Sparse high-dimensional FFT based on rank-1 lattice sampling

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Content

Introduction

- approximate high-dim. function $f\colon \mathbb{T}^d \simeq [0,1)^d \to \mathbb{C}$ by multivariate trigonometric polynomial $p\colon \mathbb{T}^d \to \mathbb{C}$ with frequencies supported on $I \subset \mathbb{Z}^d,\ |I| < \infty$,
$$p(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \hat{p}_{\boldsymbol{k}} \in \mathbb{C}$$

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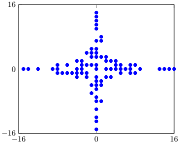
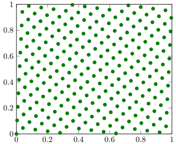
- arbitrary **known** frequency index set $I \subset \mathbb{Z}^d,\ |I| < \infty$, rank-1 lattice nodes $\boldsymbol{x}_j,\ j = 0, \ldots, M-1$
  - fast evaluation $p(\boldsymbol{x}_j)$, (e.g. [Li, Hickernell 03])
  - fast and exact reconstruction of $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, from samples $p(\boldsymbol{x}_j)$, ([Kämmerer, Kunis, Potts 12] [Kämmerer 13])
  - approximate reconstruction of f by p from samples $f(\boldsymbol{x}_j)$



$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I}$

$\longleftrightarrow$ FFT



$(p(\boldsymbol{x}_j))_{j=0}^{M-1}$

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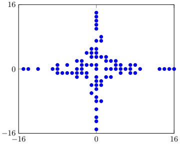

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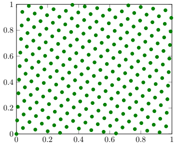


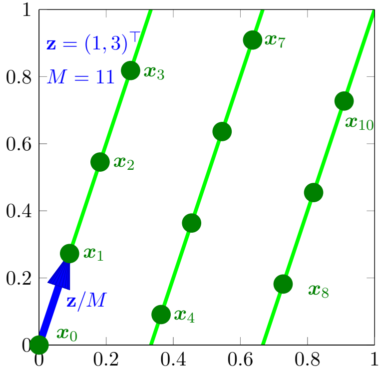
$(p(\boldsymbol{x}_j))_{j=0}^{M-1}$

- unknown frequency index set I?

Trigonometric polynomials - fast evaluation

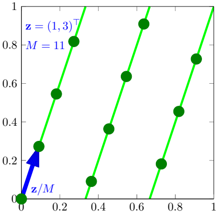
- rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, M)$: $\boldsymbol{z} \in \mathbb{N}_0^d, M \in \mathbb{N}$

$$\boldsymbol{x}_j = \frac{j}{M}\boldsymbol{z} \bmod \mathbf{1};\ j = 0, \ldots, M-1$$



Korobov 59
Maisonneuve 72
Sloan & Kachoyan 84,87,90
Temlyakov 86
Lyness 89
Sloan & Joe 94
Sloan & Reztsov 01
Li & Hickernell 03

Trigonometric polynomials - fast evaluation



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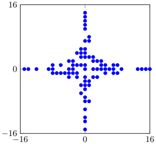
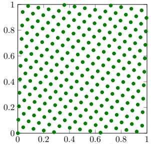
- reformulation

$$p(\boldsymbol{x}_j) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \mathrm{e}^{2\pi\mathrm{i}\frac{j\boldsymbol{k}\cdot\boldsymbol{z}}{M}} = \sum_{l=0}^{M-1} \underbrace{\left(\sum_{\substack{\boldsymbol{k}\in I \\ \boldsymbol{k}\cdot\boldsymbol{z}\equiv l \pmod M}} \hat{p}_{\boldsymbol{k}} \right)}_{\hat{g}_l} \mathrm{e}^{2\pi\mathrm{i}\frac{j\boldsymbol{k}\cdot\boldsymbol{z}}{M}} = \sum_{l=0}^{M-1} \hat{g}_l \mathrm{e}^{2\pi\mathrm{i}\frac{jl}{M}}$$



$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I}$

$\longrightarrow$



$(p(\boldsymbol{x}_j))_{j=0}^{M-1}$

Trigonometric polynomials - fast evaluation

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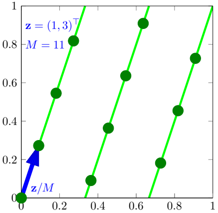

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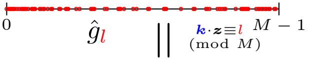

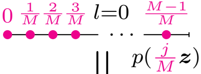

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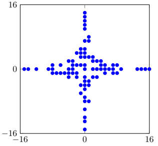
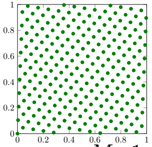


$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I}$

$$\xrightarrow[\text{FFT}]{\text{1-dim}}$$

$$\mathcal{O}(M \log M + d|I|)$$

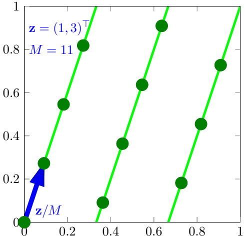
$||$



$(p(\boldsymbol{x}_j))_{j=0}^{M-1}$

Trigonometric polynomials - fast reconstruction

- rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, M)$: $\boldsymbol{z} \in \mathbb{N}_0^d, M \in \mathbb{N}$
  $$\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \mathbf{1};\ j = 0, \ldots, M-1$$
- reconstruction of Fourier coefficients $\hat{p}_{\boldsymbol{k}}$ of multivariate trigonometric polynomial $p(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



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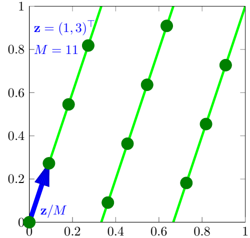
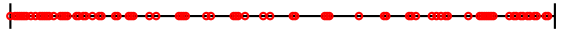


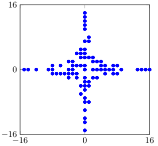
$\Rightarrow$ Definition reconstructing $\mathrm{R1L}(\boldsymbol{z}, M, I)$ for I:
$\boldsymbol{k}\cdot\boldsymbol{z} \not\equiv \boldsymbol{k'}\cdot\boldsymbol{z} \pmod{M}$ for all $\boldsymbol{k}, \boldsymbol{k'} \in I$, $\boldsymbol{k} \neq \boldsymbol{k'}$

- $|I| \leq M \leq |I|^2$, CBC construction algorithm (Kämmerer 2012)



0 $\qquad \hat{g}_l \qquad M-1$

$\Big|\Big| \quad \begin{matrix}\boldsymbol{k}\cdot\boldsymbol{z} \equiv l \\ (\mathrm{mod}\ M)\end{matrix}$

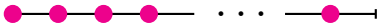


$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I}$

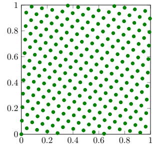
$\xleftarrow[\text{iFFT}]{\text{1-dim}}$

$\mathcal{O}(M \log M + d|I|)$

$0 \quad \frac{1}{M} \quad \frac{2}{M} \quad \frac{3}{M} \quad \ldots \quad \frac{M-1}{M}$



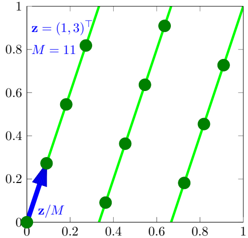
$\Big|\Big| \quad p(\frac{j}{M}\boldsymbol{z})$



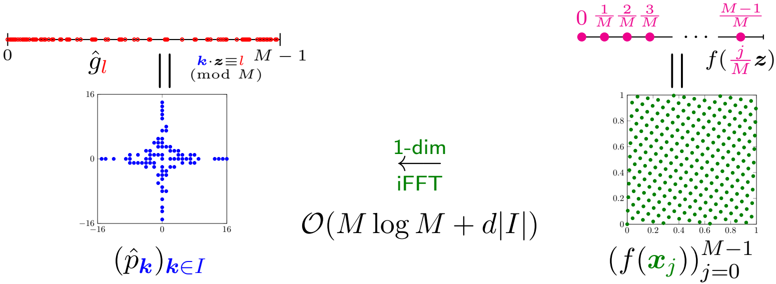
$(p(\boldsymbol{x}_j))_{j=0}^{M-1}$

Trigonometric polynomials - fast approximation

- reconstructing rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, M, I)$,
$$\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \mathbf{1};\ j = 0, \ldots, M-1$$
- approximation of function $f\colon \mathbb{T}^d \to \mathbb{C}$ by multivariate trigonometric polynomial $p(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$



- [Kuo, Sloan, Woźniakowski 06] [Kämmerer, Potts, V. 15] [Byrenheid, Kämmerer, Ullrich, V. 16]



Unknown frequency index set

until now: fast reconstruction of $p(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ or fast approximation of function $f(\boldsymbol{x}) \approx p(\boldsymbol{x})$

- given frequency index set I
- compute $\hat{p}_{\boldsymbol{k}}$ from samples along reconstructing rank-1 lattice

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  full grid $\hat{G}_N^d := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \|\boldsymbol{k}\|_\infty \le N\}$, ($|\hat{G}_{64}^{10}| \approx 1.28 \cdot 10^{21}$)

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$\Rightarrow$ multi-dimensional sparse FFT

Multi-dimensional sparse FFT

Various existing methods, e.g.

- filters [Indyk, Kapralov 14]
- Chinese remainder theorem
  - [Cuyt, Lee 08]
  - [Iwen 13]
- Prony's method
  - multiple lines [Tasche, Potts 13]
  - common zeros [Peter, Plonka, Schaback 15] [Kunis, Peter, Römer, von der Ohe 15]
- dimension-incremental projection
  - Zippel's Algorithm [Zippel 79] [Kaltofen, Lee 03] [Javadi Monagan 10]
  - via (reconstructing) rank-1 lattices [Potts, V. 15]
- randomized Kronecker substitution
  [Arnold, Roche 14] [Arnold, Giesbrecht, Roche 15]
- (reconstructing) rank-1 lattice and 1d method
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High-dim. sparse FFT via 1-dim. sparse FFT

- multivariate problem:
  - $p(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\cdot\boldsymbol{x}} = \sum_{r=1}^{|I|} \hat{p}_r \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{w}_r\cdot\boldsymbol{x}}$
  - determine: $I \subset \Gamma$, $\hat{p}_{\boldsymbol{k}} \in \mathbb{C}$ or $|I| \in \mathbb{N}$, $\boldsymbol{w}_r \in \Gamma$, $\hat{p}_r \in \mathbb{C}$

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- use reconstructing rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, M, \Gamma)$ for $\Gamma \supset I$:
  - $p\left(\frac{j}{M}\boldsymbol{z}\right) = \sum_{r=1}^{|I|} \hat{p}_r \mathrm{e}^{2\pi\mathrm{i}(\boldsymbol{w}_r\cdot\boldsymbol{z})\, j/M}$

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  $\Rightarrow$ injective mapping of multi-dim. frequencies to 1-dim.,
  $\Gamma \subset \mathbb{Z}^d \to \{0, 1, \dots, M-1\}$, $\boldsymbol{w}_r \mapsto \boldsymbol{w}_r \cdot \boldsymbol{z} \bmod M =: t_r$

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$\Rightarrow$ univariate problem (sparse FFT):

  - $\tilde{p}(x) = \sum_{r=1}^{|I|} \hat{p}_r \mathrm{e}^{2\pi\mathrm{i}t_r x}$, sample at $x = j/M$
  - determine: $|I| \in \mathbb{N}$, $t_r \equiv \boldsymbol{w}_r \cdot \boldsymbol{z} \pmod{M} \in \mathbb{Z}$, $\hat{p}_r \in \mathbb{C}$

High-dim. sparse FFT via 1-dim. sparse FFT

usage of 1-dim. sparse FFT methods based on e.g.

	samples	computational costs
compressed sensing	$\mathcal{O}\left(\lvert I\rvert \log^4(M) \log(1/\eta)\right)$	$\mathcal{O}\left(M \lvert I\rvert \log^4(M) \log(1/\eta)\right)$
	[Rauhut 07] [Kunis, Rauhut 08] [Gröchenig, Pötscher, Rauhut 10] [Foucart, Rauhut 13] ...	
filters	$\mathcal{O}\left(\lvert I\rvert(\log M)(\log(M/\lvert I\rvert))\right)$	$\mathcal{O}\left(\lvert I\rvert(\log M)(\log(M/\lvert I\rvert))\right)$
	[Hassanieh, Indyk, Katabi, Price 12] ...	
C.R.T.	$\mathcal{O}\left(\lvert I\rvert \log^4(M)\right)$	$\mathcal{O}\left(\lvert I\rvert \log^4(M)\right)$
	[Iwen 10] [Iwen 13]	
shifted sampling	$\mathcal{O}\left(\lvert I\rvert \log(M/\lvert I\rvert)\right)$	$\mathcal{O}\left(\lvert I\rvert \log(M/\lvert I\rvert)\right)$
	(on average) [Christlieb, Lawlor, Wang 15]	
ESPRIT	$\mathcal{O}(\lvert I\rvert)$	$\mathcal{O}(\lvert I\rvert^3)$
	(deterministic) [Roy, Kailath 89]	
ESPRIT + shifted samp.	$\mathcal{O}(\lvert I\rvert)$	$\mathcal{O}\left(\lvert I\rvert^{5/3}\right)$
	(1 iteration) [Potts, Tasche, V. 16]	

problems: constants?, noise / stability, “whp”, implementation

High-dim. sparse FFT via 1-dim. sparse FFT

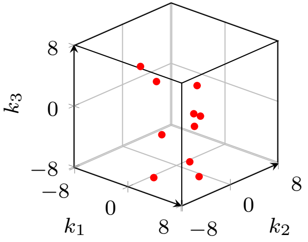
usage of 1-dim. sparse FFT methods based on e.g.

	samples	computational costs
compressed sensing	$\mathcal{O}\left(\lvert I\rvert \log^4(M) \log(1/\eta)\right)$	$\mathcal{O}\left(M \lvert I\rvert \log^4(M) \log(1/\eta)\right)$
	[Rauhut 07] [Kunis, Rauhut 08] [Gröchenig, Pötscher, Rauhut 10] [Foucart, Rauhut 13] …	
filters	$\mathcal{O}\left(\lvert I\rvert (\log M)(\log(M/\lvert I\rvert))\right)$	$\mathcal{O}\left(\lvert I\rvert (\log M)(\log(M/\lvert I\rvert))\right)$
	[Hassanieh, Indyk, Katabi, Price 12] …	
C.R.T.	$\mathcal{O}\left(\lvert I\rvert \log^4(M)\right)$	$\mathcal{O}\left(\lvert I\rvert \log^4(M)\right)$
	[Iwen 10] [Iwen 13]	
shifted sampling	$\mathcal{O}\left(\lvert I\rvert \log(M/\lvert I\rvert)\right)$	$\mathcal{O}\left(\lvert I\rvert \log(M/\lvert I\rvert)\right)$
	(on average) [Christlieb, Lawlor, Wang 15]	
ESPRIT	$\mathcal{O}(\lvert I\rvert)$	$\mathcal{O}(\lvert I\rvert^3)$
	(deterministic) [Roy, Kailath 89]	
ESPRIT + shifted samp.	$\mathcal{O}(\lvert I\rvert)$	$\mathcal{O}\left(\lvert I\rvert^{5/3}\right)$
	(1 iteration) [Potts, Tasche, V. 16]	

problems: constants?, noise / stability, “whp”, implementation

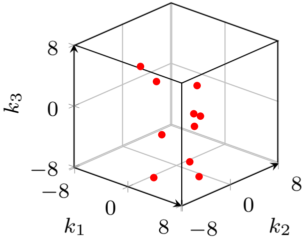
$\Rightarrow$ next: different approach based on dimension-incremental idea

Sparse dimension-incremental FFT - method

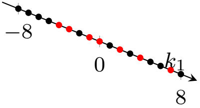


$I \subset \Gamma = \hat{G}_8^3$

Sparse dimension-incremental FFT - method

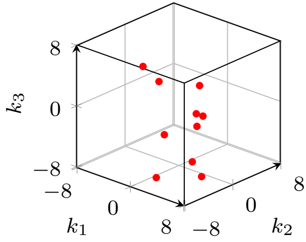


$I \subset \Gamma = \hat{G}_8^3$

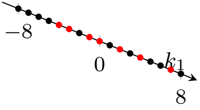


frequency candidates

Sparse dimension-incremental FFT - method

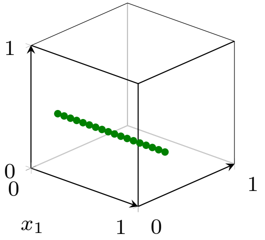


$I \subset \Gamma = \hat{G}_8^3$



frequency candidates

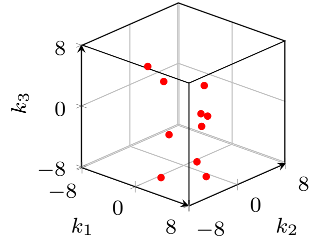
$\xrightarrow[\text{sampling set}]{\text{construct}}$



sampling nodes

$(\frac{\ell}{17}, x_2', x_3')^\top$, $\ell = 0, \ldots, 16$

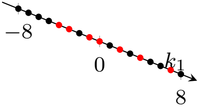
Sparse dimension-incremental FFT - method



$I \subset \Gamma = \hat{G}_8^3$

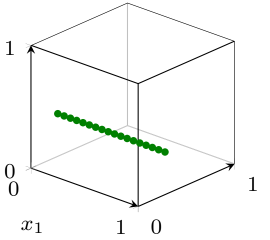
$$\hat{p}_{k_1} := \frac{1}{17}\sum_{\ell=0}^{16} p\left(\begin{pmatrix}\ell/17\\ x_2'\\ x_3'\end{pmatrix}\right) \mathrm{e}^{-2\pi\mathrm{i}\frac{\ell k_1}{17}}$$

$k_1 = -8, \ldots, 8$



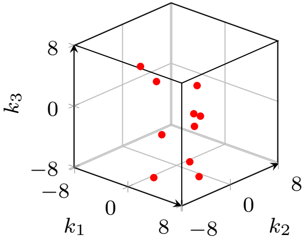
frequency candidates

$\xleftarrow[\text{iFFT}]{\text{1-dim}}$



sampling nodes
$(\frac{\ell}{17}, x_2', x_3')^\top$, $\ell = 0, \ldots, 16$

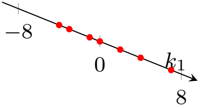
Sparse dimension-incremental FFT - method



$I \subset \Gamma = \hat{G}_8^3$

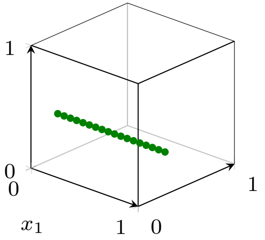
$$\hat{p}_{k_1} := \frac{1}{17} \sum_{\ell=0}^{16} p\left(\begin{pmatrix} \ell/17 \\ x_2' \\ x_3' \end{pmatrix}\right) \mathrm{e}^{-2\pi\mathrm{i}\frac{\ell k_1}{17}}$$
$$= \sum_{\substack{(h_2,h_3)\in\{-8,\ldots,8\}^2 \\ (k_1,h_2,h_3)^\top \in \operatorname{supp}\hat{p}}} \hat{p}_{\begin{pmatrix} k_1 \\ h_2 \\ h_3 \end{pmatrix}} \mathrm{e}^{2\pi\mathrm{i}(h_2 x_2' + h_3 x_3')},$$

$k_1 = -8, \ldots, 8$



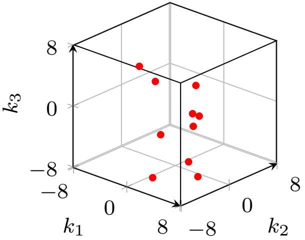
detected frequencies $I^{(1)}$

$\xleftarrow[\text{iFFT}]{\text{1-dim}}$

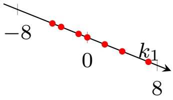


sampling nodes
$(\frac{\ell}{17}, x_2', x_3')^\top$, $\ell = 0, \ldots, 16$

Sparse dimension-incremental FFT - method

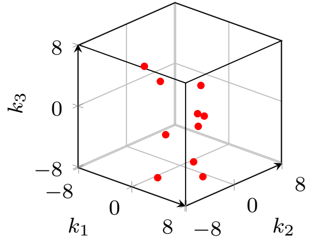


$I \subset \Gamma = \hat{G}_8^3$

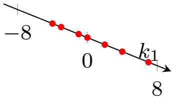


detected frequencies
$I^{(1)}$

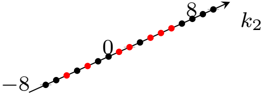
Sparse dimension-incremental FFT - method



$I \subset \Gamma = \hat{G}_8^3$

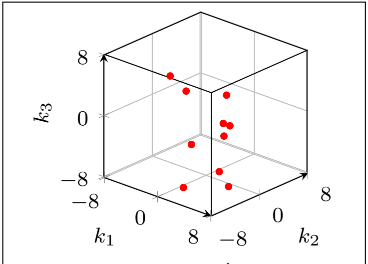


detected frequencies
$I^{(1)}$

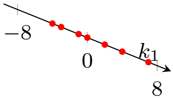


frequency candidates

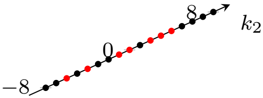
Sparse dimension-incremental FFT - method



$I \subset \Gamma = \hat{G}_8^3$

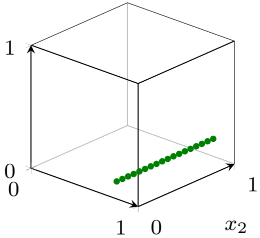


detected frequencies
$I^{(1)}$



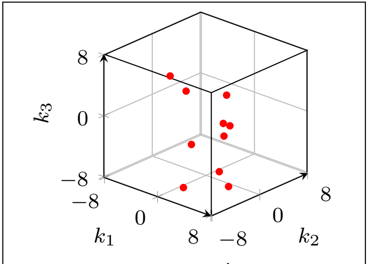
frequency candidates

$\xrightarrow[\text{sampling set}]{\text{construct}}$

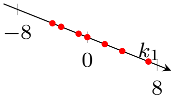


sampling nodes

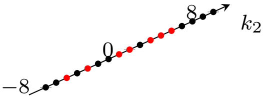
Sparse dimension-incremental FFT - method



$I \subset \Gamma = \hat{G}_8^3$

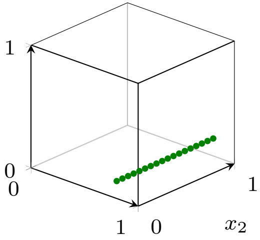


detected frequencies
$I^{(1)}$



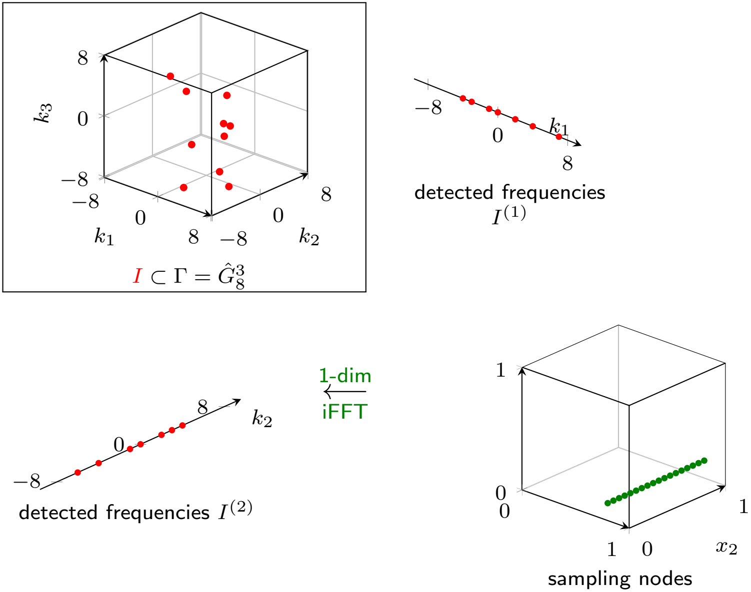
frequency candidates

1-dim $\longleftarrow$ iFFT

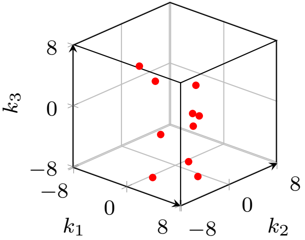


sampling nodes

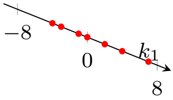
Sparse dimension-incremental FFT - method



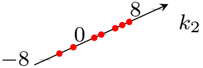
Sparse dimension-incremental FFT - method



$\color{red}{I} \subset \Gamma = \hat{G}_8^3$

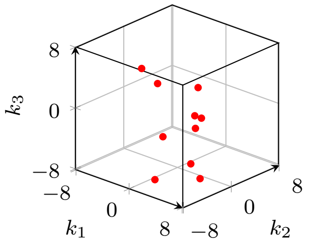


detected frequencies
$I^{(1)}$

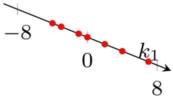


detected frequencies
$I^{(2)}$

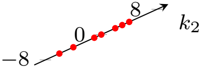
Sparse dimension-incremental FFT - method



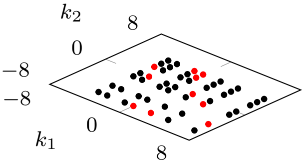
$I \subset \Gamma = \hat{G}_8^3$



detected frequencies
$I^{(1)}$

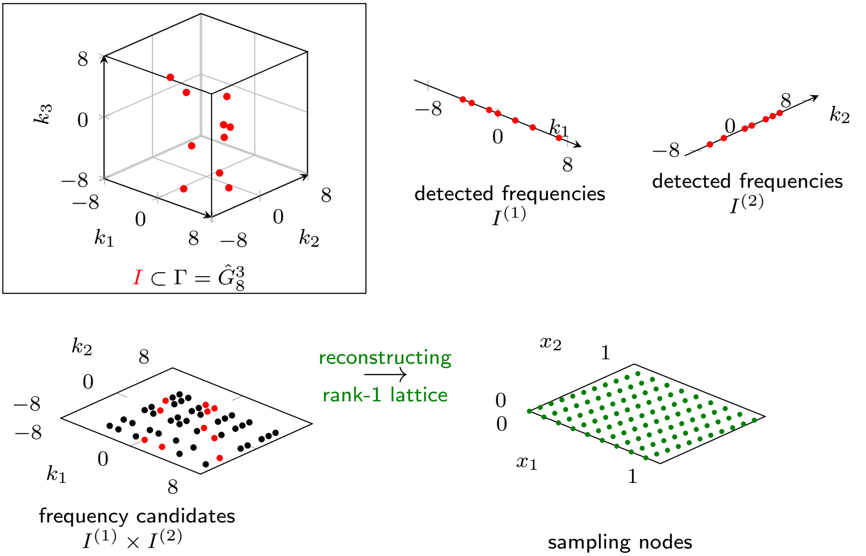


detected frequencies
$I^{(2)}$

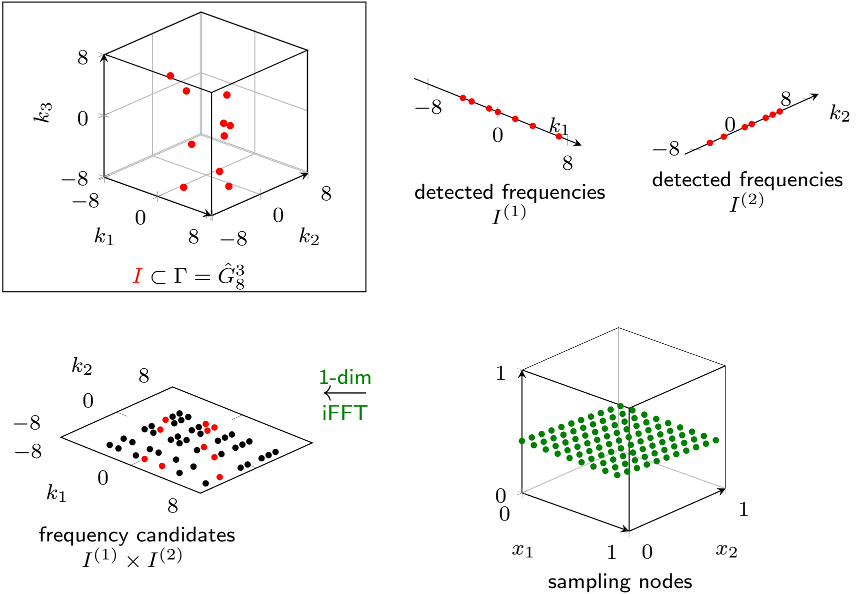


frequency candidates
$I^{(1)} \times I^{(2)}$

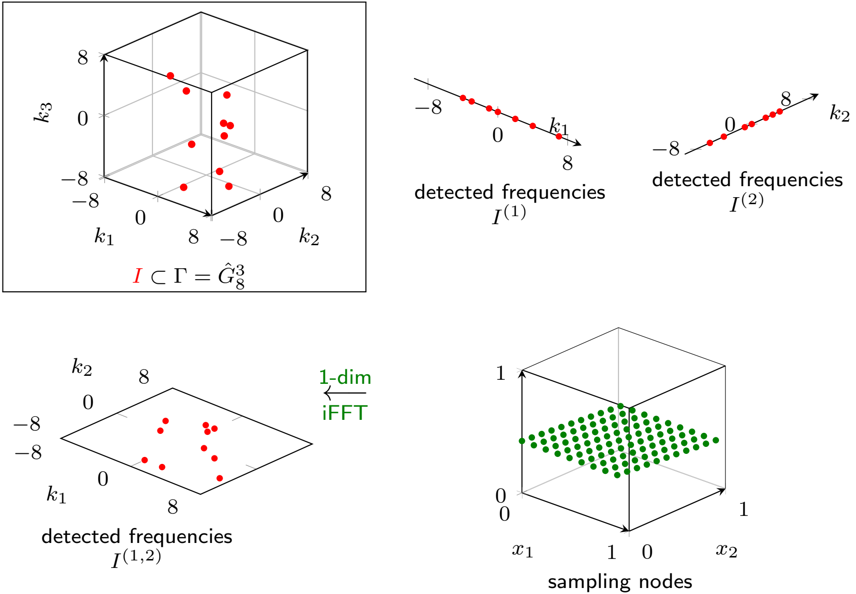
Sparse dimension-incremental FFT - method



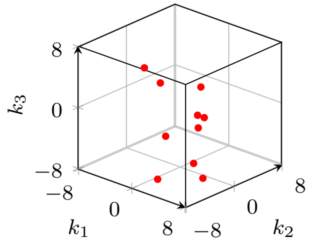
Sparse dimension-incremental FFT - method



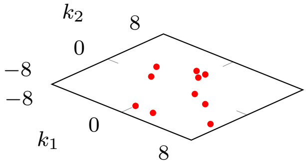
Sparse dimension-incremental FFT - method



Sparse dimension-incremental FFT - method

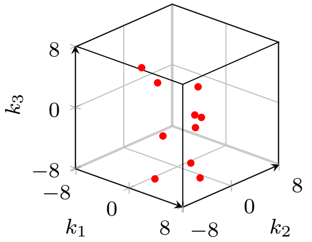


$I \subset \Gamma = \hat{G}_8^3$

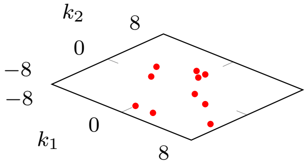


detected frequencies
$I^{(1,2)}$

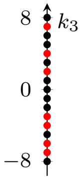
Sparse dimension-incremental FFT - method



$I \subset \Gamma = \hat{G}_8^3$

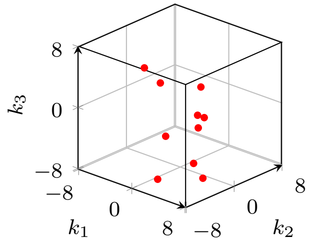


detected frequencies
$I^{(1,2)}$

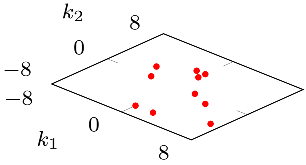


frequency candidates

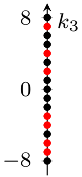
Sparse dimension-incremental FFT - method



$I \subset \Gamma = \hat{G}_8^3$

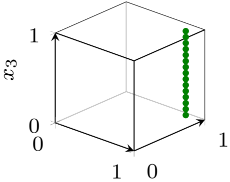


detected frequencies
$I^{(1,2)}$



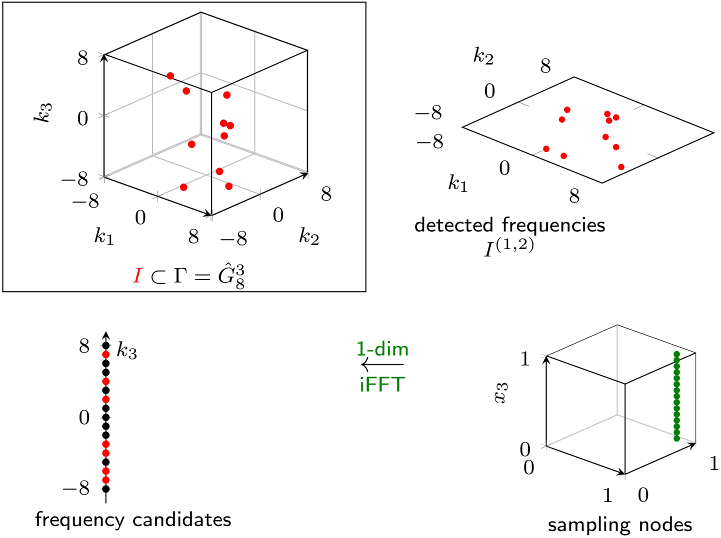
frequency candidates

construct $\longrightarrow$ sampling set

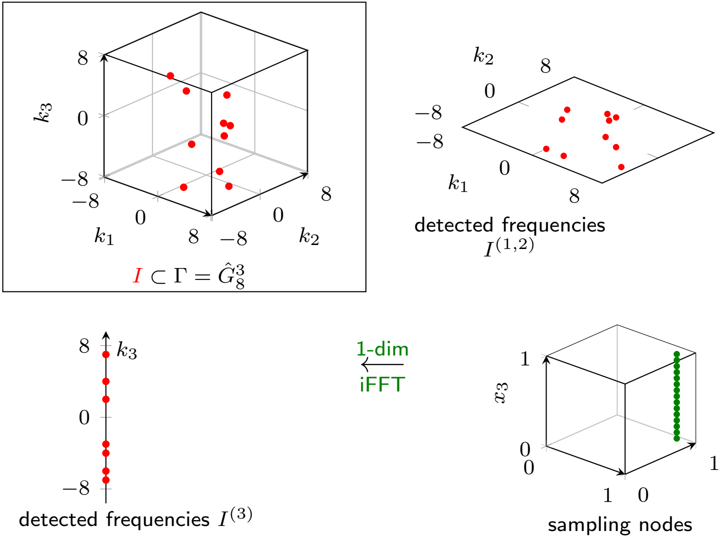


sampling nodes

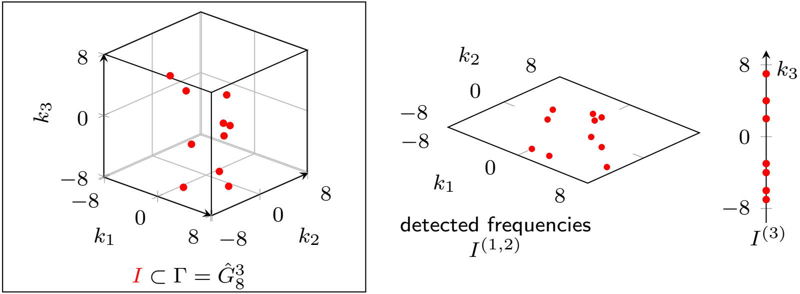
Sparse dimension-incremental FFT - method



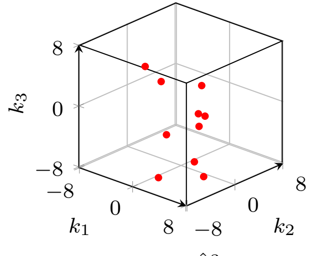
Sparse dimension-incremental FFT - method



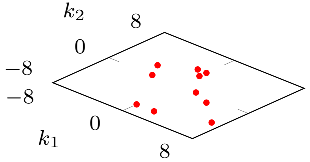
Sparse dimension-incremental FFT - method



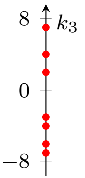
Sparse dimension-incremental FFT - method



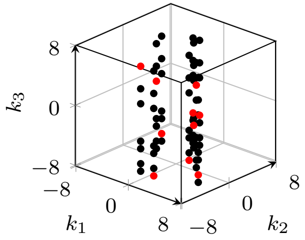
$I \subset \Gamma = \hat{G}_8^3$



detected frequencies
$I^{(1,2)}$

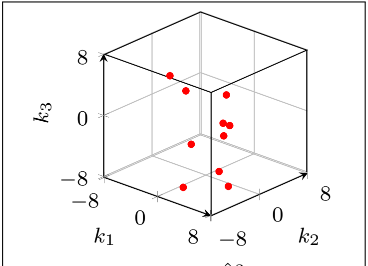


$I^{(3)}$

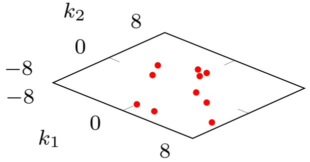


frequency candidates
$I^{(1,2)} \times I^{(3)}$

Sparse dimension-incremental FFT - method

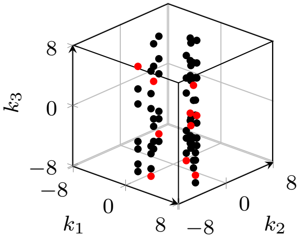


$\color{red}{I} \subset \Gamma = \hat{G}_8^3$



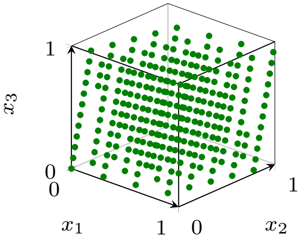
detected frequencies
$I^{(1,2)}$

$I^{(3)}$



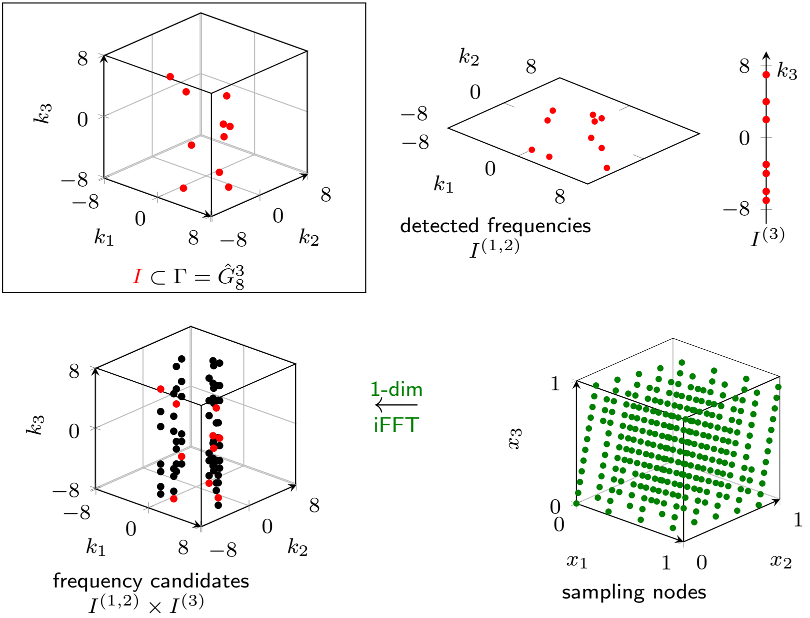
frequency candidates
$I^{(1,2)} \times I^{(3)}$

reconstructing
$\longrightarrow$
rank-1 lattice

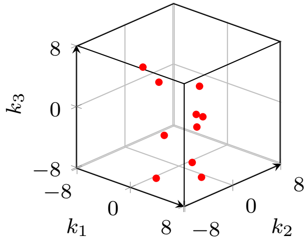


sampling nodes

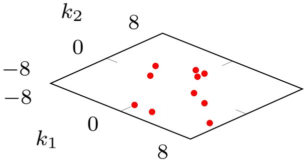
Sparse dimension-incremental FFT - method



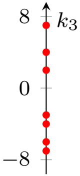
Sparse dimension-incremental FFT - method



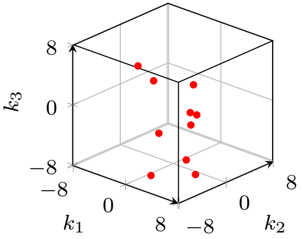
$\color{red}{I} \subset \Gamma = \hat{G}_8^3$



detected frequencies
$I^{(1,2)}$

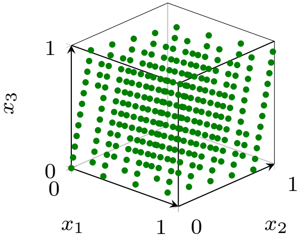


$I^{(3)}$



detected frequencies
$I^{(1,2,3)} = I$

$\xleftarrow[\text{iFFT}]{\text{1-dim}}$



sampling nodes

Sparse dimension-incremental FFT - results

search domain $\Gamma = \hat{G}_N^d$ full grid, $\sqrt{N} \lesssim |I| \lesssim N^d$

- samples: $\mathcal{O}(|I|^2 \log|\Gamma|)$
  (1 iteration)
- computational costs: $\mathcal{O}(d\,|I|^3 + |I|^2(\log|\Gamma|)\log(|I|\,\log|\Gamma|))$
  (1 iteration)
- for arbitrary Fourier coefficients $\hat{p}_{\boldsymbol{k}} \in \mathbb{C}$:
  probabilistic approach with several iterations
- if ($\mathrm{Re}(\hat{p}_{\boldsymbol{k}})$ identical sign) AND ($\mathrm{Im}(\hat{p}_{\boldsymbol{k}})$ identical sign)
  then deterministic version with 1 iteration

Sparse dimension-incremental FFT - example

- B-spline $N_m(x) := \sum_{k\in\mathbb{Z}} C_m \operatorname{sinc}\left(\frac{\pi}{m}k\right)^m \cos(\pi k)\, \mathrm{e}^{2\pi \mathrm{i} k x}$, $\|N_m|L^2(\mathbb{T})\| = 1$, $|\hat{N}_m(k)| \sim |k|^{-m}$
- $f(\boldsymbol{x}) := \prod\limits_{t\in\{1,3,8\}} N_2(x_t) + \prod\limits_{t\in\{2,5,6,10\}} N_4(x_t) + \prod\limits_{t\in\{4,7,9\}} N_6(x_t)$

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- full grid for $N = 64$, $d = 10$: $|\hat{G}_{64}^{10}| = 129^{10} \approx 1.28 \cdot 10^{21}$
- symmetric hyperbolic cross: $|I_{64}^{10}| = 696\,036\,321$
  relative $L^2(\mathbb{T}^d)$-error (best case) 4.1e-04

Sparse dimension-incremental FFT - example

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  relative $L^2(\mathbb{T}^d)$-error (best case) 4.1e-04
- results for dimension incremental algorithm with $\Gamma = \hat{G}_{64}^{10}$:

threshold	#samples	$\|I\|$	rel. L_2-error
1.0e-02	254 530	491	1.4e-01
1.0e-03	2 789 050	1 121	1.1e-02
1.0e-04	17 836 042	3 013	1.7e-03
1.0e-05	82 222 438	7 163	4.7e-04

Non-periodic case

results can be transfered from periodic to non-periodic case:

- multivariate algebraic polynomial $p : [-1,1]^d \to \mathbb{R}$ in Chebyshev form with frequencies supp. on $I \subset \mathbb{N}_0^d$, $|I| < \infty$,

$$a(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in I} \hat{a}_{\boldsymbol{k}} \prod_{t=1}^{d} T_{k_t}(x_t), \quad \hat{a}_{\boldsymbol{k}} \in \mathbb{R},$$

	periodic	non-periodic
basis function	$\mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}$	$\prod_{t=1}^{d} T_{k_t}(x_t)$

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	periodic	non-periodic
basis function	$\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$	$\prod_{t=1}^{d} T_{k_t}(x_t)$
spatial nodes	rank-1 lattice $\boldsymbol{x}_j = \frac{j}{M}\boldsymbol{z}$ mod 1	rank-1 Chebyshev lattice $\boldsymbol{x}_j = \cos\left(\frac{j}{M}\pi\boldsymbol{z}\right)$ [Poppe, Cools 12]

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evaluation / reconstruction	1-dim. FFT [Li, Hickernell 03] / [Kämmerer 13] [Kämmerer, Kunis, Potts 12]	1-dim. DCT [Cools, Poppe 11] [Potts, V. 15] / [Poppe, Cools 13] [Potts, V. 15]

Non-periodic case - related work

multivar. polynomial approx. on Lissajous curves [Bos, De Marchi, Vianello 15]

- $\ell_{\boldsymbol{z}}(t) := (\cos(z_1 t), \ldots, \cos(z_d t))$, $t \in [0, \pi]$
- rank-1 Chebyshev lattice if $t = 0, \pi/M, 2\pi/M, \ldots, \pi$
- results for algebraic polynomials of total or max. deg. $\leq n$
- hyperinterpolation for $d \geq 3$

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- hyperinterpolation for $d \geq 3$
- case $d = 2$ interpolation
  - Padua point set,
    e.g. [Bos, Caliari, Marchi, Vianello, Xu 06]
    $\mathcal{A}_n := \{\boldsymbol{x}_j := (\cos(j\pi/(n+1)), \cos(j\pi/n))^\top : j = 0, \ldots, M\}$
  - $\mathcal{A}_n = \mathrm{CL}(\boldsymbol{z}, M)$, where $\boldsymbol{z} := (n, n+1)^\top$ and $M := n\,(n+1)$
  - exact reconstruction of arbitrary 2d algebraic polynomial of total degree $\leq n$

multivar. polynomial interpolation on Lissajous-Chebyshev nodes
[Dencker, Erb 15]

Summary

- **known** (arbitrary) frequency index set via rank-1 lattices
  - fast evaluation / reconstruction of trigonometric polynomials

    [Li, Hickernell 03] / [Kämmerer, Kunis, Potts 12] [Kämmerer 13]
  - approximation of periodic functions

    [Kuo, Sloan, Woźniakowski 06] [Kämmerer, Potts, V. 15] [Byrenheid, Kämmerer, Ullrich, V. 16]
  - fast evaluation / reconstruction of algebraic polynomials

    [Potts, V. 15] / [Potts, V. 15]
  - *approximation of non-periodic functions (not in this talk)*

    see e.g. [Dick, Nuyens, Pillichshammer 14] [Suryanarayana, Nuyens, Cools 15]

    [Cools, Kuo, Nuyens, Suryanarayana 16]
- **unknown** frequency index set
  - multivariate sparse FFT via C.R.T. / Prony

    e.g. [Cuyt, Lee 08] [Peter, Plonka, Schaback 15] [Kunis, Peter, Römer, von der Ohe 15]
  - high-dim. sparse FFT via rank-1 lattice and 1-dim. sparse FFT

    e.g. [Potts, Tasche, V. 16]
  - sparse dimension-incremental FFT based on rank-1 lattices (periodic and non-periodic)

    [Potts, V. 15] [V. 17]