Sparse high-dimensional FFT with applications to data mining

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Content

Introduction

- approximate high-dim. function $f\colon \mathbb{T}^d \simeq [0,1)^d \to \mathbb{C}$ by multivariate trigonometric polynomial $p\colon \mathbb{T}^d \to \mathbb{C}$ with frequencies supported on $I \subset \mathbb{Z}^d,\ |I| < \infty$,
$$p(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \hat{p}_{\boldsymbol{k}} \in \mathbb{C}$$

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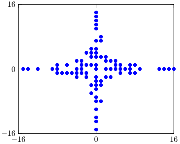
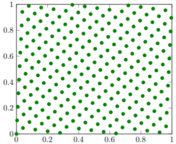
- arbitrary **known** frequency index set $I \subset \mathbb{Z}^d$, $|I| < \infty$, rank-1 lattice nodes $\boldsymbol{x}_j$, $j = 0, \ldots, M-1$
  - fast evaluation $p(\boldsymbol{x}_j)$, (e.g. [Li, Hickernell 03])
  - fast and exact reconstruction of $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, from samples $p(\boldsymbol{x}_j)$, ([Kämmerer, Kunis, Potts 12] [Kämmerer 13])
  - approximate reconstruction of f by p from samples $f(\boldsymbol{x}_j)$



$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I}$

$\longleftrightarrow$ FFT



$(p(\boldsymbol{x}_j))_{j=0}^{M-1}$

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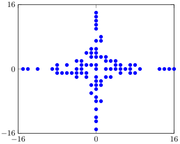

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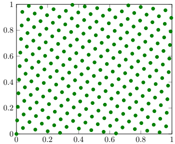


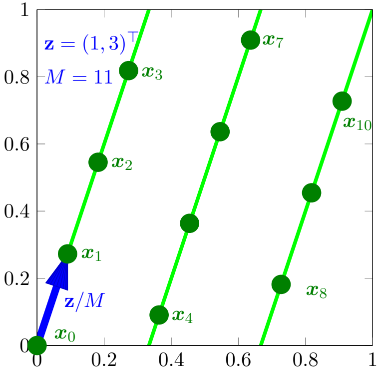
$(p(\boldsymbol{x}_j))_{j=0}^{M-1}$

- unknown frequency index set I?

Trigonometric polynomials - fast evaluation

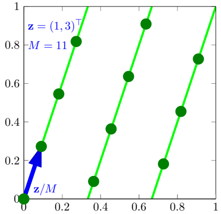
- rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, M)$: $\boldsymbol{z} \in \mathbb{N}_0^d, M \in \mathbb{N}$

$$\boldsymbol{x}_j = \frac{j}{M}\boldsymbol{z} \bmod \mathbf{1};\ j = 0, \ldots, M-1$$



Korobov 59
Maisonneuve 72
Sloan & Kachoyan 84,87,90
Temlyakov 86
Lyness 89
Sloan & Joe 94
Sloan & Reztsov 01
Li & Hickernell 03

Trigonometric polynomials - fast evaluation



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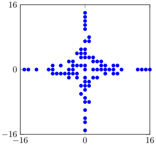
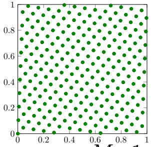
- reformulation

$$p(\boldsymbol{x}_j) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \mathrm{e}^{2\pi\mathrm{i}\frac{j\boldsymbol{k}\cdot\boldsymbol{z}}{M}} = \sum_{l=0}^{M-1} \underbrace{\left(\sum_{\substack{\boldsymbol{k}\in I \\ \boldsymbol{k}\cdot\boldsymbol{z} \equiv l \pmod M}} \hat{p}_{\boldsymbol{k}} \right)}_{\hat{g}_l} \mathrm{e}^{2\pi\mathrm{i}\frac{j\boldsymbol{k}\cdot\boldsymbol{z}}{M}} = \sum_{l=0}^{M-1} \hat{g}_l \mathrm{e}^{2\pi\mathrm{i}\frac{jl}{M}}$$



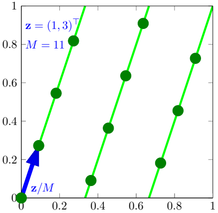
$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I}$

$\longrightarrow$



$(p(\boldsymbol{x}_j))_{j=0}^{M-1}$

Trigonometric polynomials - fast evaluation



- rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, M)$: $\boldsymbol{z} \in \mathbb{N}_0^d, M \in \mathbb{N}$

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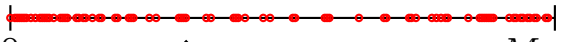

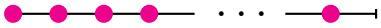

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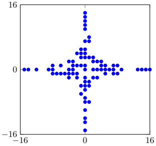
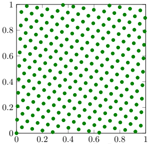


$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I}$

$$\xrightarrow[\text{FFT}]{\text{1-dim}}$$

$$\mathcal{O}(M \log M + d|I|)$$

||



$(p(\boldsymbol{x}_j))_{j=0}^{M-1}$

Trigonometric polynomials - fast reconstruction

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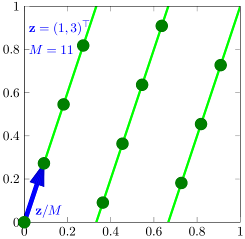
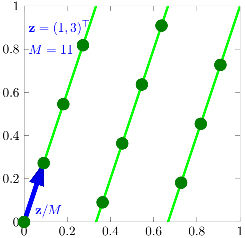
- reconstruction of Fourier coefficients $\hat{p}_{\boldsymbol{k}}$ of multivariate trigonometric polynomial $p(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}$



Trigonometric polynomials - fast reconstruction



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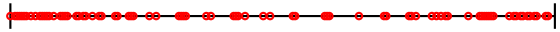
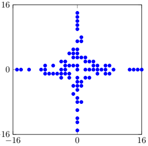
$\Rightarrow$ Definition reconstructing $\mathrm{R1L}(\boldsymbol{z}, M, I)$ for I:
$\boldsymbol{k} \cdot \boldsymbol{z} \not\equiv \boldsymbol{k}' \cdot \boldsymbol{z} \pmod{M}$ for all $\boldsymbol{k}, \boldsymbol{k}' \in I$, $\boldsymbol{k} \neq \boldsymbol{k}'$

- $|I| \leq M \leq |I|^2$, CBC construction algorithm [Kämmerer 2012]



$\boldsymbol{k}\cdot\boldsymbol{z} \equiv l \pmod{M}$

$||$

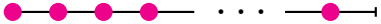


$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k} \in I}$

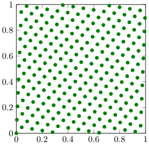
$\overset{\text{1-dim}}{\underset{\text{iFFT}}{\longleftarrow}}$

$\mathcal{O}(M \log M + d|I|)$

$0 \quad \frac{1}{M} \quad \frac{2}{M} \quad \frac{3}{M} \quad \dots \quad \frac{M-1}{M}$



$||$ $p(\frac{j}{M}\boldsymbol{z})$



$(p(\boldsymbol{x}_j))_{j=0}^{M-1}$

Trigonometric polynomials - fast approximation

- fast approximation of high-dimensional function $f\colon \mathbb{T}^d \to \mathbb{C}$
  by multivariate trigonometric polynomial $p(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$
  from samples at reconstructing rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, M, I)$

Kämmerer, L., Potts, D., Volkmer, T.
Approximation of multivariate periodic functions by trigonometric polynomials based on rank-1 lattice sampling.
J. Complexity 31, 543 – 576, 2015.

Byrenheid, G., Kämmerer, L., Ullrich, T., Volkmer, T.
Tight error bounds for rank-1 lattice sampling in spaces of hybrid mixed smoothness.
arXiv:1510.08336, Preprint, 2016.

(http://www.tu-chemnitz.de/∼tovo)

Unknown frequency index set

until now: fast reconstruction of $p(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ or
fast approximation of function $f(\boldsymbol{x}) \approx p(\boldsymbol{x})$

- given frequency index set I
- compute $\hat{p}_{\boldsymbol{k}}$ from samples along reconstructing rank-1 lattice

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  full grid $\hat{G}_N^d := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \|\boldsymbol{k}\|_\infty \leq N\}$, ($|\hat{G}_{64}^{10}| \approx 1.28 \cdot 10^{21}$)

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$\Rightarrow$ multi-dimensional sparse FFT

Multi-dimensional sparse FFT

Various existing methods, e.g.

- filters [Indyk, Kapralov 14]
- Chinese remainder theorem
  - [Cuyt, Lee 08]
  - [Iwen 13]
- Prony's method
  - multiple lines [Tasche, Potts 13]
  - common zeros [Peter, Plonka, Schaback 15] [Kunis, Peter, Römer, von der Ohe 15]
- dimension-incremental projection
  - Zippel's Algorithm [Zippel 79] [Kaltofen, Lee 03] [Javadi Monagan 10]
  - via (reconstructing) rank-1 lattices [Potts, V. 15]
- randomized Kronecker substitution
  [Arnold, Roche 14] [Arnold, Giesbrecht, Roche 15]
- (reconstructing) rank-1 lattice and 1d method
  [Potts, Tasche, V. 16]

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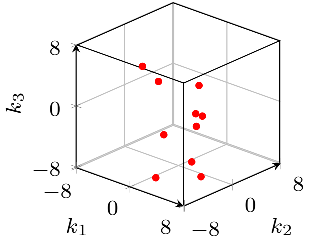
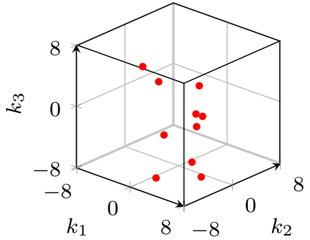
  [Potts, Tasche, V. 16]

Sparse dimension-incremental FFT - method



$\color{red}{I} \subset \Gamma = \hat{G}_8^3$

Sparse dimension-incremental FFT - method



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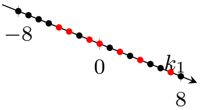
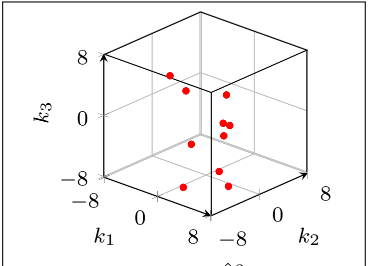


frequency candidates

Sparse dimension-incremental FFT - method



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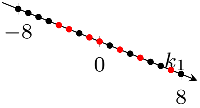
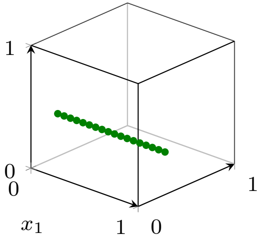


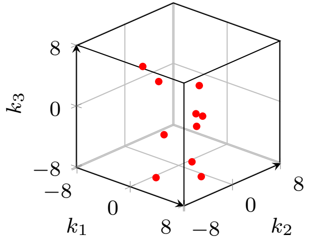
frequency candidates

construct
$\longrightarrow$
sampling set



sampling nodes
$(\frac{\ell}{17}, x_2', x_3')^\top$, $\ell = 0, \ldots, 16$

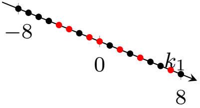
Sparse dimension-incremental FFT - method



$I \subset \Gamma = \hat{G}_8^3$

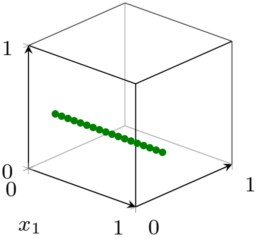
$$\hat{p}_{k_1} := \frac{1}{17} \sum_{\ell=0}^{16} p\left(\begin{pmatrix} \ell/17 \\ x_2' \\ x_3' \end{pmatrix}\right) \mathrm{e}^{-2\pi\mathrm{i}\frac{\ell k_1}{17}}$$

$k_1 = -8, \ldots, 8$



frequency candidates

1-dim
⟵
iFFT



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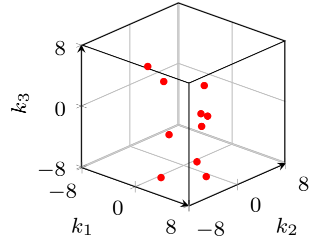
Sparse dimension-incremental FFT - method



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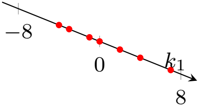
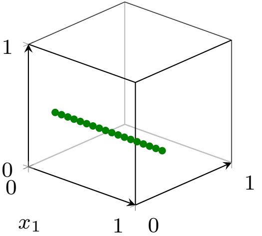
$$\hat{p}_{k_1} := \frac{1}{17}\sum_{\ell=0}^{16} p\left(\begin{pmatrix}\ell/17\\ x_2'\\ x_3'\end{pmatrix}\right) \mathrm{e}^{-2\pi\mathrm{i}\frac{\ell k_1}{17}} = \sum_{\substack{(h_2,h_3)\in\{-8,\ldots,8\}^2\\ (k_1,h_2,h_3)^\top\in\operatorname{supp}\hat{p}}} \hat{p}_{\begin{pmatrix}k_1\\ h_2\\ h_3\end{pmatrix}} \mathrm{e}^{2\pi\mathrm{i}(h_2x_2'+h_3x_3')},$$

$k_1 = -8, \ldots, 8$



detected frequencies $I^{(1)}$

1-dim
$\longleftarrow$
iFFT



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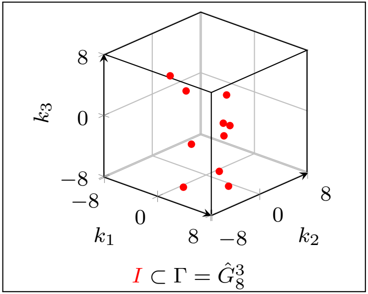
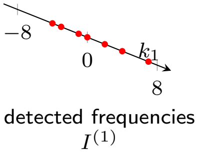
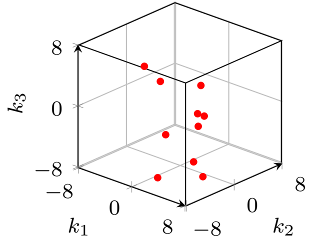
Sparse dimension-incremental FFT - method

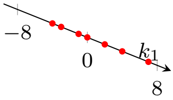




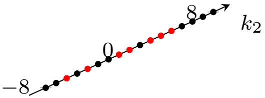
Sparse dimension-incremental FFT - method



$I \subset \Gamma = \hat{G}_8^3$

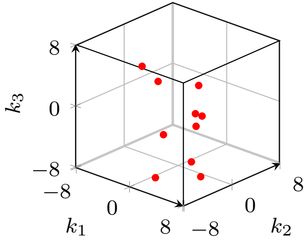


detected frequencies
$I^{(1)}$



frequency candidates

Sparse dimension-incremental FFT - method



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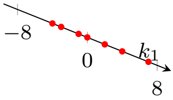

detected frequencies

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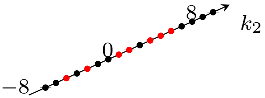
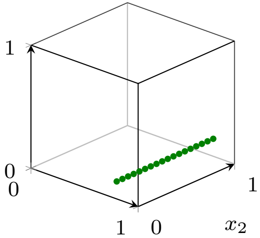


frequency candidates

construct

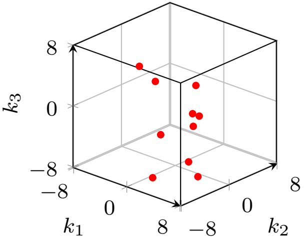
⟶

sampling set



sampling nodes

Sparse dimension-incremental FFT - method



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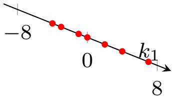

detected frequencies
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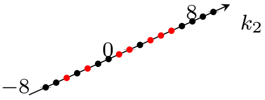
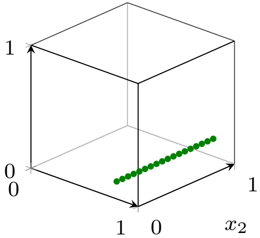


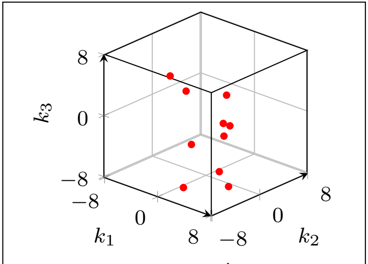
frequency candidates

1-dim
←
iFFT



sampling nodes

Sparse dimension-incremental FFT - method



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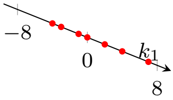
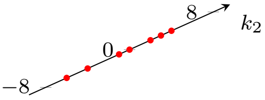
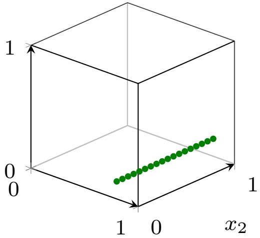


detected frequencies $I^{(1)}$



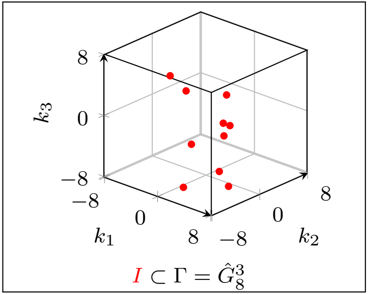
detected frequencies $I^{(2)}$

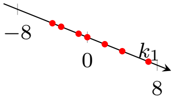
1-dim ⟵ iFFT



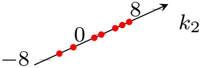
sampling nodes

Sparse dimension-incremental FFT - method



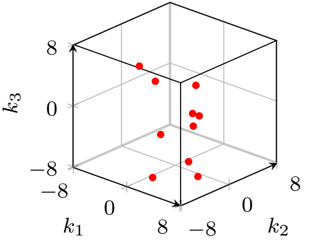


detected frequencies
$I^{(1)}$



detected frequencies
$I^{(2)}$

Sparse dimension-incremental FFT - method



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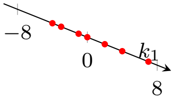

detected frequencies
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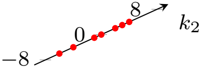
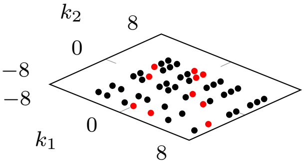
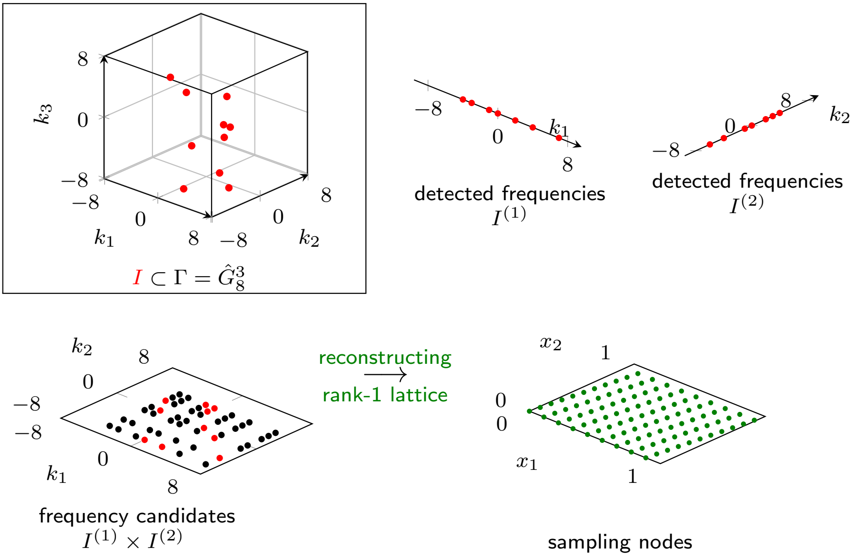


detected frequencies
$I^{(2)}$



frequency candidates
$I^{(1)} \times I^{(2)}$

Sparse dimension-incremental FFT - method



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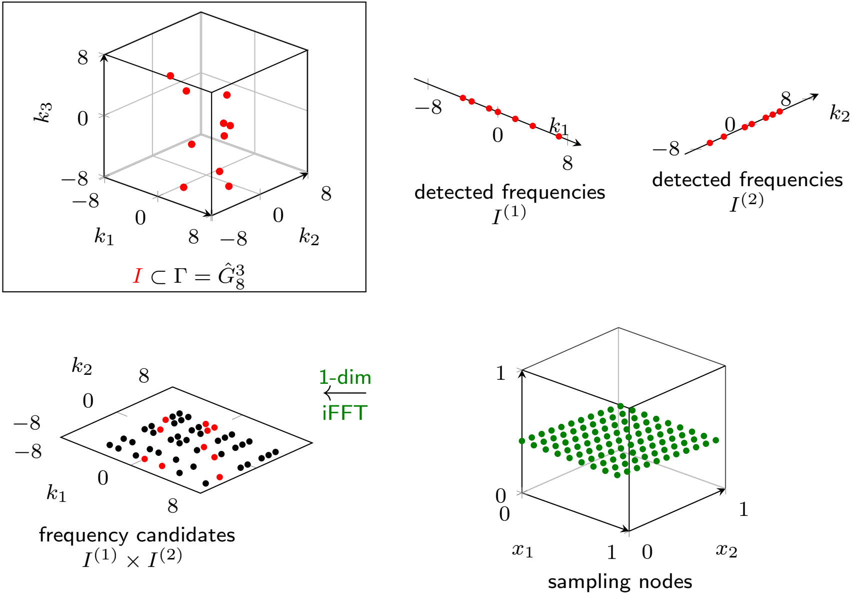
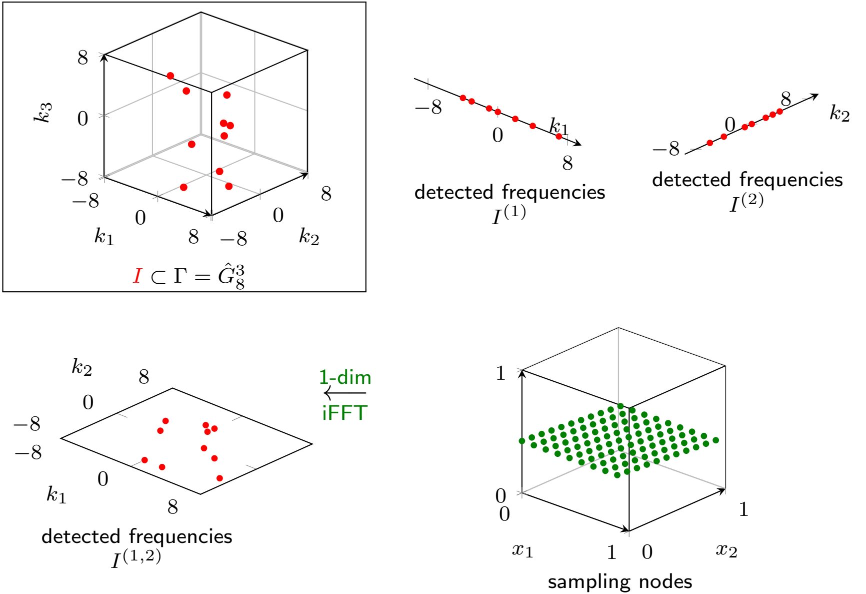


Sparse dimension-incremental FFT - method



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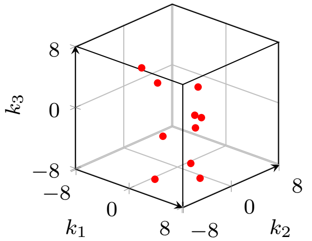

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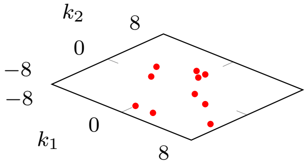
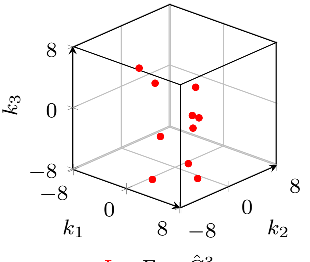
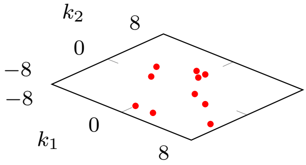


detected frequencies
$I^{(1,2)}$

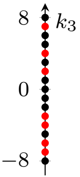
Sparse dimension-incremental FFT - method



$I \subset \Gamma = \hat{G}_8^3$

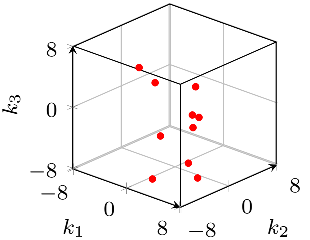


detected frequencies $I^{(1,2)}$



frequency candidates

Sparse dimension-incremental FFT - method



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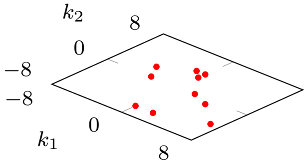

detected frequencies
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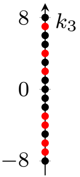
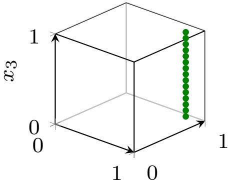


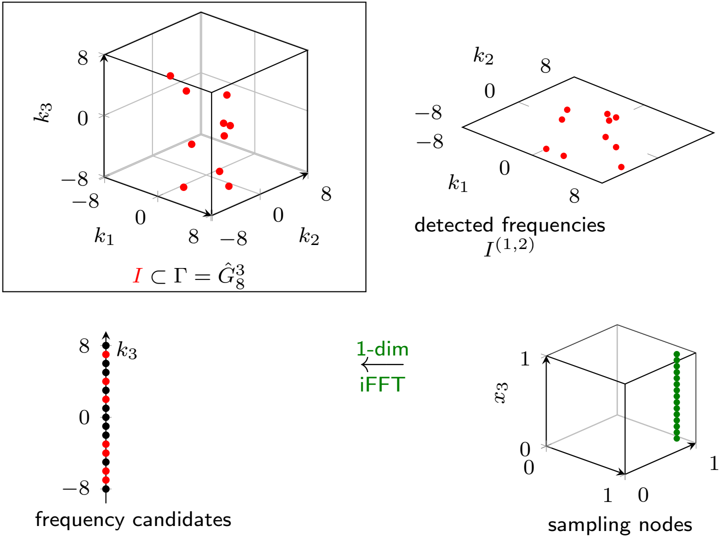
frequency candidates

construct $\longrightarrow$ sampling set

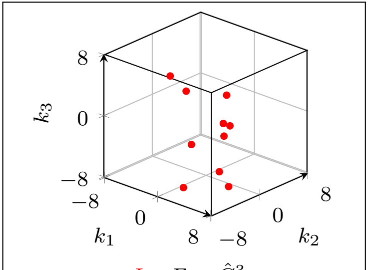


sampling nodes

Sparse dimension-incremental FFT - method



Sparse dimension-incremental FFT - method



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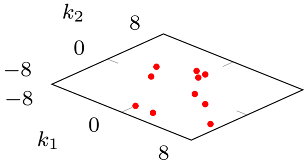
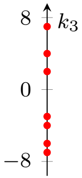
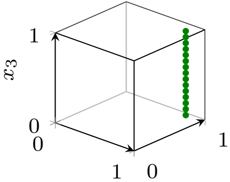


detected frequencies $I^{(1,2)}$



detected frequencies $I^{(3)}$

1-dim ⟵ iFFT



sampling nodes

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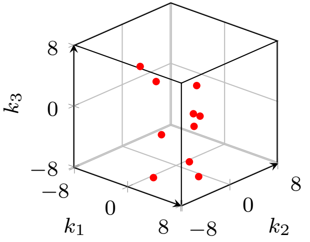

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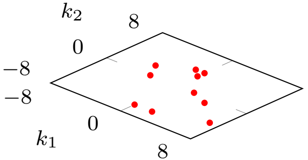
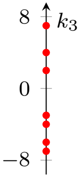
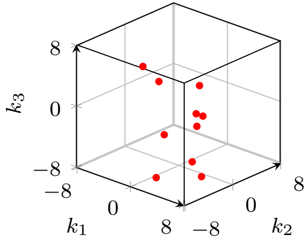


detected frequencies
$I^{(1,2)}$



$I^{(3)}$

Sparse dimension-incremental FFT - method



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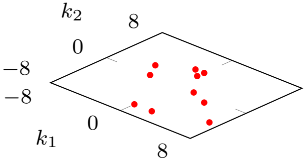

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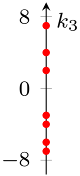
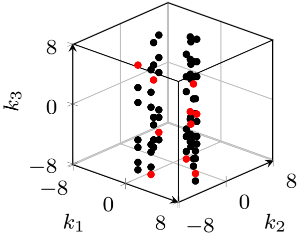
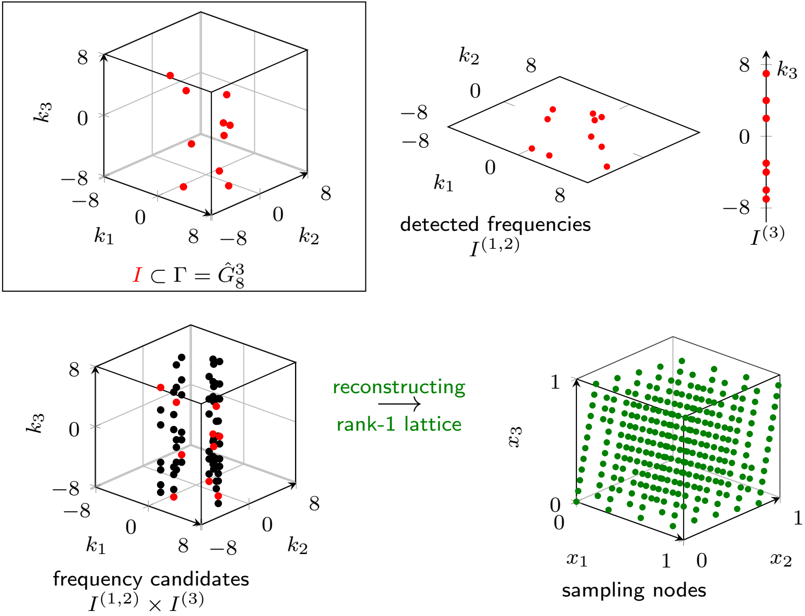


$I^{(3)}$



frequency candidates
$I^{(1,2)} \times I^{(3)}$

Sparse dimension-incremental FFT - method



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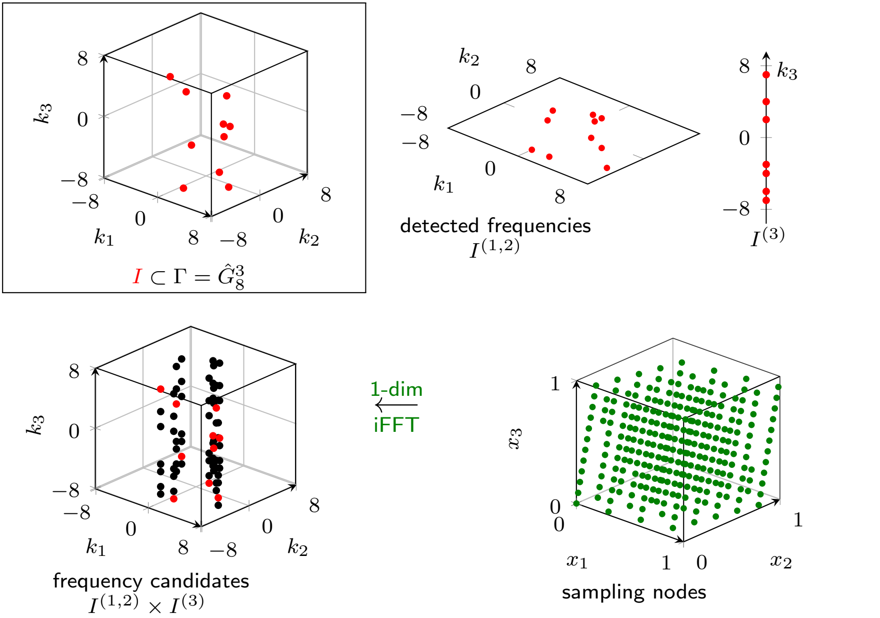

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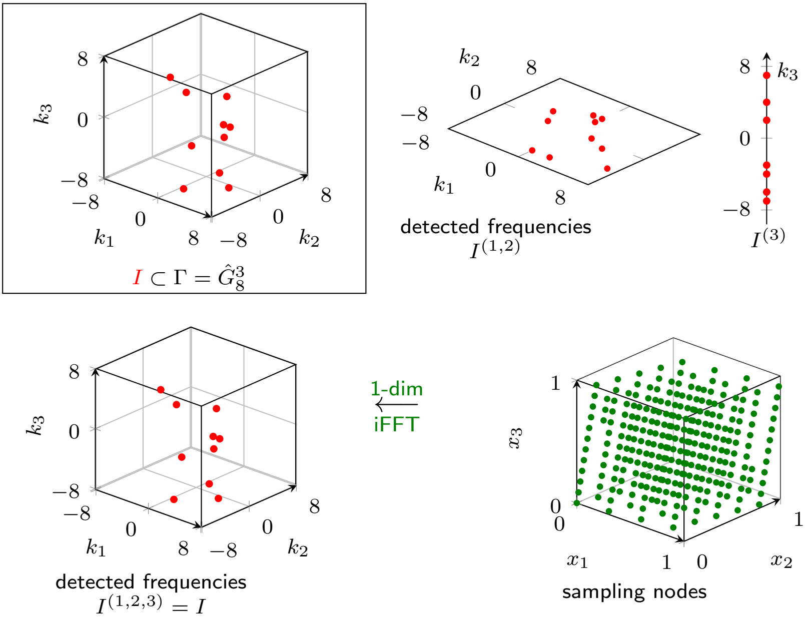

Sparse dimension-incremental FFT - results

search domain $\Gamma = \hat{G}_N^d$ full grid, $\sqrt{N} \lesssim |I| \lesssim N^d$

- samples: $\mathcal{O}(|I|^2 \log |\Gamma|)$
- computational costs: $\mathcal{O}(d\,|I|^3 + |I|^2(\log |\Gamma|) \log(|I|\, \log |\Gamma|))$
- for arbitrary Fourier coefficients $\hat{p}_{\boldsymbol{k}} \in \mathbb{C}$:
  probabilistic approach with several iterations
- if $(\mathrm{Re}(\hat{p}_{\boldsymbol{k}})$ identical sign) AND $(\mathrm{Im}(\hat{p}_{\boldsymbol{k}})$ identical sign)
  then deterministic version with 1 iteration

Potts, D., Volkmer, T.
Sparse high-dimensional FFT based on rank-1 lattice sampling.
Appl. Comput. Harm. Anal. 41, 713 – 748, 2016.
(http://www.tu-chemnitz.de/∼tovo)

Sparse dimension-incremental FFT - example

- B-spline $N_m(x) := \sum_{k\in\mathbb{Z}} C_m \operatorname{sinc}\left(\frac{\pi}{m}k\right)^m \cos(\pi k)\, \mathrm{e}^{2\pi \mathrm{i} k x}$, $\|N_m|L^2(\mathbb{T})\| = 1$, $|\hat{N}_m(k)| \sim |k|^{-m}$
- $f(\boldsymbol{x}) := \prod_{t\in\{1,3,8\}} N_2(x_t) + \prod_{t\in\{2,5,6,10\}} N_4(x_t) + \prod_{t\in\{4,7,9\}} N_6(x_t)$

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- full grid for $N = 64$, $d = 10$: $|\hat{G}_{64}^{10}| = 129^{10} \approx 1.28 \cdot 10^{21}$
- symmetric hyperbolic cross: $|I_{64}^{10}| = 696\,036\,321$
  relative $L^2(\mathbb{T}^d)$-error (best case) 4.1e-04

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- symmetric hyperbolic cross: $|I_{64}^{10}| = 696\,036\,321$
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- results for dimension incremental algorithm with $\Gamma = \hat{G}_{64}^{10}$:

threshold	#samples	$\lvert I\rvert$	rel. L_2-error
1.0e-02	254 530	491	1.4e-01
1.0e-03	2 789 050	1 121	1.1e-02
1.0e-04	17 836 042	3 013	1.7e-03
1.0e-05	82 222 438	7 163	4.7e-04

Regression

sparse dimension-incremental FFT:

- method required function f to be evaluated at arbitrary (rank-1 lattice) points
- What if sampling points $\boldsymbol{y}_\ell \in \mathbb{T}^d$ are given a priori?

Regression

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task:

- Given a set of data $S := \{(\boldsymbol{y}_\ell, f_\ell)\}_{\ell=0,\dots,L-1}$
  - with nodes $\boldsymbol{y}_\ell \in \mathbb{T}^d$ and
  - function values $f_\ell := f(\boldsymbol{y}_\ell) \in \mathbb{R}$,
- determine
  - frequency index set $I \subset \Gamma \subset \mathbb{Z}^d$ and
  - Fourier coefficients $\hat{p}_{\boldsymbol{k}}$

  of approximant $p(\boldsymbol{x}) := \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$

Regression - dimension-incremental

ansatz: for set of data $S := \{(\boldsymbol{y}_\ell, f_\ell)\}_{\ell=0,\ldots,L-1}$

- use regularization network approach and consider the regularized least squares problem

  e.g. [Garcke, Griebel, Thess 01]

$$\frac{1}{L}\sum_{\ell=0}^{L-1}(f_\ell - p(\boldsymbol{y}_\ell))^2 + \lambda\Phi(p) \to \min$$

with regularization parameter $\lambda \geq 0$ and
e.g. with $\Phi(p) := \|\nabla p\|_2^2$

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$\Rightarrow$ solve $\left(\frac{1}{L}\boldsymbol{A}^*\boldsymbol{A} + \lambda\boldsymbol{C}\right)(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in\Gamma} = \frac{1}{L}\boldsymbol{A}^*(f_\ell)_{\ell=0}^{L-1}$,

$$\boldsymbol{A} = \left(\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{y}_\ell}\right)_{\ell=0,\dots,L-1;\ \boldsymbol{k}\in\Gamma}$$

$$\boldsymbol{C} = \left(\langle\nabla\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \nabla\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{h}\cdot\boldsymbol{x}}\rangle_2\right)_{\boldsymbol{k},\boldsymbol{h}\in\Gamma} = \mathrm{diag}\left((4\pi^2\|\boldsymbol{k}\|_2^2)_{\boldsymbol{k}\in\Gamma}\right)$$

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- use regularization network approach and consider the regularized least squares problem

  e.g. [Garcke, Griebel, Thess 01]

$$\frac{1}{L}\sum_{\ell=0}^{L-1}(f_\ell - p(\boldsymbol{y}_\ell))^2 + \lambda\Phi(p) \to \min$$

  with regularization parameter $\lambda \geq 0$ and e.g. with $\Phi(p) := \|\nabla p\|_2^2$

$\Rightarrow$ solve $\left(\frac{1}{L}\boldsymbol{A}^*\boldsymbol{A} + \lambda\boldsymbol{C}\right)(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in\Gamma} = \frac{1}{L}\boldsymbol{A}^*(f_\ell)_{\ell=0}^{L-1}$,

$$\boldsymbol{A} = \left(\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{y}_\ell}\right)_{\ell=0,\ldots,L-1;\ \boldsymbol{k}\in\Gamma}$$

$$\boldsymbol{C} = \left(\langle\nabla\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \nabla\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{h}\cdot\boldsymbol{x}}\rangle_2\right)_{\boldsymbol{k},\boldsymbol{h}\in\Gamma} = \mathrm{diag}\left((4\pi^2\|\boldsymbol{k}\|_2^2)_{\boldsymbol{k}\in\Gamma}\right)$$

- combine with dimension-incremental idea

Regression - dimension-incremental

idea of the approach step-by-step:

- start with 1 attribute

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idea of the approach step-by-step:

- start with 1 attribute
  - compute projected Fourier coefficients from given samples by solving the regularized least squares problem
  - keep largest Fourier coefficients
  - obtain (one-dimensional) approximant

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  - obtain (low dimensional) approximant
  - evaluate approximant on train / eval data set
  - if no improvement compared to previous case, then reject newly added attribute

  end

Regression - example

- B-spline $N_m(x) := \sum_{k\in\mathbb{Z}} C_m \operatorname{sinc}\left(\frac{\pi}{m}k\right)^m \cos(\pi k)\, \mathrm{e}^{2\pi \mathrm{i} k x}$,
  $\|N_m|L^2(\mathbb{T})\| = 1$, $|\hat{N}_m(k)| \sim |k|^{-m}$
- $f(\boldsymbol{x}) := \prod_{t\in\{1,3,8\}} N_2(x_t) + \prod_{t\in\{2,5,6,10\}} N_4(x_t) + \prod_{t\in\{4,7,9\}} N_6(x_t)$

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- hyperbolic cross I_N^d for $N = 16$, $d = 10$: $|I_{16}^{10}| = 45\,548\,649$

index set I	$\lvert I \rvert$	rel. L_2-error (best case)
I_{16}^{10}	45 548 649	3.1e-03
$I \subset I_{16}^{10}$	2 000	4.0e-03
$I \subset I_{16}^{10}$	1 000	1.2e-02

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- results for dimension incremental algorithm with $I \subset \Gamma = I_{16}^{10}$ for given set of data $S := \{(\boldsymbol{y}_\ell, f_\ell)\}_{\ell=0,\ldots,L-1}$, $\boldsymbol{C} := \operatorname{diag}\left(\left(4\pi^2 \prod_{s=1}^d \max(1, |k_s|)\right)_{\boldsymbol{k}\in\Gamma}\right)$

#samples L	$\lvert I\rvert$	rel. L_2-error
400 000	2 000	2.5e-02

Classification as regression

common approach for two-class problem:

- map classes to $f_\ell \in \{0, 1\}$ or $\in \{-1, 1\}$
- set of data $S := \{(\boldsymbol{y}_\ell, f_\ell)\}_{\ell=0,\ldots,L-1}$
- solve the regularized least squares problem

  e.g. [Garcke, Griebel, Thess 01]

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- for data point $\boldsymbol{y}$
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$\Rightarrow$ apply dimension-incremental method

Classification as regression - example

- DMC2013 data set (shopping cart cancellation)
- target attribute order $\in \{0, 1\}$
- 21 attributes
- 429 000 lines of train/eval data (split 50/50)
- 45 068 lines of test data

method	classification rate train	eval	test
order $:= 1$	0.6763	0.6758	0.6806
Decision Tree	0.7787	0.7718	0.7547
dimension incremental	0.7522	0.7519	0.7562

Summary

- **known** (arbitrary) frequency index set via rank-1 lattices
  - fast evaluation / reconstruction of trigonometric polynomials

    [Li, Hickernell 03] / [Kämmerer, Kunis, Potts 12] [Kämmerer 13]
  - approximation of periodic functions
- **unknown** frequency index set
  - sparse dimension-incremental FFT based on rank-1 lattices

    Potts, D., Volkmer, T.
    Sparse high-dimensional FFT based on rank-1 lattice sampling.
    Appl. Comput. Harm. Anal. 41, 713 – 748, 2016.
    (http://www.tu-chemnitz.de/∼tovo)
- dimension-incremental method for regression / classification

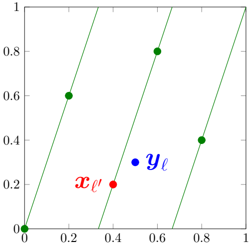
Taylor and rank-1 lattice based NFFT

- dimension incremental method uses evaluations $p(\boldsymbol{y}_\ell) := \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}_\ell}$, $\ell = 0, \ldots, L-1$.

$\Rightarrow$ fast version desired

- approximate $p(\boldsymbol{y}_\ell)$ by Taylor expansion $s_m(\boldsymbol{y}_\ell)$ at closest rank-1 lattice point $\boldsymbol{x}_{\ell'}$,

$$s_m(\boldsymbol{y}_\ell) = p(\boldsymbol{x}_{\ell'}) + \sum_{0<|\boldsymbol{\nu}|<m} \frac{(\boldsymbol{y}_\ell - \boldsymbol{x}_{\ell'})^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} (D^{\boldsymbol{\nu}} p)(\boldsymbol{x}_{\ell'})$$



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$$= p(\boldsymbol{x}_{\ell'}) + \sum_{0<|\boldsymbol{\nu}|<m} \frac{(\boldsymbol{y}_\ell - \boldsymbol{x}_{\ell'})^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \left(\sum_{\boldsymbol{k}\in I} (2\pi\mathrm{i}\boldsymbol{k})^{\boldsymbol{\nu}} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_{\ell'}} \right)$$

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- for fixed $\boldsymbol{\nu} \in \mathbb{N}_0^d$, compute $(D^{\boldsymbol{\nu}} p)(\boldsymbol{x}_j)$ for all $\boldsymbol{x}_j$, $j = 0, \ldots, M-1$, with 1-dim FFT(M) in $\mathcal{O}(M \log M + d|I|)$

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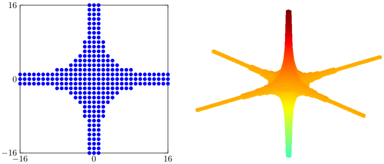
$\Rightarrow$ in total $\mathcal{O}\left(m^d (L + M \log M + d|I|)\right)$ arithmetic operations

Taylor and R1L based NFFT - error estimates

Lemma (V. 13)

Let $I = I_N^d$ hyperbolic cross,



I_N^d

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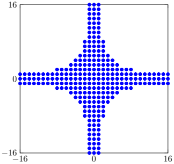

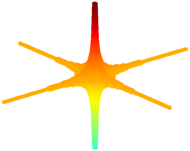

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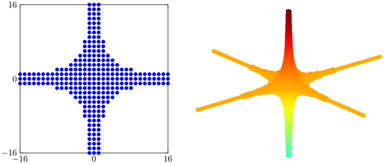
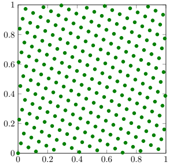


I_N^d



$\mathrm{R1L}(\boldsymbol{z}, M)$

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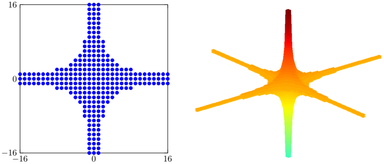
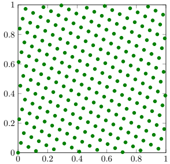


I_N^d



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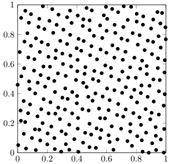

$(p(\boldsymbol{y}_\ell))_{\ell=0}^{L-1}$

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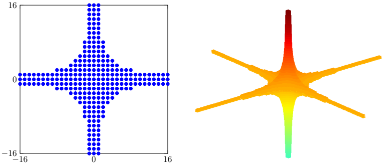
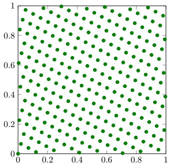
$$|(p - s_m)(\boldsymbol{y}_\ell)| \quad \le \quad \frac{(2\pi)^m}{m!}\varepsilon^m \sum_{\boldsymbol{k}\in I_N^d} |\hat{p}_{\boldsymbol{k}}| \|\boldsymbol{k}\|_1^m$$



I_N^d



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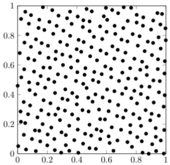

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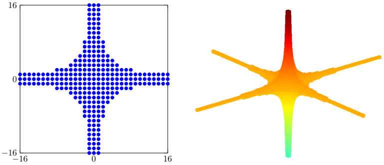

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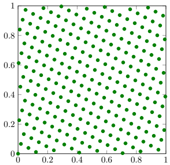

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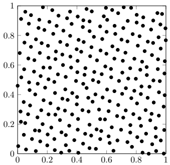

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\end{aligned}
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- error estimates can be generalized to other finite frequency index sets $I \subset \mathbb{Z}^d$, e.g. ℓ_1 balls, energy-based hyperbolic crosses, ...

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- decay properties of Fourier coefficients may be included