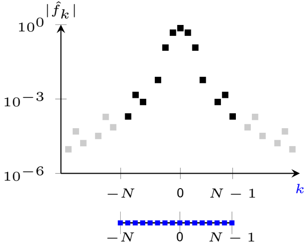

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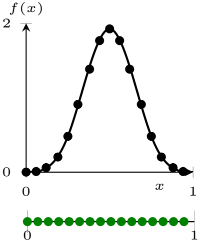

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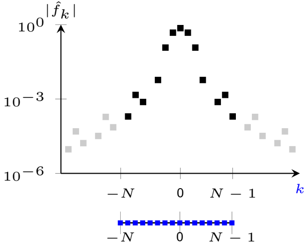

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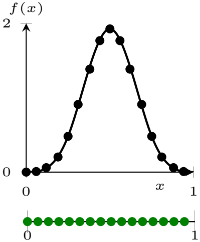
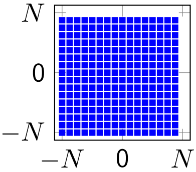


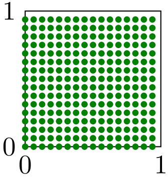
- full grid in frequency domain, equispaced grid in spatial domain



$$(\hat{\tilde{\hat{f}}}_{\boldsymbol{k}})_{\boldsymbol{k}\in I} \xleftrightarrow[\text{FFT}]{d\text{-dim.}} (f(\boldsymbol{x}_j))_{j=0}^{|I|-1}$$

$$\mathcal{O}(N^d \log N)$$

curse of dimensionality



? high-dimensional case $\Longrightarrow$ assumption: sparsity or smoothness

- hyperbolic cross in frequency domain, sparse grid in spatial domain

[Smolyak '63; Temlyakov '85; Baszenski, Delvos '89; Hallatschek '92; Gradinaru '07; Griebel, Hamaekers '14]

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condition number of Fourier matrix
[Kämmerer, Kunis '11]

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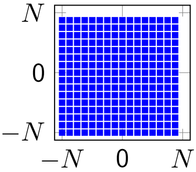

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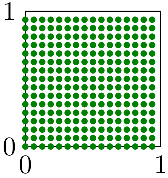

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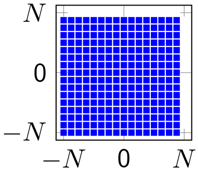

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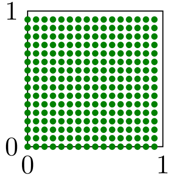

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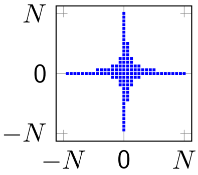

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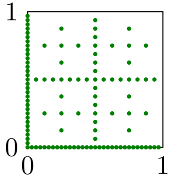

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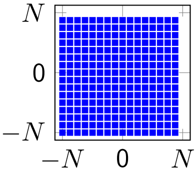

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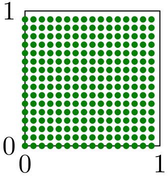

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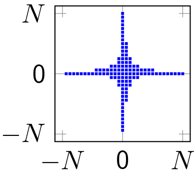

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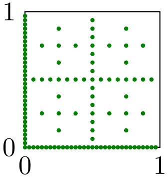

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Outline

1. Introduction

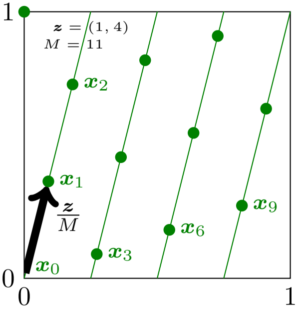
2. Fast reconstruction and approximation of multivariate periodic functions using rank-1 lattice sampling

3. Reconstructing multiple rank-1 lattices

4. High-dimensional dimension-incremental sparse FFT

5. Conclusion

- $f(\boldsymbol{x}) = p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, arbitrary index set $I \subset \mathbb{Z}^d$, $|I| < \infty$
- rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, M) := \{\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1}\}_{j=0}^{M-1}$, $\boldsymbol{z} \in \mathbb{Z}^d$, $M \in \mathbb{N}$, as discretization in spatial domain



Korobov '59
Maisonneuve '72
Sloan & Kachoyan '84,'87,'90
Temlyakov '86
Lyness '89
Sloan & Joe '94
Sloan & Reztsov '01
Li & Hickernell '03
Kämmerer & Kunis & Potts '12

$$\hat{p}_{\boldsymbol{0}} = \int_{\mathbb{T}^d} p_I(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \sum_{j=0}^{M-1} \frac{1}{M}\, p_I(\boldsymbol{x}_j)$$

$$\hat{p}_{\boldsymbol{k}} = \int_{\mathbb{T}^d} p_I(\boldsymbol{x})\,\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x},\ \boldsymbol{k} \in I$$

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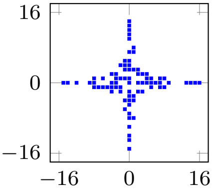
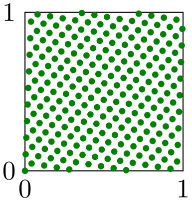


$$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I} \stackrel{?}{\longleftarrow} (p_I(\boldsymbol{x}_j))_{j=0}^{M-1}$$

$$\mathcal{O}(M \log M + d\,|I|)$$



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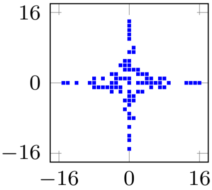
  $\Rightarrow$ reconstruction property: [Kämmerer,Kunis,Potts '12]
  $\boldsymbol{k} \cdot \boldsymbol{z} \not\equiv \boldsymbol{k}' \cdot \boldsymbol{z} \pmod{M}$ for all $\boldsymbol{k}, \boldsymbol{k}' \in I$, $\boldsymbol{k} \neq \boldsymbol{k}'$

  - $|I| \le M \le |I|^2$, simple CBC construction method [Kämmerer '12]



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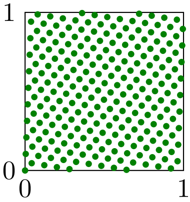

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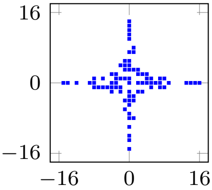
- fast approximation of $f \in L_2(\mathbb{T}^d) \cap C(\mathbb{T}^d)$, [Byrenheid,Kämmerer,Ullrich,V. '16] [V. '17]
  $f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, by $S_I^{\mathrm{R1L}} f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{f}_{\boldsymbol{k}}^{\mathrm{R1L}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



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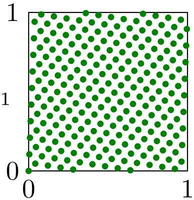
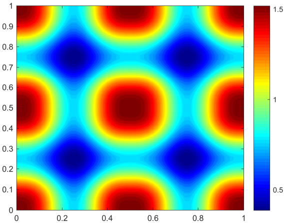
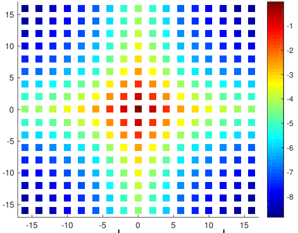


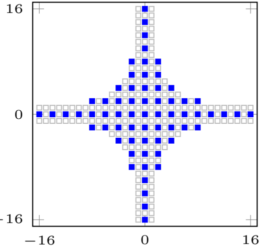
- $f(\boldsymbol{x}) := \prod_{s=1}^{d} \left(2 + \operatorname{sgn}((x_s \bmod 1) - \frac{1}{2})\, \sin(2\pi x_s)^3\right),\ f \in \mathcal{H}^{\frac{7}{2}-\epsilon}_{\mathrm{mix}}(\mathbb{T}^d),\ \epsilon > 0$
- $\mathcal{H}^{\beta}_{\mathrm{mix}}(\mathbb{T}^d) := \left\{ f \in L_2 \colon \|f\| := \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}|^2 \prod_{s=1}^{d} \max(1,|k_s|)^{2\beta}} < \infty \right\},\ \beta \geq 0$
- hyperbolic cross $I := \{\boldsymbol{k} \in 2\,\mathbb{Z}^d \colon \prod_{s=1}^{d} \max(1,|k_s|) \leq N\}$
- sparse grids: $\|f - S_I^{\mathrm{SG}} f | L_2(\mathbb{T}^d)\| \lesssim N^{-\beta} \log^{\frac{d-1}{2}} N \, \|f|\mathcal{H}^{\beta}_{\mathrm{mix}}(\mathbb{T}^d)\|$ [Sickel, Ullrich '07]
- rank-1 lattice: $\|f - S_I^{\mathrm{R1L}} f | L_2(\mathbb{T}^d)\| \lesssim N^{-\beta} \log^{\frac{d-1}{2}} N \, \|f|\mathcal{H}^{\beta}_{\mathrm{mix}}(\mathbb{T}^d)\|$
  special case of [Byrenheid, Kämmerer, Ullrich, V. '16] [V. '17]
- general approach: $\|f - S_I^{\mathrm{R1L}} f\| \leq \|f - S_I f\| + \|S_I f - S_I^{\mathrm{R1L}} f\|$



$f\left((x_1, x_2)^\top\right)$



$\log_{10} \left| \hat{f}_{(k_1,k_2)^\top} \right|$



I

Example – function from Sobolev space of dominating mixed smoothness

- $f(\boldsymbol{x}) := \prod_{s=1}^{d} \left(2 + \operatorname{sgn}((x_s \bmod 1) - \frac{1}{2})\, \sin(2\pi x_s)^3\right)$, $f \in \mathcal{H}^{\frac{7}{2}-\epsilon}_{\mathrm{mix}}(\mathbb{T}^d)$, $\epsilon > 0$
- $\mathcal{H}^{\beta}_{\mathrm{mix}}(\mathbb{T}^d) := \left\{ f \in L_2 \colon \|f\| := \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}|^2 \prod_{s=1}^{d} \max(1,|k_s|)^{2\beta}} < \infty \right\}$, $\beta \geq 0$
- hyperbolic cross $I := \{\boldsymbol{k} \in 2\,\mathbb{Z}^d \colon \prod_{s=1}^{d} \max(1,|k_s|) \leq N\}$
- sparse grids: $\|f - S_I^{\mathrm{SG}} f | L_2(\mathbb{T}^d)\| \lesssim N^{-\beta} \log^{\frac{d-1}{2}} N \; \|f|\mathcal{H}^{\beta}_{\mathrm{mix}}(\mathbb{T}^d)\|$ [Sickel, Ullrich '07]
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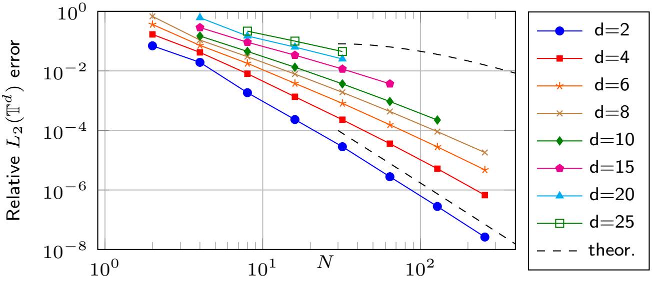
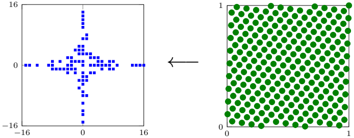
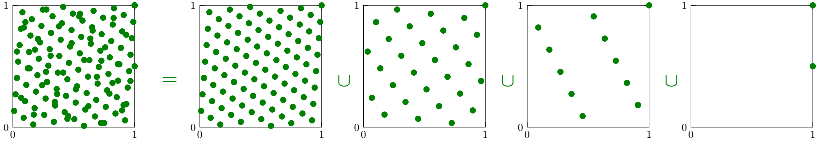


- reconstructing $\mathrm{R1L}(\boldsymbol{z}, M, I)$ has size M with $|I| \leq M \leq |I|^2$
  - number of samples: $\mathcal{O}(|I|^2)$
  - construction complexity: $\mathcal{O}(d\,|I|^3)$



$\Rightarrow$ use more than one rank-1 lattice [Kämmerer '16] [Kämmerer '17]

- perform greedy search for sequence of rank-1 lattices which reconstruct as many as possible frequencies from I



$\Rightarrow$ number of samples: $\mathcal{O}\big(|I| \log |I|\big)$,
construction complexity: $\mathcal{O}\big(d\,|I| \log^2 |I|\big)$ (w.h.p.)

Unknown frequency index set I

until now:

- fast reconstruction / approximation from samples
  for arbitrary given frequency index set $I \subset \mathbb{Z}^d$, $|I| < \infty$

next: unknown frequency index set I / weights / function space
$\Rightarrow$ multi-dimensional sparse FFT

- task: determine frequency index set I from samples
  belonging to ≈largest Fourier coefficients $\hat{f}_{\boldsymbol{k}}$ or to $\hat{f}_{\boldsymbol{k}} \neq 0$
- search domain $\Gamma \subset \mathbb{Z}^d$, e.g. full grid $\hat{G}_N^d := \{-N, -N+1, \ldots, N\}^d$, $N \in \mathbb{N}$
- various existing methods, e.g., based on
  - filters [Indyk, Kapralov '14]
  - Chinese Remainder Theorem [Cuyt, Lee '08] [Iwen '13]
  - Prony's method [Tasche, Potts '13] [Peter, Plonka, Schaback '15] [Kunis, Peter, Römer, von der Ohe '15]
- problems: non-sparsity, implementations?, stability, many frequencies

$\Rightarrow$ dimension-incremental sparse FFT based on rank-1 lattices [Potts, V. '15] [V. '17]
(similar basic idea without rank-1 lattices:
[Zippel '79] [Kaltofen, Lee '03] [Javadi Monagan '10] [Potts, Tasche '13])

until now:

- fast reconstruction / approximation from samples for arbitrary given frequency index set $I \subset \mathbb{Z}^d$, $|I| < \infty$

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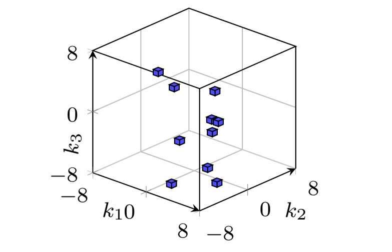
High-dimensional dimension-incremental sparse FFT [Potts, V. '15] [V. '17]
Method



$I \subset \Gamma = \hat{G}_8^3$

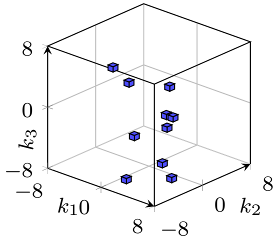
$$\hat{p}_{k_1} := \frac{1}{17} \sum_{\ell=0}^{16} p\left(\begin{pmatrix} \ell/17 \\ x_2' \\ x_3' \end{pmatrix}\right) \mathrm{e}^{-2\pi\mathrm{i}\frac{\ell k_1}{17}}$$

$k_1 = -8, \ldots, 8$

frequency candidates

sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$

High-dimensional dimension-incremental sparse FFT [Potts, V. '15] [V. '17]
Method



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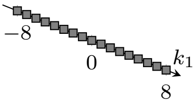

frequency candidates

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High-dimensional dimension-incremental sparse FFT [Potts, V. '15] [V. '17]
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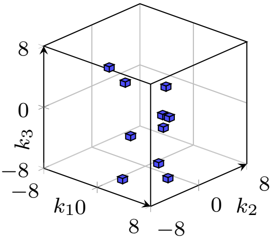

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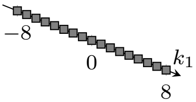
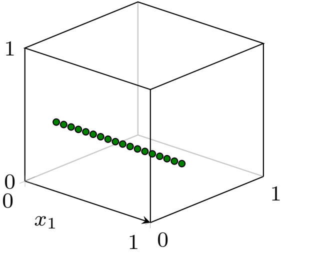


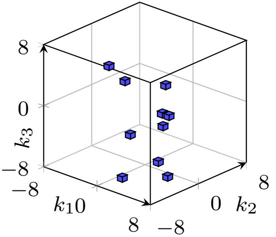
frequency candidates

construct $\longrightarrow$ sampling set



sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$

High-dimensional dimension-incremental sparse FFT [Potts, V. '15] [V. '17]
Method



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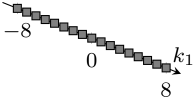
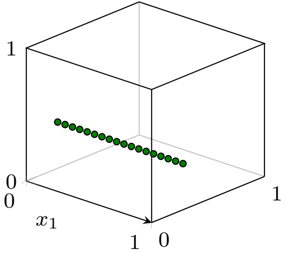


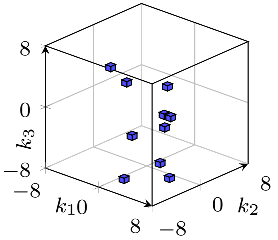
frequency candidates

1-dim. ⟵ FFT



sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$

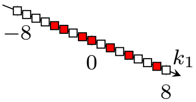
High-dimensional dimension-incremental sparse FFT [Potts, V. '15] [V. '17]
Method



$I \subset \Gamma = \hat{G}_8^3$

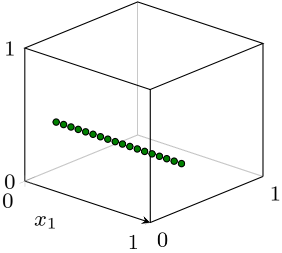
$$\hat{p}_{k_1} := \frac{1}{17}\sum_{\ell=0}^{16} p\left(\begin{pmatrix}\ell/17\\ x_2'\\ x_3'\end{pmatrix}\right)\mathrm{e}^{-2\pi\mathrm{i}\frac{\ell k_1}{17}}$$
$$= \sum_{\substack{(h_2,h_3)\in\{-8,\ldots,8\}^2\\ (k_1,h_2,h_3)^\top\in\operatorname{supp}\hat{p}}} \hat{p}_{\begin{pmatrix}k_1\\h_2\\h_3\end{pmatrix}} \mathrm{e}^{2\pi\mathrm{i}(h_2x_2'+h_3x_3')},$$

$k_1 = -8, \ldots, 8$

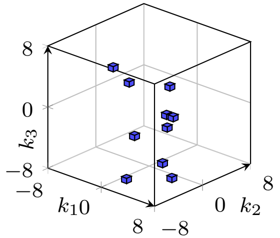


detected frequencies $I^{(1)}$

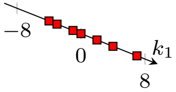
1-dim. $\longleftarrow$ FFT



sampling nodes $\{(\frac{\ell}{17}, x_2', x_3')\}_{\ell=0}^{16}$



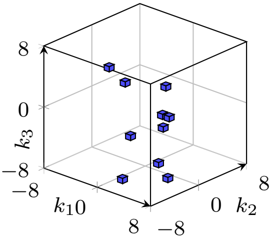
$I \subset \Gamma = \hat{G}_8^3$



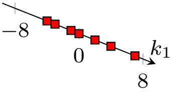
detected frequencies $I^{(1)}$

frequency candidates

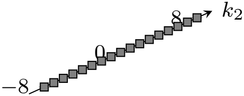
sampling nodes



$I \subset \Gamma = \hat{G}_8^3$

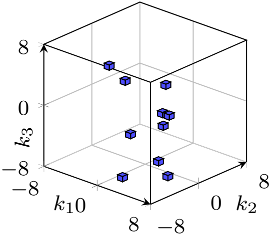


detected frequencies
$I^{(1)}$



frequency candidates

sampling nodes



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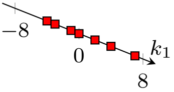
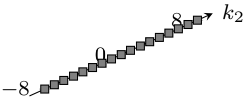
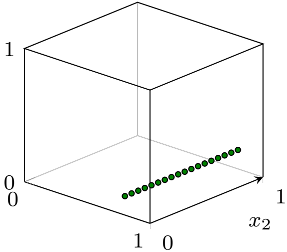


detected frequencies $I^{(1)}$

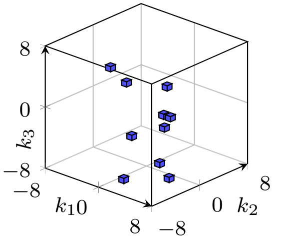


frequency candidates

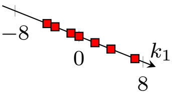
construct $\longrightarrow$ sampling set



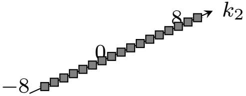
sampling nodes



$I \subset \Gamma = \hat{G}_8^3$

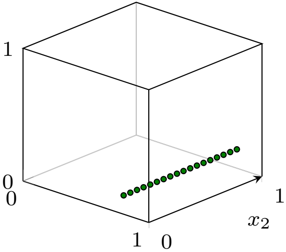


detected frequencies $I^{(1)}$

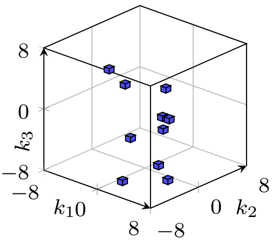


frequency candidates

1-dim. ← FFT



sampling nodes



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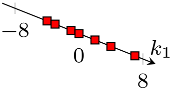
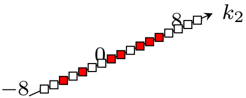
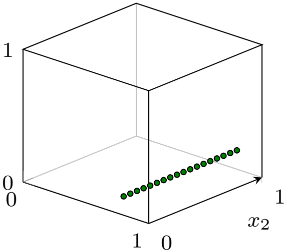


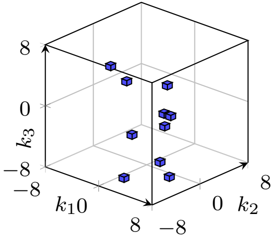
detected frequencies $I^{(1)}$



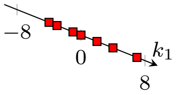
detected frequencies $I^{(2)}$



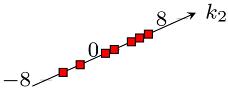
sampling nodes



$I \subset \Gamma = \hat{G}_8^3$

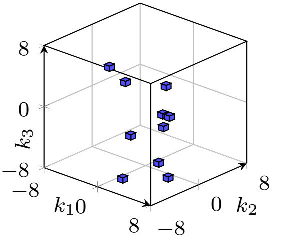


detected frequencies $I^{(1)}$



detected frequencies $I^{(2)}$

frequency candidates $I^{(1)} \times I^{(2)}$



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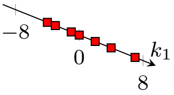
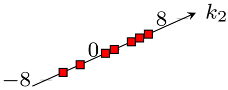
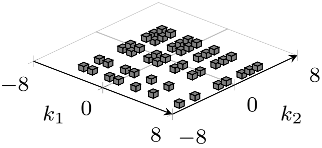


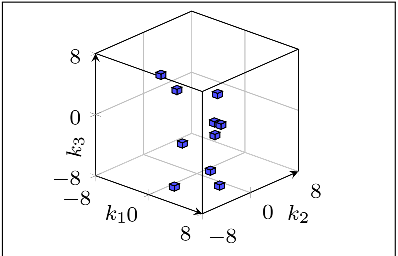
detected frequencies
$I^{(1)}$



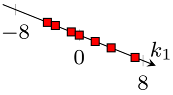
detected frequencies
$I^{(2)}$



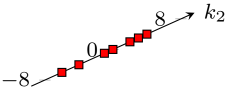
frequency candidates $I^{(1)} \times I^{(2)}$



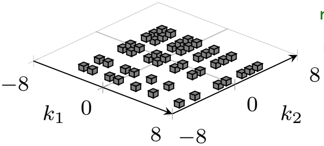
$I \subset \Gamma = \hat{G}_8^3$



detected frequencies $I^{(1)}$

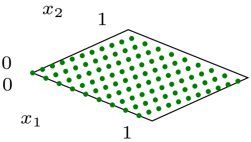


detected frequencies $I^{(2)}$



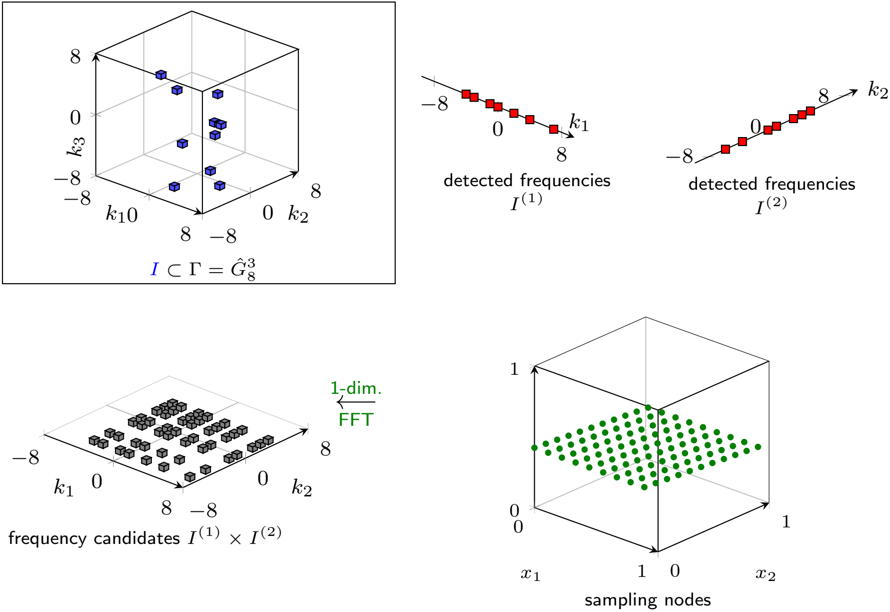
frequency candidates $I^{(1)} \times I^{(2)}$

reconstructing → rank-1 lattice



sampling nodes

High-dimensional dimension-incremental sparse FFT [Potts, V. '15] [V. '17]
Method



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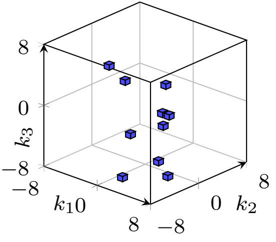
High-dimensional dimension-incremental sparse FFT [Potts, V. '15] [V. '17]
Method



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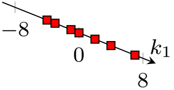
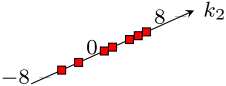
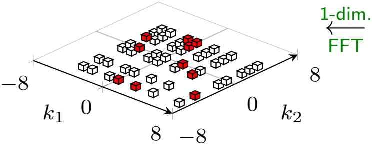


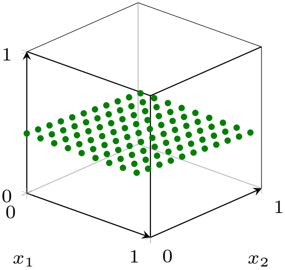
detected frequencies $I^{(1)}$



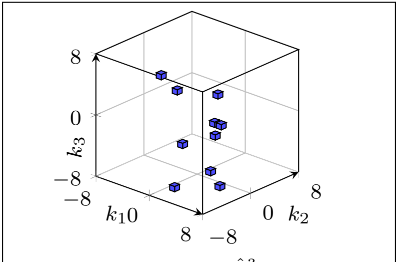
detected frequencies $I^{(2)}$



detected frequencies $I^{(1,2)}$



sampling nodes



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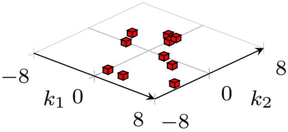
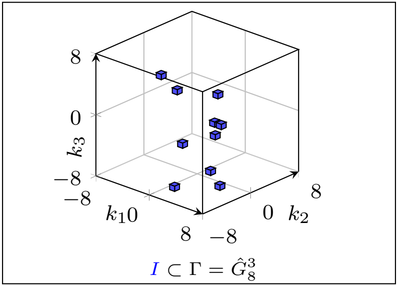


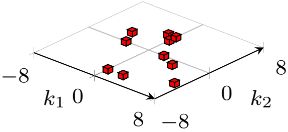
detected frequencies $I^{(1,2)}$

frequency candidates

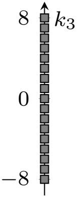
sampling nodes



$I \subset \Gamma = \hat{G}_8^3$

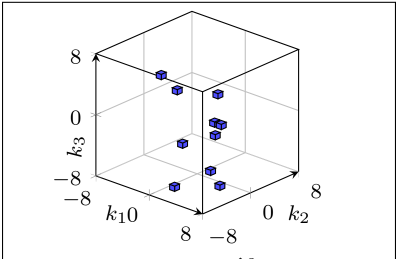


detected frequencies $I^{(1,2)}$



frequency candidates

sampling nodes



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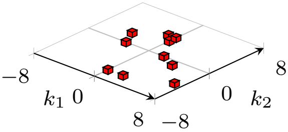
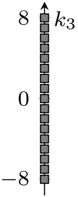
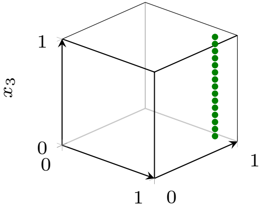


detected frequencies $I^{(1,2)}$

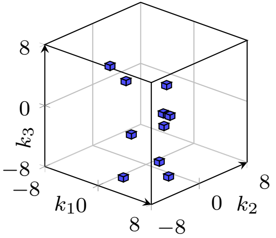


frequency candidates

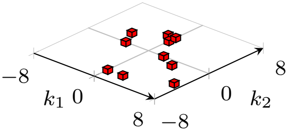
construct $\longrightarrow$ sampling set



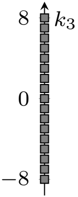
sampling nodes



$I \subset \Gamma = \hat{G}_8^3$

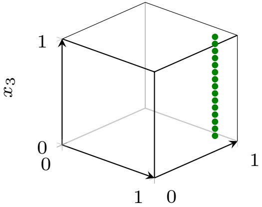


detected frequencies $I^{(1,2)}$

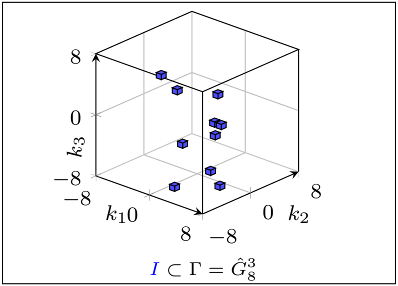


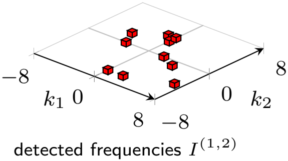
frequency candidates

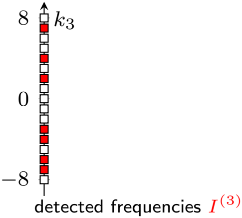
1-dim.
$\longleftarrow$
FFT



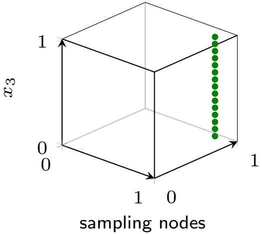
sampling nodes

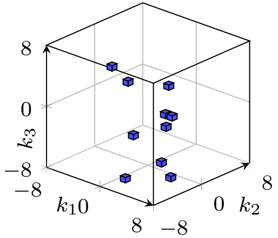

8
0
−8
k_3
−8
$k_1$0
8
−8
0
k_2
8
$I \subset \Gamma = \hat{G}_8^3$


−8
k_1 0
8
−8
0
k_2
8
detected frequencies $I^{(1,2)}$

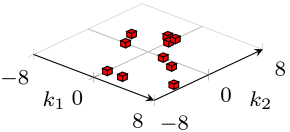

8
k_3
0
−8
detected frequencies $I^{(3)}$


1-dim.
FFT

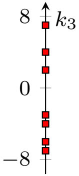

1
x_3
0
0
1
0
1
sampling nodes



$I \subset \Gamma = \hat{G}_8^3$



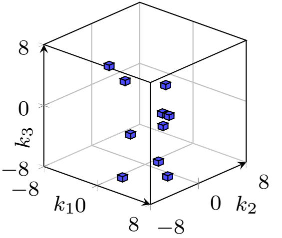
detected frequencies $I^{(1,2)}$



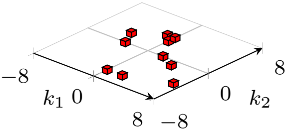
$I^{(3)}$

frequency candidates $I^{(1,2)} \times I^{(3)}$

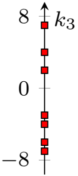
sampling nodes



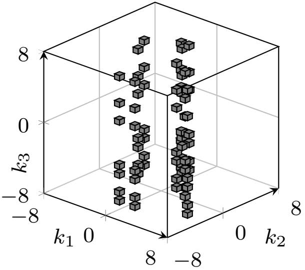
$I \subset \Gamma = \hat{G}_8^3$



detected frequencies $I^{(1,2)}$

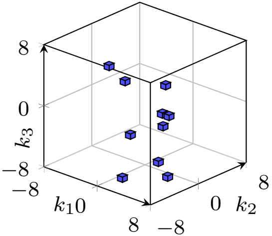


$I^{(3)}$



frequency candidates $I^{(1,2)} \times I^{(3)}$

sampling nodes



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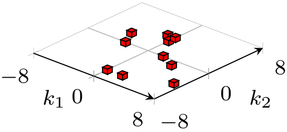
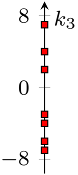
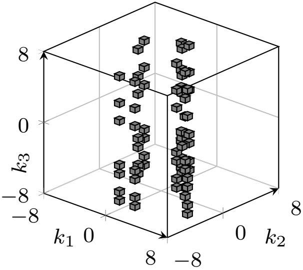


detected frequencies $I^{(1,2)}$

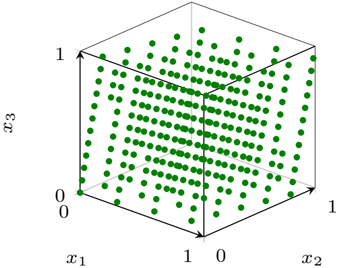


$I^{(3)}$

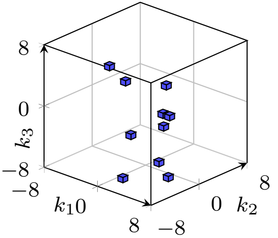


frequency candidates $I^{(1,2)} \times I^{(3)}$

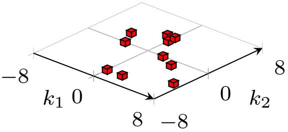
reconstructing
$\longrightarrow$
rank-1 lattice



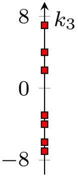
sampling nodes



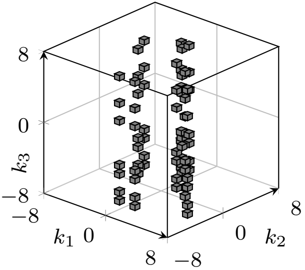
$I \subset \Gamma = \hat{G}_8^3$



detected frequencies $I^{(1,2)}$

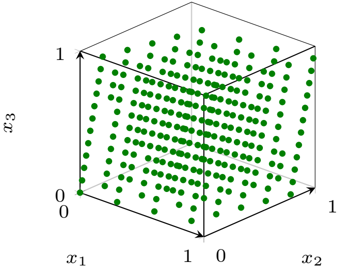


$I^{(3)}$



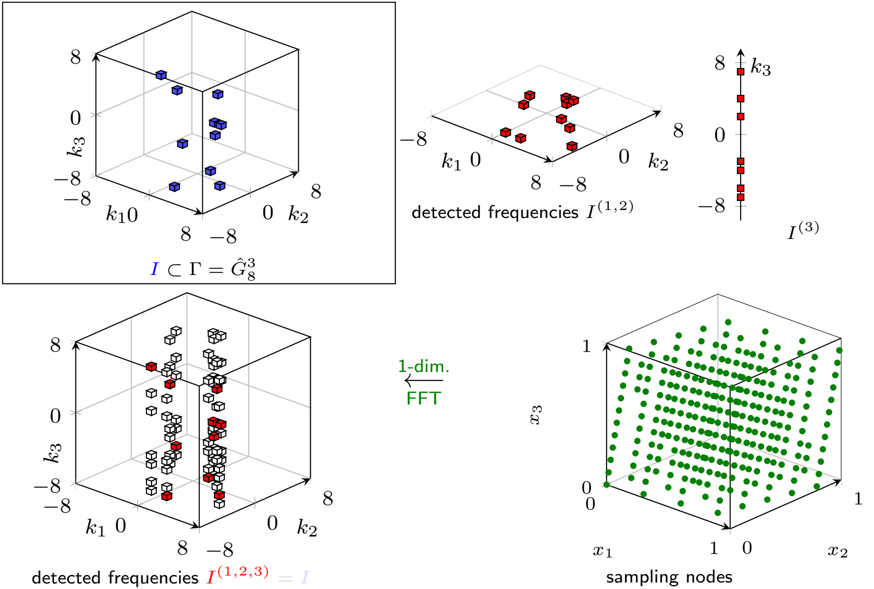
frequency candidates $I^{(1,2)} \times I^{(3)}$

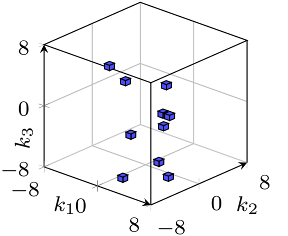
1-dim.
⟵
FFT



sampling nodes

High-dimensional dimension-incremental sparse FFT [Potts, V. '15] [V. '17]
Method





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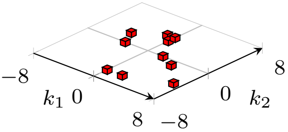
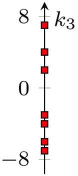
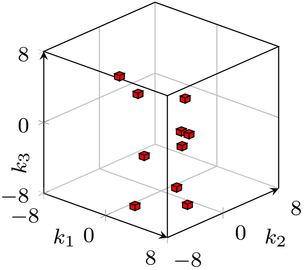


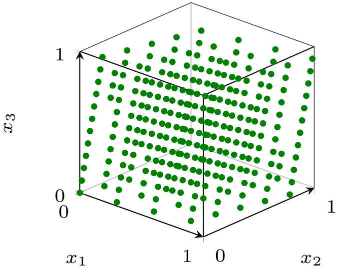
detected frequencies $I^{(1,2)}$



$I^{(3)}$



detected frequencies $I^{(1,2,3)} = I$



sampling nodes

complexity of dimension-incremental sparse FFT using rank-1 lattices:

- sparsity $s = |I|$, search domain $\Gamma = \hat{G}_N^d := \{-N, \ldots, N\}^d \supset I$, number of detection iterations r
  - if $(\mathrm{Re}(\hat{p}_{\boldsymbol{k}})$ identical sign) AND $(\mathrm{Im}(\hat{p}_{\boldsymbol{k}})$ identical sign) $\Rightarrow r = 1$

	single rank-1 lattices	multiple rank-1 lattices
samples	$\mathcal{O}(d\, r^3 s^2 N)$	$\mathcal{O}(d\, r^2 s N \log^2(rsN))$
arithmetic op.	$\mathcal{O}(d\, r^3 s^3 + d\, r^3 s^2 N \log(rsN))$	$\mathcal{O}\big(d^2 r^2 s N \log^4(rsN)\big)$ (w.h.p.)

example: $p_I(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}$, $I \subset \Gamma = \hat{G}_{32}^5 := \{-32, \ldots, 32\}^5$, $|\Gamma| \approx 1.16 \cdot 10^9$

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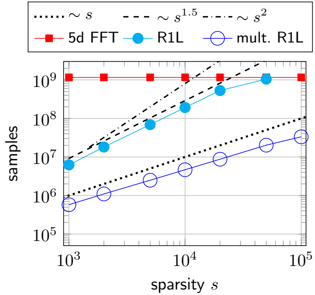

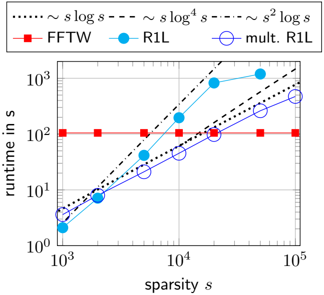

complexity of dimension-incremental sparse FFT using rank-1 lattices:

- sparsity $s = |I|$, search domain $\Gamma = \hat{G}_N^d := \{-N, \ldots, N\}^d \supset I$, number of detection iterations r
  - if ($\mathrm{Re}(\hat{p}_{\boldsymbol{k}})$ identical sign) AND ($\mathrm{Im}(\hat{p}_{\boldsymbol{k}})$ identical sign) $\Rightarrow r = 1$

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example:

- B-spline $N_m(x) := \sum_{k\in\mathbb{Z}} C_m \operatorname{sinc}\left(\frac{\pi}{m}k\right)^m (-1)^k \mathrm{e}^{2\pi\mathrm{i}kx}$
- $f(\boldsymbol{x}) := \prod_{t\in\{1,3,8\}} N_2(x_t) + \prod_{t\in\{2,5,6,10\}} N_4(x_t) + \prod_{t\in\{4,7,9\}} N_6(x_t)$
- dimension-incremental method for $\Gamma = \hat{G}_{64}^{10}$: ($|\hat{G}_{64}^{10}| \approx 1.28 \cdot 10^{21}$)

threshold	single rank-1 lattices #samples	$\lvert I\rvert$	rel. L_2 error	multiple rank-1 lattices #samples	$\lvert I\rvert$	rel. L_2 error
1.0e-02	327 689	493	1.3e-01	246 681	501	5.5e-01
1.0e-03	2 551 143	1 109	1.1e-02	1 441 455	1 205	1.1e-02
1.0e-04	17 198 228	3 009	2.0e-03	7 473 447	3 463	2.1e-03
1.0e-05	132 285 922	7 435	4.8e-04	37 056 491	11 053	4.9e-04

Conclusion

- multivariate periodic functions and rank-1 lattices
  - fast reconstruction of multivariate trigonometric polynomials p_I for arbitrary frequency index sets I [Kämmerer '14]
  - fast approximation [Kämmerer '14], error estimates in [Kämmerer '14] [Kämmerer, Potts, V. '15] [Byrenheid, Kämmerer, Ullrich, V. '16] [V. '17]
- similar results for multivariate non-periodic functions and rank-1 Chebyshev lattices (not in this talk) [Potts, V. '15] [V. '17]
- reconstructing multiple rank-1 lattices [Kämmerer '16] [Kämmerer '17]
  - distinct reduction of number of samples and arithmetic operations
- high-dimensional dimension-incremental sparse FFT and rank-1 lattices [Potts, V. '16] [V. '17]
  - determination of unknown frequency index set I
  - very good numerical results
    for high-dimensional sparse trigonometric polynomials and
    for high-dimensional functions (non-sparse in frequency domain)

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Software: MATLAB toolboxes (for single rank-1 lattices) `http://www.tu-chemnitz.de/~tovo`