Multivariate sparse FFT based on rank-1 Chebyshev lattice sampling

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TECHNISCHE UNIVERSITÄT CHEMNITZ

joint work with Daniel Potts

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Introduction

- $T_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{s=1}^{d} T_{k_s}(x_s) := \prod_{s=1}^{d} \cos(k_s \, \arccos(x_s))$, $\boldsymbol{k} \in \mathbb{N}_0^d$,
  orthogonal basis of
  $L_{2,w}([-1,1]^d) := \left\{ f\colon [-1,1]^d \to \mathbb{R}, \|f\| := \sqrt{\int_{[-1,1]^d} |f(\boldsymbol{x})|^2 \, w(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}} < \infty \right\}$,
  w.r.t. $w(\boldsymbol{x}) := \prod_{s=1}^{d} 1/\sqrt{1-x_s^2}$

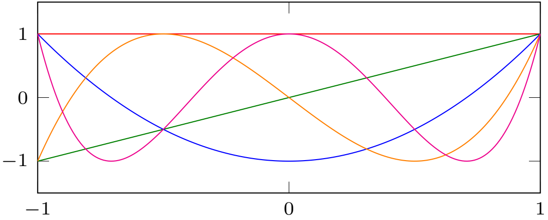
(T_m is algebraic polynomial of degree m restricted to domain $[-1,1]$)

$T_0(x) = 1$ $\quad$ $T_1(x) = x$

$T_2(x) = 2x^2 - 1$ $\quad$ $T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 + 1$

$T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x) \quad m \in \mathbb{N}$



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  w.r.t. $w(\boldsymbol{x}) := \prod_{s=1}^{d} 1/\sqrt{1-x_s^2}$
- multivariate algebraic polynomial $a_I : [-1,1]^d \to \mathbb{R}$
  in Chebyshev form with frequencies supp. on $I \subset \mathbb{N}_0^d$, $|I| < \infty$,

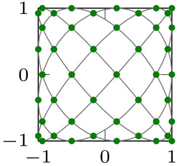
$$a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k} \in I} \hat{a}_{\boldsymbol{k}} \, T_{\boldsymbol{k}}(\boldsymbol{x}), \; \hat{a}_{\boldsymbol{k}} \in \mathbb{R},$$

- reconstruction of $f = a_I$
  or approximation of function $f \in L_{2,w}([-1,1]^d)$, $f \approx a_I(\boldsymbol{x})$
- Aim: Determine unknown I based on samples
  - multivariate case, especially $d > 4$, curse of dimensionality
  - suitable sampling set?
  - fast method?

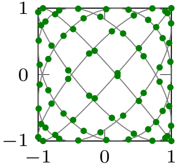
Content

Rank-1 Chebyshev lattices

- consider $f(\boldsymbol{x}) = a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x}),\ \hat{a}_{\boldsymbol{k}} \in \mathbb{R}$
- **known arbitrary** discretization in frequency domain $I \subset \mathbb{N}_0^d$, $|I| < \infty$
- discretization in spatial domain: rank-1 Chebyshev lattice [Cools, Poppe '11]
  $\mathrm{CL}(\boldsymbol{z}, M) := \left\{ \boldsymbol{x}_j := \cos\left(\frac{j}{M}\pi\boldsymbol{z}\right) : j = 0, \ldots, M \right\} \subset [-1,1]^d$, $\boldsymbol{z} \in \mathbb{N}_0^d$, $M \in \mathbb{N}_0$



$\boldsymbol{z} := (8,9)^\top$, $M := 72$, $|\mathrm{CL}(\boldsymbol{z}, M)| = 45$
Padua points [Caliari, De Marchi, Vianello '05]



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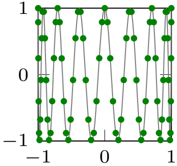

$\boldsymbol{z} := (1,16)^\top$, $M := 76$, $|\mathrm{CL}(\boldsymbol{z}, M)| = 77$

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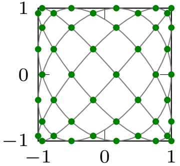

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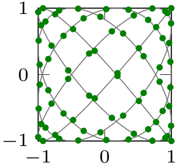

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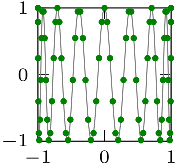

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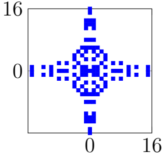
? conditions on $\mathrm{CL}(\boldsymbol{z}, M)$ for **fast reconstruction** of a_I using DCT

$\Rightarrow$ [Potts, V. '15]

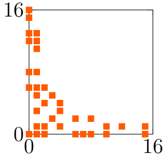
Rank-1 Chebyshev lattices - reconstruction of a_I [Potts, V. '15]

- $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$, **arbitrary known I**
- reconstructing rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, M, I)$:
  $\boldsymbol{k} \cdot \boldsymbol{z} \,\mathrm{emod}\, M \neq \boldsymbol{h} \cdot \boldsymbol{z} \,\mathrm{emod}\, M$
  for all $\boldsymbol{k} \in I$ and $\boldsymbol{h} \in \mathcal{M}(I)$, $\boldsymbol{k} \neq (|h_1|, \ldots, |h_d|)$

$$l \,\mathrm{emod}\, M := \begin{cases} l \bmod (2M), & l \bmod (2M) \leq M, \\ 2M - (l \bmod (2M)) & \text{else}, \end{cases}$$

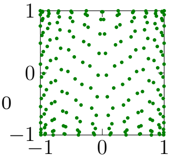


$\mathcal{M}(I)$



I

$(\hat{a}_{\boldsymbol{k}})_{\boldsymbol{k}\in I} \longleftarrow (a_I(\boldsymbol{x}_j))_{j=0}^{M}$



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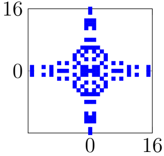

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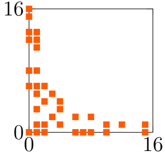

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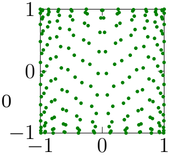

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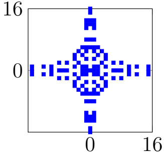
- construction with simple component-by-component (CBC) approach
- reconstruct $\hat{a}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$:
  - $\tilde{\hat{a}}_l := \sum_{j=0}^{M}{}'' a_I(\boldsymbol{x}_j) \cos\left(\frac{jl}{M}\pi\right), \quad l = 0, \ldots, M$, by 1d DCT($M$)
  - $\hat{a}_{\boldsymbol{k}} = \tilde{\hat{a}}_l \dfrac{2^{|\boldsymbol{k}|_0 + 1 - \delta_{0,l} - \delta_{M,l}}}{M\, |\{\boldsymbol{h} \in \mathcal{M}(\{\boldsymbol{k}\}) \colon \boldsymbol{h} \cdot \boldsymbol{z} \,\mathrm{emod}\, M = l\}|}$, $l := \boldsymbol{k} \cdot \boldsymbol{z} \,\mathrm{emod}\, M$



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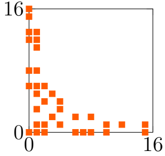

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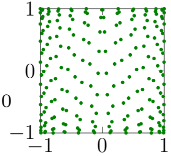
$(\hat{a}_{\boldsymbol{k}})_{\boldsymbol{k}\in I} \xleftarrow[\text{DCT-I}]{\text{1-dim.}} (a_I(\boldsymbol{x}_j))_{j=0}^{M}$

$\mathcal{O}(M \log M + d\,|\mathcal{M}(I)|)$



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Unknown frequency index set I

until now: $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$

- fast reconstruction / approximation from samples
  for arbitrary known $I \subset \mathbb{N}_0^d$, $|I| < \infty$, (works also for periodic case)

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next: unknown frequency index set $I \Rightarrow$ multi-dimensional sparse FFT

- task: determine I containing $\approx$largest Fourier/Chebyshev coefficients $\hat{a}_{\boldsymbol{k}}$
- search domain $\Gamma \subset \mathbb{Z}^d$ or $\mathbb{N}_0^d$, e.g. full grid $\hat{G}_N^d$ with $\|\boldsymbol{k}\|_\infty \leq N$

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- various existing methods, e.g., based on
  - filters [Indyk, Kapralov '14]
  - Chinese remainder theorem [Cuyt, Lee '08] [Iwen '13]
  - Prony's method [Tasche, Potts '13] [Peter, Plonka, Schaback '15]
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- Problems: non-sparsity, implementations?, stability, many frequencies

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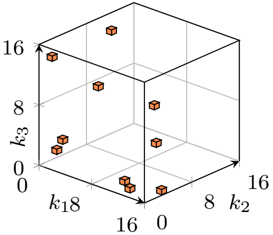
$\Rightarrow$ dimension-incremental sparse FFT using rank-1 lattices [Potts, V. '15]
(similar idea without rank-1 lattices:
[Zippel '79] [Mansour '95] [Kaltofen, Lee '03] [Javadi Monagan '10] [Potts, Tasche '13])

$\Rightarrow$ adapted for non-periodic case

Multivariate sparse FFT - method $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$



$I \subset \Gamma = \hat{G}_{16}^3$

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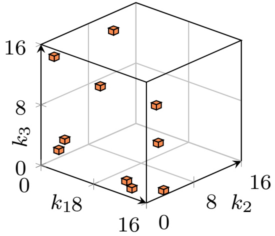

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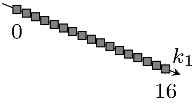
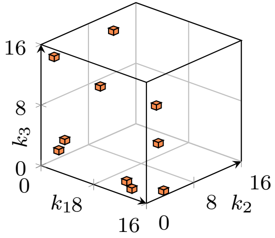


frequency candidates

Multivariate sparse FFT - method $a_I(x) := \sum_{k \in I} \hat{a}_k T_k(x)$



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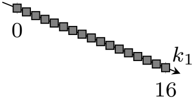
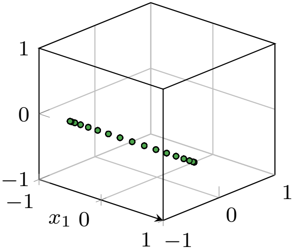


frequency candidates

construct $\longrightarrow$ sampling set



sampling nodes $\left\{(\cos(\frac{\ell\pi}{16}), x_2', x_3')\right\}_{\ell=0}^{16}$

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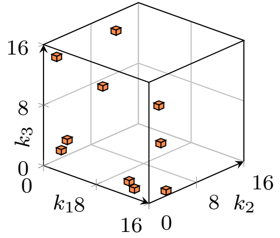

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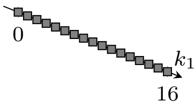
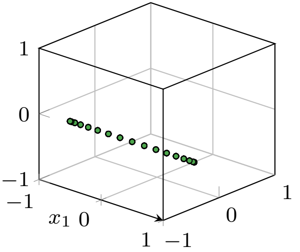


frequency candidates

1-dim. $\longleftarrow$ DCT



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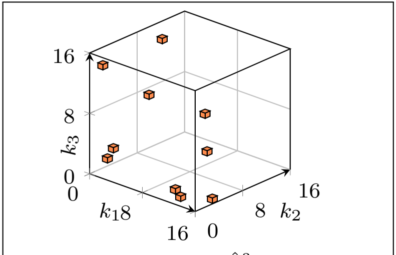

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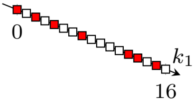

detected frequencies $I^{(1)}$

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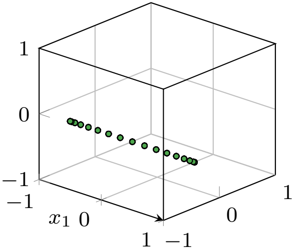

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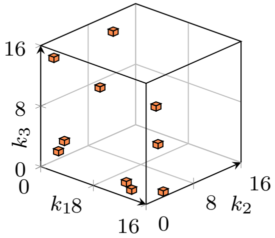

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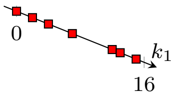

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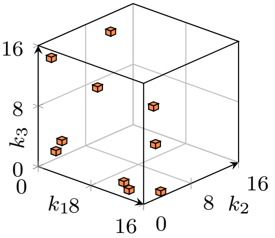

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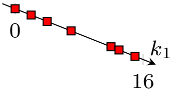

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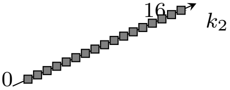

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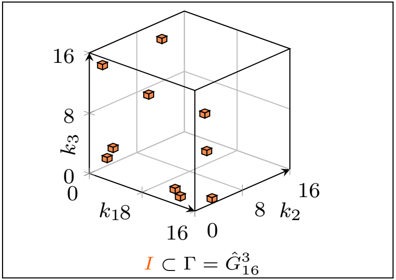
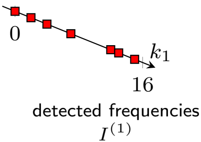
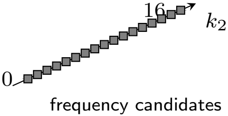
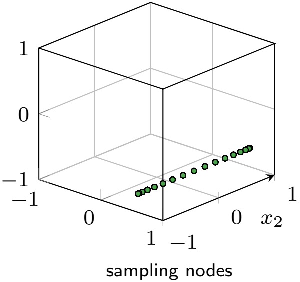


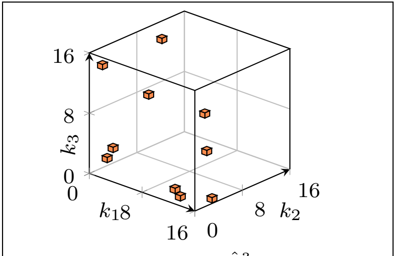




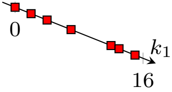
construct
sampling set



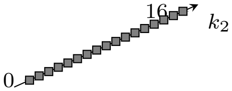
Multivariate sparse FFT - method $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$



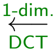
$I \subset \Gamma = \hat{G}^3_{16}$

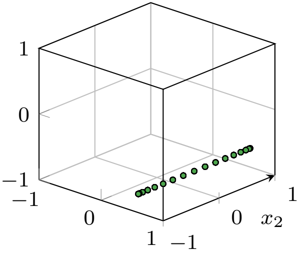


detected frequencies
$I^{(1)}$



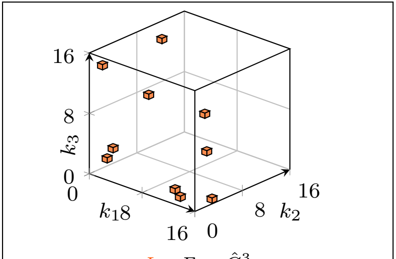
frequency candidates



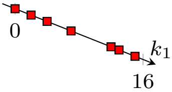


sampling nodes

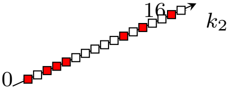
Multivariate sparse FFT - method $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$



$I \subset \Gamma = \hat{G}^3_{16}$

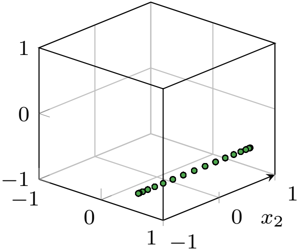


detected frequencies $I^{(1)}$



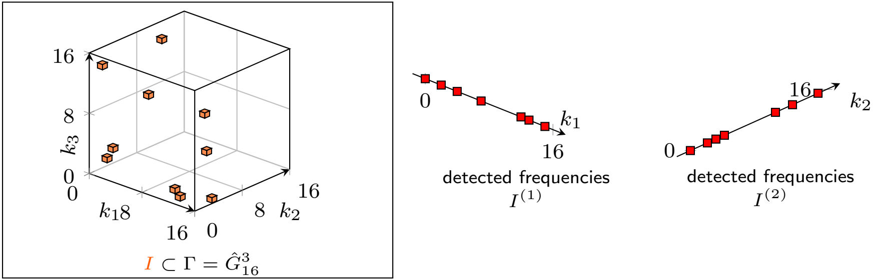
detected frequencies $I^{(2)}$

1-dim. $\longleftarrow$ DCT

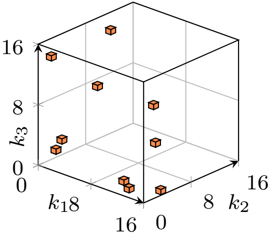


sampling nodes

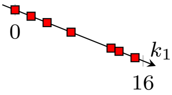
Multivariate sparse FFT - method $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$



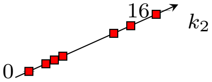
Multivariate sparse FFT - method $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$



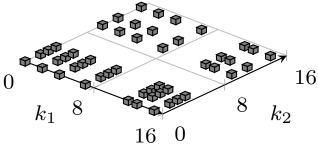
$I \subset \Gamma = \hat{G}_{16}^3$



detected frequencies $I^{(1)}$

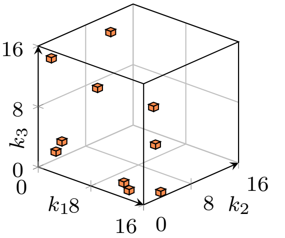


detected frequencies $I^{(2)}$

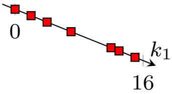


frequency candidates $I^{(1)} \times I^{(2)}$

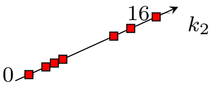
Multivariate sparse FFT - method $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k} \in I} \hat{a}_{\boldsymbol{k}} \, T_{\boldsymbol{k}}(\boldsymbol{x})$



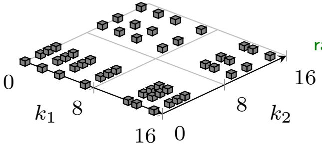
$I \subset \Gamma = \hat{G}^3_{16}$



detected frequencies $I^{(1)}$

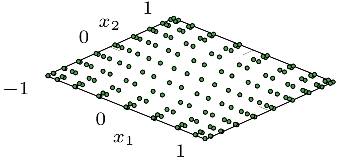


detected frequencies $I^{(2)}$



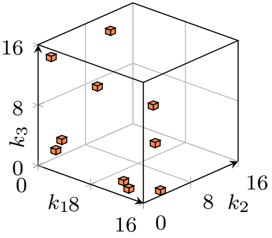
frequency candidates $I^{(1)} \times I^{(2)}$

reconstructing
$\longrightarrow$
rank-1 Chebyshev lattice

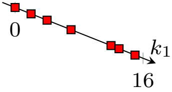


sampling nodes

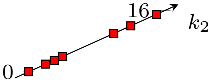
Multivariate sparse FFT - method $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$



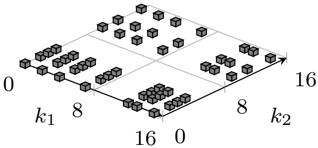
$I \subset \Gamma = \hat{G}^3_{16}$



detected frequencies $I^{(1)}$

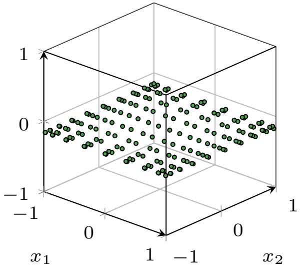


detected frequencies $I^{(2)}$



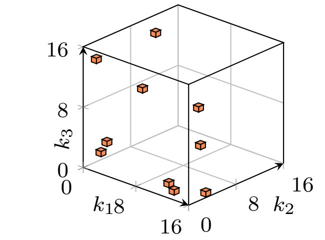
frequency candidates $I^{(1)} \times I^{(2)}$

1-dim. DCT

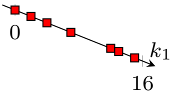


sampling nodes

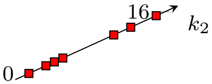
Multivariate sparse FFT - method $a_I(x) := \sum_{k \in I} \hat{a}_k T_k(x)$



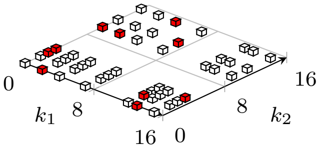
$I \subset \Gamma = \hat{G}_{16}^3$



detected frequencies $I^{(1)}$

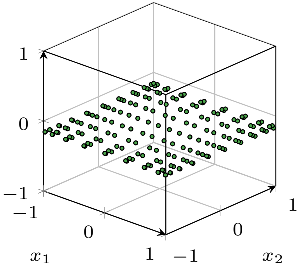


detected frequencies $I^{(2)}$



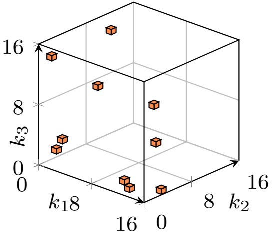
detected frequencies $I^{(1,2)}$

1-dim. ← DCT

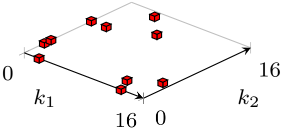


sampling nodes

Multivariate sparse FFT - method $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$

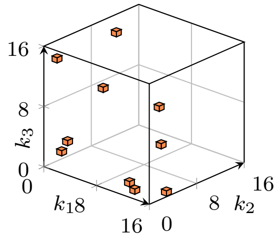


$I \subset \Gamma = \hat{G}_{16}^3$

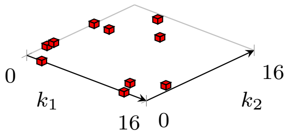


detected frequencies $I^{(1,2)}$

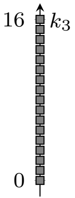
Multivariate sparse FFT - method $a_I(x) := \sum_{k \in I} \hat{a}_k T_k(x)$



$I \subset \Gamma = \hat{G}_{16}^3$

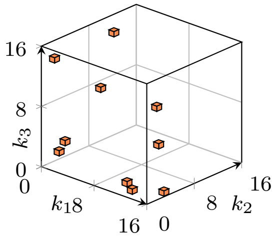


detected frequencies $I^{(1,2)}$

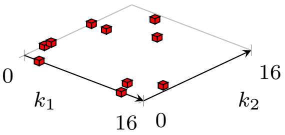


frequency candidates

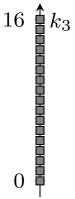
Multivariate sparse FFT - method $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$



$I \subset \Gamma = \hat{G}_{16}^3$

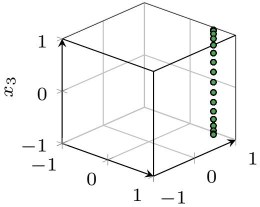


detected frequencies $I^{(1,2)}$



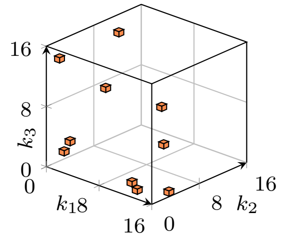
frequency candidates

construct $\longrightarrow$ sampling set

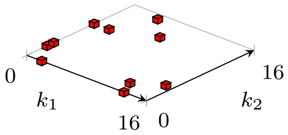


sampling nodes

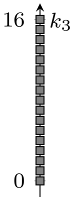
Multivariate sparse FFT - method $a_I(x) := \sum_{k \in I} \hat{a}_k T_k(x)$



$I \subset \Gamma = \hat{G}_{16}^3$

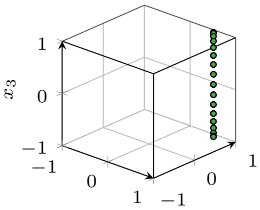


detected frequencies $I^{(1,2)}$



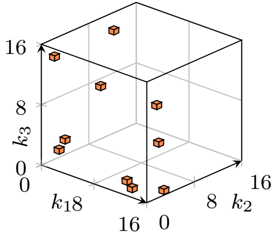
frequency candidates

1-dim. DCT

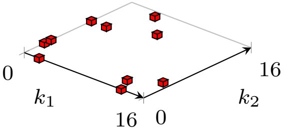


sampling nodes

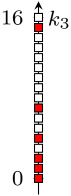
Multivariate sparse FFT - method $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$



$I \subset \Gamma = \hat{G}^3_{16}$

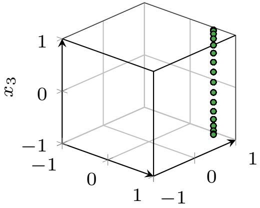


detected frequencies $I^{(1,2)}$



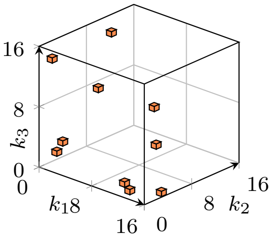
detected frequencies $I^{(3)}$

1-dim. ⟵ DCT

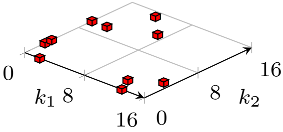


sampling nodes

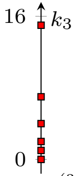
Multivariate sparse FFT - method $a_I(x) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$



$I \subset \Gamma = \hat{G}_{16}^{3}$

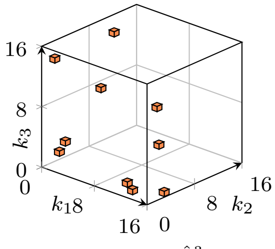


detected frequencies $I^{(1,2)}$

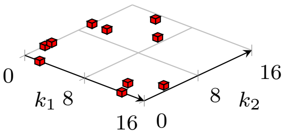


$I^{(3)}$

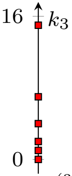
Multivariate sparse FFT - method $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$



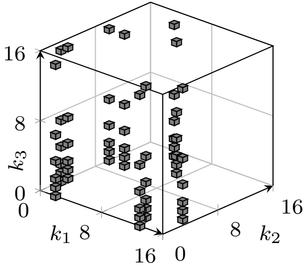
$I \subset \Gamma = \hat{G}_{16}^3$



detected frequencies $I^{(1,2)}$

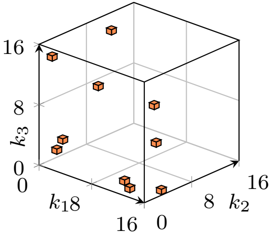


$I^{(3)}$

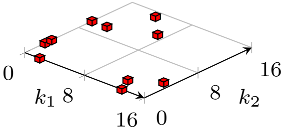


frequency candidates $I^{(1,2)} \times I^{(3)}$

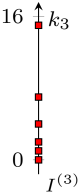
Multivariate sparse FFT - method $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$

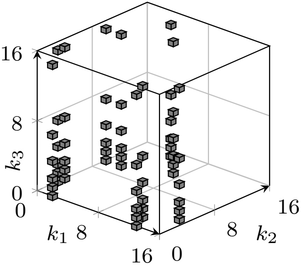


$I \subset \Gamma = \hat{G}^3_{16}$



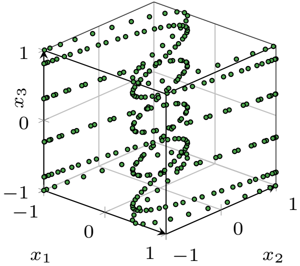
detected frequencies $I^{(1,2)}$





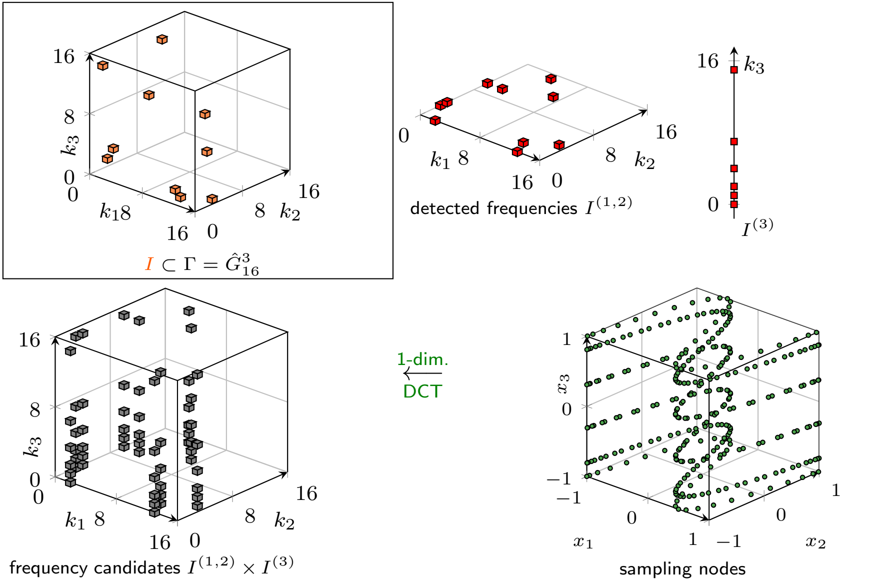
frequency candidates $I^{(1,2)} \times I^{(3)}$

reconstructing
$\longrightarrow$
rank-1 Chebyshev lattice

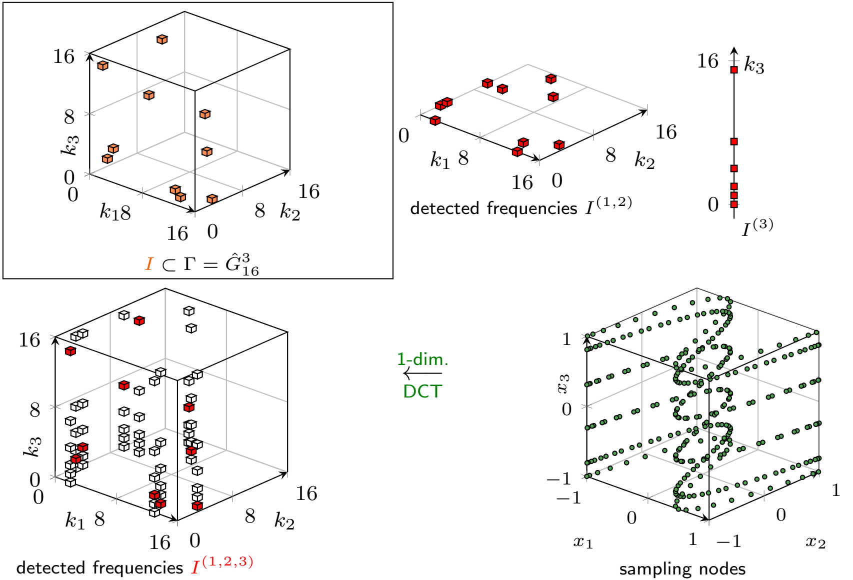


sampling nodes

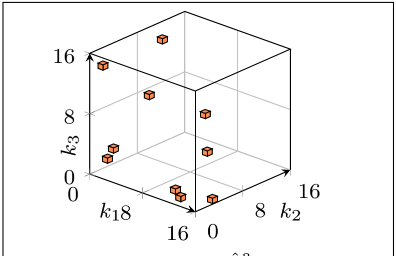
Multivariate sparse FFT - method $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$



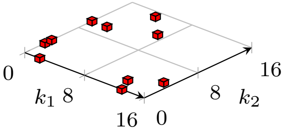
Multivariate sparse FFT - method $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$



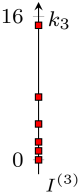
Multivariate sparse FFT - method $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$

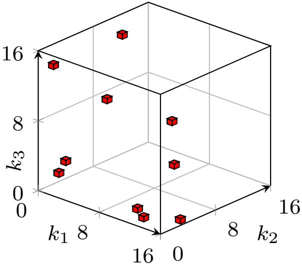


$I \subset \Gamma = \hat{G}_{16}^3$



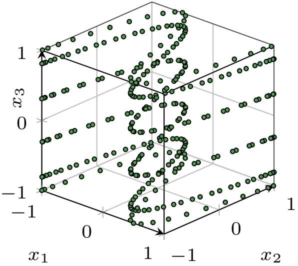
detected frequencies $I^{(1,2)}$





detected frequencies $I^{(1,2,3)} = I$

1-dim. ⟵ DCT



sampling nodes

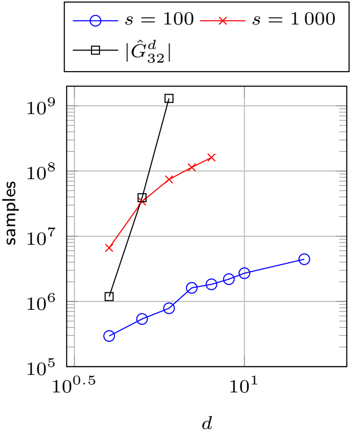
Multivariate sparse FFT - complexity

- sparsity $s = |I|$,
  frequency search domain $\Gamma = \hat{G}_N^d := \{0, 1, \ldots, N\}^d \supset I$,
  $\sqrt{N} \lesssim s \lesssim N^d$
- samples: $\mathcal{O}(s^2 N)$
- arithmetic operations: $\mathcal{O}(s^3 + s^2 N \log(s\, N))$
- for arbitrary Chebyshev coefficients $\hat{a}_{\boldsymbol{k}} \in \mathbb{R}$:
  probabilistic approach with several inner iterations
- if (all $\hat{a}_{\boldsymbol{k}} > 0$) OR (all $\hat{a}_{\boldsymbol{k}} < 0$)
  then deterministic version with 1 iteration

Numerical results - multivar. algebraic polynomial

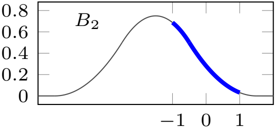
- $f(\boldsymbol{x}) = a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$
- random Chebyshev coefficients $\hat{a}_{\boldsymbol{k}} \in [-1,1]$, $|\hat{a}_{\boldsymbol{k}}| \geq 10^{-6}$
- search domain $\Gamma = \hat{G}_{32}^d := \{0,1,\ldots,32\}^d$, random $I \subset \Gamma$
- multivariate sparse FFT for unknown I
- 10 repetitions

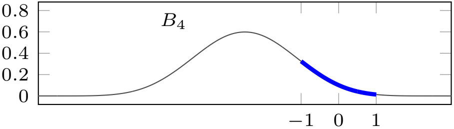
d	$\lvert I\rvert$	$\lvert\Gamma\rvert = \lvert\hat{G}_{32}^d\rvert \approx$	max. total #samples	max. rel. ℓ_2-error
4	100	1.2e+06	295 118	7.2e-16
5	100	3.9e+07	537 964	5.5e-16
6	100	1.3e+09	785 671	1.2e-15
7	100	4.3e+10	1 614 677	9.4e-16
8	100	1.4e+12	1 828 842	6.4e-16
9	100	4.6e+13	2 195 804	7.3e-16
10	100	1.5e+15	2 710 158	1.8e-15
15	100	6.0e+22	4 439 451	4.2e-14
4	1 000	1.2e+06	6 630 162	6.7e-16
5	1 000	3.9e+07	34 116 319	7.4e-16
6	1 000	1.3e+09	74 215 472	1.5e-15
7	1 000	4.3e+10	113 804 504	8.0e-16
8	1 000	1.4e+12	161 481 230	1.5e-15



Numerical results - 9-dim. test function

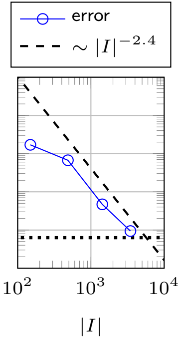
- 9-dimensional test function $f\colon [-1,1]^9 \to \mathbb{R}$,
  $f(\boldsymbol{x}) := \prod_{t\in\{1,3,4,7\}} B_2(x_t) + \prod_{t\in\{2,5,6,8,9\}} B_4(x_t)$
  - B_m is a shifted, scaled and dilated B-spline of order m
  - Chebyshev coefficients of B_2 and B_4 decay like $\sim k^{-3}$ and $\sim k^{-5}$

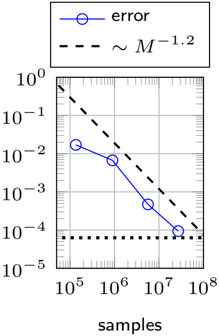




$\Gamma = \hat{G}^9_{32}$, $|\Gamma| \approx$ 4.6e13, 10 repetitions

θ	$\lvert I\rvert$	max. total #samples	max. rel. $L_{2,w}$-error
1e-2	149	135 216	1.7e-02
1e-3	485	898 310	6.7e-03
1e-4	1 431	5 662 360	4.7e-04
1e-5	3 465	27 009 528	9.4e-05





Summary

- **known** (arbitrary) frequency index set I, rank-1 Chebyshev lattice
  - fast reconstruction of multivariate algebraic polynomials $a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$

D. Potts and T. V. **Fast and exact reconstruction of arbitrary multivariate algebraic polynomials in Chebyshev form.** In *11th international conference on Sampling Theory and Applications (SampTA 2015)*, pages 392–396, Washington D.C., 2015.

- **unknown** frequency index set I
  - sparse dimension-incremental FFT based on rank-1 Chebyshev lattices
    - fast reconstruction of high-dimensional sparse algebraic polynomials a_I / fast approximation of multivariate non-periodic functions
    - numerical results for up to 15-/9-dimensional case presented
    - sparsity $s = |I|$, frequency search domain $\Gamma = \{0, 1, \ldots, N\}^d$, $\sqrt{N} \lesssim s \lesssim N^d$
      samples: $\mathcal{O}(s^2 N)$
      arithmetic operations: $\mathcal{O}(s^3 + s^2 N \log(s\, N))$

D. Potts and T. V. **Multivariate sparse FFT based on rank-1 Chebyshev lattice sampling.** In *12th international conference on Sampling Theory and Applications (SampTA 2017)*, Tallinn, 2017.