# Fast and exact reconstruction of arbitrary multivariate algebraic polynomials in Chebyshev form

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joint work with L. Kämmerer and D. Potts

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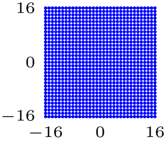
# Content

## Introduction

## Fast evaluation

periodic case $\tilde{p}\colon \mathbb{T}^d \to \mathbb{C}$, $\tilde{p}(\boldsymbol{x}) := \sum_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}} \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}}\, \mathrm{e}^{2\pi \mathrm{i} \tilde{\boldsymbol{k}} \cdot \boldsymbol{x}}$

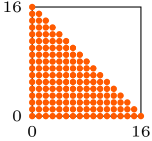



non-periodic case $p\colon [-1,1]^d \to \mathbb{R}$, $p(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$

## Fast and exact reconstruction

periodic case

non-periodic case




## Multiple lattices

## Approximation

## Summary

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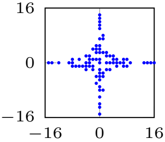


## Fast evaluation

periodic case $\tilde{p}\colon \mathbb{T}^d \to \mathbb{C}$, $\tilde{p}(\boldsymbol{x}) := \sum_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}} \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}} \, \mathrm{e}^{2\pi \mathrm{i} \tilde{\boldsymbol{k}} \cdot \boldsymbol{x}}$

non-periodic case $p\colon [-1,1]^d \to \mathbb{R}$, $p(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}} \, T_{\boldsymbol{k}}(\boldsymbol{x})$

## Fast and exact reconstruction

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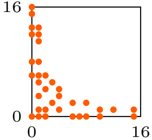



## Multiple lattices

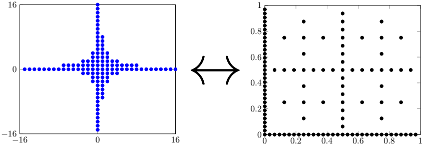
## Approximation

## Summary

- $d$-dimensional torus $\mathbb{T}^d \simeq [0,1)^d$
- multivariate trigonometric polynomial $\tilde{p}\colon \mathbb{T}^d \to \mathbb{C}$ with frequencies supported on $\tilde{\mathrm{I}} \subset \mathbb{Z}^d$, $|\tilde{\mathrm{I}}| < \infty$,

$$\tilde{p}(\boldsymbol{x}) := \sum_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}} \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}} \mathrm{e}^{2\pi \mathrm{i} \tilde{\boldsymbol{k}} \cdot \boldsymbol{x}}, \quad \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}} \in \mathbb{C}$$

- hyperbolic cross frequency index sets $\tilde{I}$




$\tilde{I}$ sparse grid

([Baszenski, Delvos 89] [Hallatschek 92] [Gradinaru 07] [Griebel, Hamaekers 14])

# Introduction - periodic case

- $d$-dimensional torus $\mathbb{T}^d \simeq [0,1)^d$
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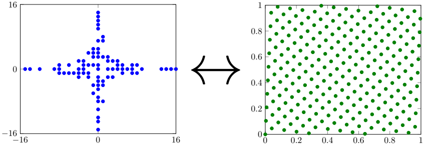
- arbitrary frequency index sets $\tilde{I}$
  - fast evaluation $\tilde{p}(\boldsymbol{x}_j)$, $j = 0, \ldots, M-1$ (e.g. [Li, Hickernell 03])
  - fast and exact reconstruction of $\hat{\tilde{p}}_{\tilde{\boldsymbol{k}}}$, $\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}$, from samples $\tilde{p}(\boldsymbol{x}_j)$, $j = 0, \ldots, M-1$ ([Kämmerer, Kunis, Potts 12] [Kämmerer 13])




$\tilde{I}$ $\qquad$ $\boldsymbol{x}_j$

# Introduction - periodic case

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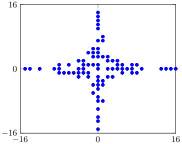
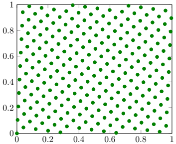



$\tilde{I}$

$\leftrightarrow$




$\boldsymbol{x}_j$

Aim: transfer results to non-periodic case

# Introduction - non-periodic case

- multivariate algebraic polynomial $p : [-1,1]^d \to \mathbb{R}$ in Chebyshev form with frequencies supp. on $I \subset \mathbb{N}_0^d$, $|I| < \infty$,

$$p(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}} \, T_{\boldsymbol{k}}(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}} \prod_{t=1}^{d} T_{k_t}(x_t), \quad \hat{p}_{\boldsymbol{k}} \in \mathbb{R}, \qquad (1)$$

$T_m : [-1,1] \to [-1,1]$, $T_m(x) := \cos(m \arccos x)$, $m \in \mathbb{N}_0$

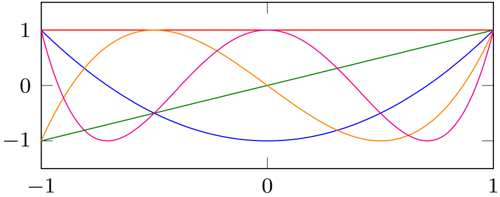
($T_m$ is algebraic polynomial of degree $m$ restricted to domain $[-1,1]$)

$T_0(x) = 1$ $\quad$ $T_1(x) = x$

$T_2(x) = 2x^2 - 1$ $\quad$ $T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 + 1$

$T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x) \qquad m \in \mathbb{N}$



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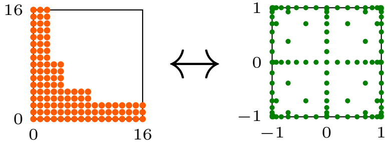
- e.g. hyperbolic cross index sets $I$

$\Rightarrow$ fast evaluation/reconstruction at Chebyshev sparse grid nodes

[Barthelmann, Novak, Ritter 00] [Shen, Yu 10]



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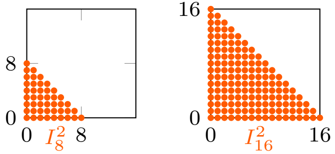
- e.g. $\ell_1$-ball index set $I = I_n^d := \{\boldsymbol{k} \in \mathbb{N}_0^d \colon \|\boldsymbol{k}\|_1 \leq n\}$, $n \in \mathbb{N}_0$

$\Rightarrow$ each algebraic polynomial of total degree $\leq n$ (restricted to the domain $[-1,1]^d$) can be written as (1) with $I = I_n^d$



# Introduction - non-periodic case

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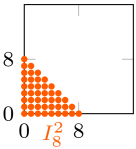
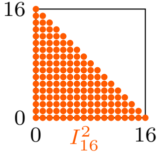
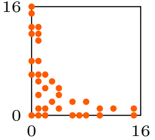



- Aim:
  - fast evaluation of high-dimensional arbitrary algebraic polynomials (given in Chebyshev form)
  - fast and exact reconstruction from samples

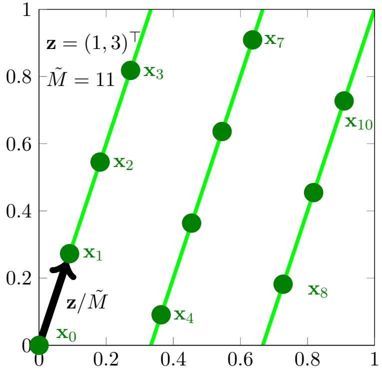
(special case of $\ell_1$-ball index sets $I = I_n^d$ e.g. [Poppe,Cools 12])



# Fast evaluation - periodic case

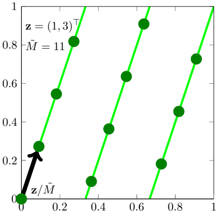
- rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, \tilde{M})$: $\boldsymbol{z} \in \mathbb{N}^d, \tilde{M} \in \mathbb{N}$

$$\boldsymbol{x}_j = \frac{j}{\tilde{M}} \boldsymbol{z} \bmod \mathbf{1};\ j = 0, \ldots, \tilde{M} - 1$$




Korobov 59
Maisonneuve 72
Sloan & Kachoyan 84,87,90
Temlyakov 86
Lyness 89
Sloan & Joe 94
Sloan & Reztsov 01
Li & Hickernell 03

# Fast evaluation - periodic case

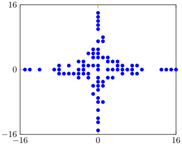



- rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, \tilde{M})$: $\boldsymbol{z} \in \mathbb{N}^d, \tilde{M} \in \mathbb{N}$

$$\boldsymbol{x}_j = \frac{j}{\tilde{M}} \boldsymbol{z} \text{ mod } \mathbf{1};\ j = 0, \dots, \tilde{M} - 1$$

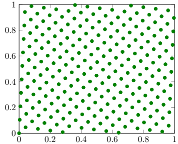
- multivariate trigonometric polynomial $\tilde{p}(\boldsymbol{x}) = \sum_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}} \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}} \mathrm{e}^{2\pi\mathrm{i}\tilde{\boldsymbol{k}}\cdot\boldsymbol{x}}$
- reformulation

$$\tilde{p}(\boldsymbol{x}_j) = \sum_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}} \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}} \mathrm{e}^{2\pi\mathrm{i}\frac{j\tilde{\boldsymbol{k}}\cdot\boldsymbol{z}}{\tilde{M}}} = \sum_{l=0}^{\tilde{M}-1} \underbrace{\left( \sum_{\substack{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}} \\ \tilde{\boldsymbol{k}}\cdot\boldsymbol{z} \equiv l \pmod{\tilde{M}}}} \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}} \right)}_{\hat{g}_l} \mathrm{e}^{2\pi\mathrm{i}\frac{j\tilde{\boldsymbol{k}}\cdot\boldsymbol{z}}{\tilde{M}}} = \sum_{l=0}^{\tilde{M}-1} \hat{g}_l \mathrm{e}^{2\pi\mathrm{i}\frac{jl}{\tilde{M}}}$$



$(\hat{\tilde{p}}_{\tilde{\boldsymbol{k}}})_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}}$

$\longrightarrow$

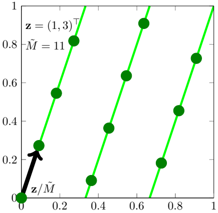


$(\tilde{p}(\boldsymbol{x}_j))_{j=0}^{\tilde{M}-1}$

# Fast evaluation - periodic case

- rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, \tilde{M})$: $\boldsymbol{z} \in \mathbb{N}^d, \tilde{M} \in \mathbb{N}$

$$\boldsymbol{x}_j = \frac{j}{\tilde{M}} \boldsymbol{z} \bmod \mathbf{1};\ j = 0, \dots, \tilde{M} - 1$$




- multivariate trigonometric polynomial $\tilde{p}(\boldsymbol{x}) = \sum_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}} \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}} \mathrm{e}^{2\pi \mathrm{i} \tilde{\boldsymbol{k}} \cdot \boldsymbol{x}}$
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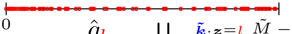
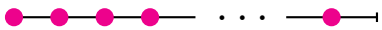
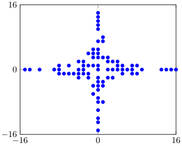



$\hat{g}_l$ || $\tilde{\boldsymbol{k}} \cdot \boldsymbol{z} \equiv l \pmod{\tilde{M}}$

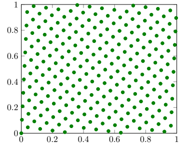



|| $\tilde{p}(\frac{j}{\tilde{M}} \boldsymbol{z})$



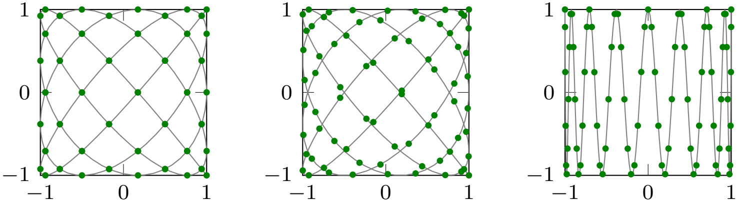

$(\hat{\tilde{p}}_{\tilde{\boldsymbol{k}}})_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}}$ $\xrightarrow[\text{FFT}]{\text{1-dim}}$ $(\tilde{p}(\boldsymbol{x}_j))_{j=0}^{\tilde{M}-1}$

$$\mathcal{O}(\tilde{M} \log \tilde{M} + d|\tilde{\mathrm{I}}|)$$



# Fast evaluation - non-periodic case

- $d$-dimensional rank-1 Chebyshev lattice [Cools, Poppe 2011]
  $\mathrm{CL}(\boldsymbol{z}, M) := \{\boldsymbol{x}_j : j = 0, \ldots, M\} \subset [-1,1]^d$
  with nodes $\boldsymbol{x}_j := \cos\left(\frac{j}{M}\pi\boldsymbol{z}\right)$, $j = 0, \ldots, M$
  - generating vector $\boldsymbol{z} \in \mathbb{N}_0^d$
  - size parameter $M \in \mathbb{N}_0$




$\boldsymbol{z} := (8,9)^\top$, $M := 72$, $|\mathrm{CL}(\boldsymbol{z}, M)| = 45$

$\boldsymbol{z} := (8,9)^\top$, $M := 73$, $|\mathrm{CL}(\boldsymbol{z}, M)| = 74$

$\boldsymbol{z} := (1,16)^\top$, $M := 76$, $|\mathrm{CL}(\boldsymbol{z}, M)| = 77$

# Fast evaluation - non-periodic case

- with Chebyshev nodes $\boldsymbol{x}_j := \cos\left(\frac{j}{M}\pi\boldsymbol{z}\right)$, $j = 0, \dots, M$,

$$p(\boldsymbol{x}_j) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} T_{\boldsymbol{k}}(\boldsymbol{x}_j)$$

$$= \sum_{l=0}^{M} \underbrace{\left( \sum_{\substack{\boldsymbol{k}\in I,\ \boldsymbol{s}\in\{-1,1\}^d \\ (\boldsymbol{s}\odot\boldsymbol{k})\cdot\boldsymbol{z} \operatorname{emod} M = l}} \frac{\hat{p}_{\boldsymbol{k}}}{2^d} \right)}_{\hat{g}_l} \cos\left(\frac{jl}{M}\pi\right) = \sum_{l=0}^{M} \hat{g}_l \cos\left(\frac{jl}{M}\pi\right)$$

with even-mod relation

$$l \operatorname{emod} M := \begin{cases} l \bmod (2M) & \text{for } l \bmod (2M) \leq M, \\ 2M - (l \bmod (2M)) & \text{else} \end{cases}$$

# Fast evaluation - non-periodic case

- with Chebyshev nodes $\boldsymbol{x}_j := \cos\left(\frac{j}{M}\pi\boldsymbol{z}\right)$, $j = 0, \ldots, M$,

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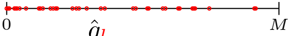
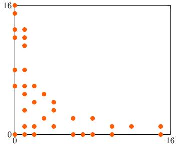



$\hat{g}_l$

$||$ $\boldsymbol{k}\cdot\boldsymbol{z} \,\mathrm{emod}\, M = l$

$p\left(\cos\left(\frac{j}{M}\pi\boldsymbol{z}\right)\right)$

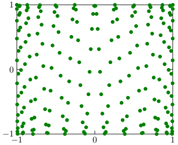
$||$




$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I}$

$\xrightarrow[\text{DCT-I}]{\text{1-dim}}$

$$\mathcal{O}(M \log M + d\, 2^d |I|)$$



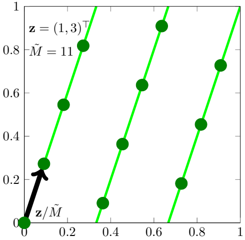

$(p(\boldsymbol{x}_j))_{j=0}^{M}$

# Fast and exact reconstruction - periodic case

- rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, \tilde{M})$: $\boldsymbol{z} \in \mathbb{N}^d, \tilde{M} \in \mathbb{N}$

$$\boldsymbol{x}_j := \frac{j}{\tilde{M}} \boldsymbol{z} \bmod \mathbf{1};\ j = 0, \ldots, \tilde{M} - 1$$

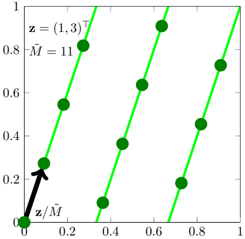
- reconstruction of Fourier coefficients $\hat{\tilde{p}}_{\tilde{\boldsymbol{k}}}$ of multivariate trigonometric polynomial $\tilde{p}(\boldsymbol{x}) = \sum_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}} \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}} \mathrm{e}^{2\pi \mathrm{i} \tilde{\boldsymbol{k}} \cdot \boldsymbol{x}}$



# Fast and exact reconstruction - periodic case

- rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, \tilde{M})$: $\boldsymbol{z} \in \mathbb{N}^d, \tilde{M} \in \mathbb{N}$

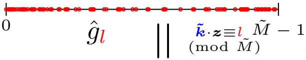
$$\boldsymbol{x}_j := \frac{j}{\tilde{M}} \boldsymbol{z} \bmod \mathbf{1};\ j = 0, \ldots, \tilde{M} - 1$$

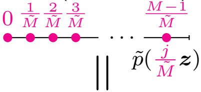


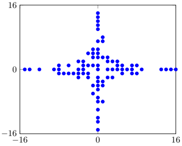

- reconstruction of Fourier coefficients $\hat{\tilde{p}}_{\tilde{\boldsymbol{k}}}$ of multivariate trigonometric polynomial $\tilde{p}(\boldsymbol{x}) = \sum_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}} \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}} \mathrm{e}^{2\pi \mathrm{i} \tilde{\boldsymbol{k}} \cdot \boldsymbol{x}}$

$\Rightarrow$ Definition reconstructing $\mathrm{R1L}(\boldsymbol{z}, \tilde{M})$ for $\tilde{\mathrm{I}}$:
$\tilde{\boldsymbol{k}} \cdot \boldsymbol{z} \not\equiv \tilde{\boldsymbol{k}}' \cdot \boldsymbol{z} \pmod{\tilde{M}}$ for all $\tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}}' \in \tilde{\mathrm{I}}$, $\tilde{\boldsymbol{k}} \neq \tilde{\boldsymbol{k}}'$

- $|\tilde{\mathrm{I}}| \leq \tilde{M} \leq |\tilde{\mathrm{I}}|^2$, CBC construction algorithm (Kämmerer 2012)



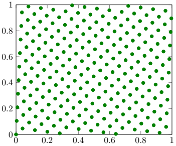



$(\hat{\tilde{p}}_{\tilde{\boldsymbol{k}}})_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}}$

$\xleftarrow[\text{iFFT}]{\text{1-dim}}$

$\mathcal{O}(M \log M + d|\tilde{\mathrm{I}}|)$



$(\tilde{p}(\boldsymbol{x}_j))_{j=0}^{M-1}$

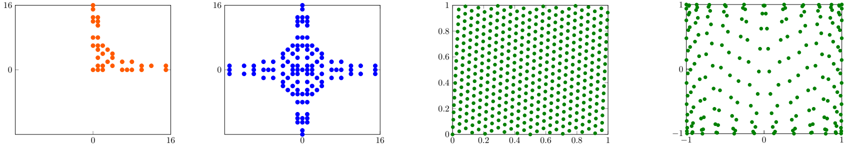
# Fast and exact reconstruction - non-periodic case

## Theorem (Potts, V. 15)

*Let $I \subset \mathbb{N}_0^d$ be an arbitrary index set with $|I| < \infty$. Moreover, let* $\mathrm{R1L}(\boldsymbol{z}, \tilde{M}) := \{\boldsymbol{x}_j := \frac{j}{\tilde{M}}\boldsymbol{z} \bmod \boldsymbol{1} \colon j = 0, \ldots, \tilde{M}-1\}$ *be a reconstructing rank-1 lattice with even rank-1 lattice size $\tilde{M} \in 2\mathbb{N}$ for the extended symmetric index set $\mathcal{M}(I)$, i.e.,*

$$\tilde{\boldsymbol{k}} \cdot \boldsymbol{z} \not\equiv \tilde{\boldsymbol{k}}' \cdot \boldsymbol{z} \pmod{\tilde{M}} \text{ for all } \tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}}' \in \mathcal{M}(I),\ \tilde{\boldsymbol{k}} \neq \tilde{\boldsymbol{k}}'.$$

*Then, the rank-1 Chebyshev lattice* $\mathrm{CL}(\boldsymbol{z}, M = \frac{\tilde{M}}{2}) := \{\boldsymbol{x}_j := \cos(\frac{2j}{\tilde{M}}\pi\boldsymbol{z}) \colon j = 0, \ldots, \frac{\tilde{M}}{2}\}$ *allows the reconstruction of the coefficients $\hat{p}_{\boldsymbol{k}}$ of $p(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$.*




$\boldsymbol{k} \in I$ | $\tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}}' \in \mathcal{M}(I)$ | reco. $\mathrm{R1L}(\boldsymbol{z}, \tilde{M})$ | reco. $\mathrm{CL}(\boldsymbol{z}, \frac{\tilde{M}}{2})$

# Fast and exact reconstruction - non-periodic case

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- under mild assumptions: $\tilde{M} \leq 2^{2d+1}|I|^2$, $M \leq 2^{2d}|I|^2$

  [Potts, V. 14] as a consequence of [Kämmerer 13]

# Fast and exact reconstruction - non-periodic case

- compute with DCT-I coefficients

$$\hat{a}_l := \sum_{j=0}^{M} (\varepsilon_j^M)^2 \, p(\boldsymbol{x}_j) \, \cos\left(\frac{jl}{M}\pi\right), \; l \in I_M^1, \; \varepsilon_l^M = \begin{cases} 1/\sqrt{2}, & l = 0, M \\ 1, & \text{else} \end{cases}$$

- If rank-1 Chebyshev lattice allows reconstruction, then for $\boldsymbol{k} \in I$

$$\hat{p}_{\boldsymbol{k}} = \frac{2(\varepsilon^M_{\boldsymbol{k}\cdot\boldsymbol{z} \,\mathrm{emod}\, M})^2}{M} \, \hat{a}_{\boldsymbol{k}\cdot\boldsymbol{z} \,\mathrm{emod}\, M} \cdot \frac{2^d}{|\{\boldsymbol{s} \in \{-1,1\}^d \colon (\boldsymbol{s} \odot \boldsymbol{k}) \cdot \boldsymbol{z} \,\mathrm{emod}\, M = \boldsymbol{k} \cdot \boldsymbol{z} \,\mathrm{emod}\, M\}|}$$

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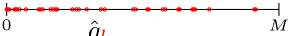
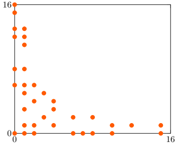



$\hat{g}_l$

$||$ $\boldsymbol{k}\cdot\boldsymbol{z} \operatorname{emod} M = l$

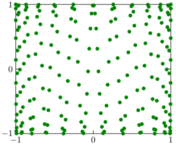
$p\left(\cos\left(\frac{j}{M}\pi\boldsymbol{z}\right)\right)$

$||$




$\xleftarrow[\text{DCT-I}]{\text{1-dim}}$

$\mathcal{O}(M \log M + d\,2^d |I|)$

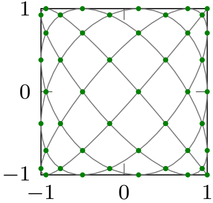



$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I}$

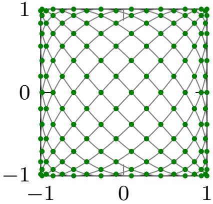
$(p(\boldsymbol{x}_j))_{j=0}^M$

# Fast and exact reconstruction - Padua points

- Padua point set (special 2-dim. rank-1 Chebyshev lattice), e.g. [Bos, Caliari, Marchi, Vianello, Xu 2006]
  $\mathcal{A}_n := \{\boldsymbol{x}_j := (\cos(j\pi/(n+1)), \cos(j\pi/n))^\top : j = 0, \ldots, M\}$
- $\mathcal{A}_n = \mathrm{CL}(\boldsymbol{z}, M)$, where $\boldsymbol{z} := (n, n+1)^\top$ and $M := n\,(n+1)$



$\mathcal{A}_8$



$\mathcal{A}_{16}$

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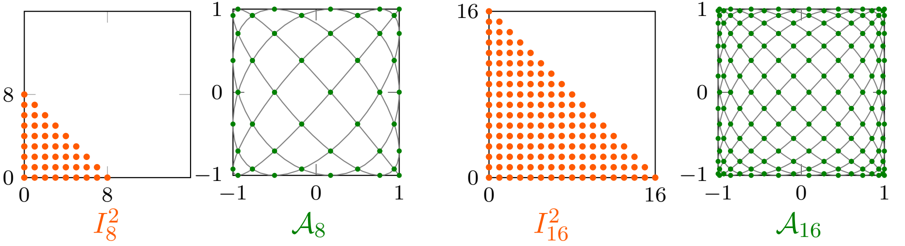
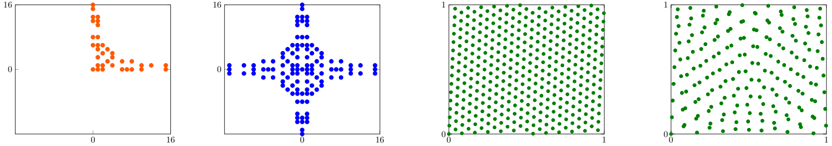
- $\mathcal{A}_n = \mathrm{CL}(\boldsymbol{z}, M)$, where $\boldsymbol{z} := (n, n+1)^\top$ and $M := n\,(n+1)$
- allows exact reconstruction of all coefficients $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I_n^2$, of an arbitrary 2d algebraic polynomial of total degree $\leq n$ (restricted to the domain $[-1,1]^2$)
- $|I_n^2| = |\mathcal{A}_n| = \binom{n+2}{2} = \frac{n^2}{2} + \frac{3}{2}n + 1$, whereas $M = n^2 + n$.



# Relation to tent-transformed rank-1 lattices

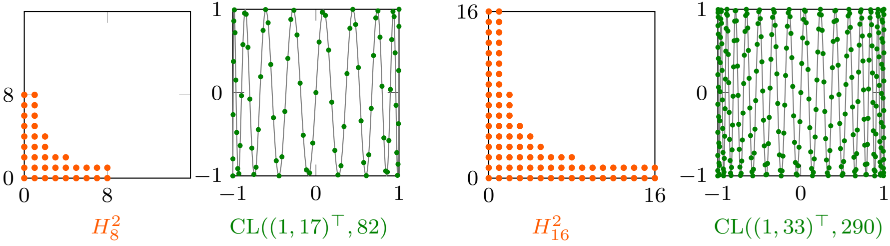
- tent-transformed rank-1 lattices
  $P_\psi(\boldsymbol{z},\tilde{M}) := \{\psi(j\boldsymbol{z}/\tilde{M} \bmod \mathbf{1})\colon j=0,\ldots,\tilde{M}-1\}$
  [Dick, Nuyens, Pillichshammer 14], [Suryanarayana, Nuyens, Cools 15]
- tent transform $\psi\colon [0,1]\to[0,1]$, $\psi(x) := 1-|2x-1|$
- functions $\tilde{g}\colon [0,1]^d \to \mathbb{R}$, $\tilde{g}(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \tilde{g}_{\boldsymbol{k}}\phi_{\boldsymbol{k}}(\boldsymbol{x})$, $I\subset\mathbb{N}_0^d$,
- half-periodic cosine functions $\phi_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{t=1}^{d} \sqrt{2}^{\delta_{0,k_t}} \cos(\pi k_t x_t)$,
  $\boldsymbol{k}\in\mathbb{N}_0^d$, ONB of $L_2([0,1]^d)$
- reconstructing rank-1 lattice $\mathrm{R1L}(\boldsymbol{z},\tilde{M})$ for $\mathcal{M}(I)$
- reconstruction from samples $\tilde{g}\left(P_\psi(\boldsymbol{z},\tilde{M})\right)$ with 1-dim FFT
- polynomials $\tilde{g}$ are not algebraic polynomials in general



$\boldsymbol{k}\in I$ | $\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}'\in\mathcal{M}(I)$ | reco. $\mathrm{R1L}(\boldsymbol{z},\tilde{M})$ | $P_\psi(\boldsymbol{z},\tilde{M})$

# Examples of reconstr. rank-1 Chebyshev lattices

- hyperbolic cross index sets
  $I = H_n^d := \left\{ \boldsymbol{k} \in \mathbb{N}_0^d \colon \prod_{t=1}^d \max(1, k_t) \leq n \right\}, \, n, d \in \mathbb{N}$
- $|H_n^d| = \mathcal{O}(2^d \, n \log^{d-1} n)$
- search for rank-1 Chebyshev lattices $\mathrm{CL}(\boldsymbol{z}, M)$
  that allow reconstruction of coefficients $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$,
  of polynomial $p(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}} \, T_{\boldsymbol{k}}(\boldsymbol{x})$

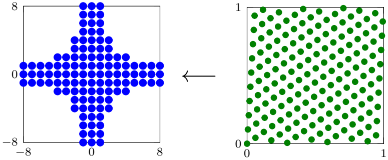


# Examples of reconstr. rank-1 Chebyshev lattices

| param. | | card. | reco. R1L | | direct search | |
|---|---|---|---|---|---|---|
| $d$ | $n$ | $\lvert H_n^d\rvert$ | $M = \frac{\tilde{M}}{2}$ | $\frac{\frac{\tilde{M}}{2}+1}{\lvert H_n^d\rvert}$ | $M$ | $\frac{M+1}{\lvert H_n^d\rvert}$ |
| 2 | 256 | 1 979 | 66 050 | 33.38 | 66 050 | 33.38 |
| 2 | 512 | 4 305 | 263 170 | 61.13 | 263 170 | 61.13 |
| 2 | 1024 | 9 311 | 1 050 626 | 112.84 | 1 050 626 | 112.84 |
| 3 | 256 | 10 303 | 359 075 | 34.85 | 302 883 | 29.40 |
| 3 | 512 | 23 976 | 1 424 662 | 59.42 | 1 424 613 | 59.42 |
| 3 | 1024 | 55 202 | 5 560 838 | 100.74 | 4 600 672 | 83.34 |
| 6 | 16 | 8 684 | 557 773 | 64.23 | 303 396 | 34.94 |
| 6 | 32 | 26 088 | 2 867 903 | 109.93 | 1 751 513 | 67.14 |
| 6 | 64 | 76 433 | 13 603 339 | 177.98 | 8 979 932 | 117.49 |
| 10 | 2 | 6 144 | 2 157 672 | 351.18 | 495 451 | 80.64 |
| 10 | 4 | 27 904 | 15 390 479 | 551.55 | 3 083 988 | 110.52 |
| 10 | 8 | 109 824 | 88 580 127 | 806.56 | 25 099 619 | 228.54 |

# Multiple rank-1 lattices

- reconstructing R1L has size $\tilde{M}$ with $|\tilde{I}| \leq \tilde{M} \leq |\tilde{I}|^2$



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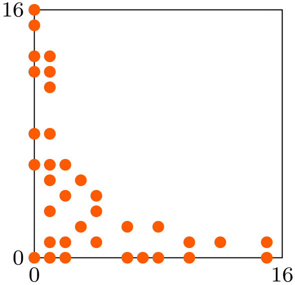
- use more than one rank-1 lattice [Kämmerer 15]

# Multiple rank-1 lattices

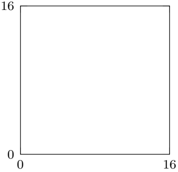
- reconstructing R1L has size $\tilde{M}$ with $|\tilde{I}| \leq \tilde{M} \leq |\tilde{I}|^2$
- use more than one rank-1 lattice [Kämmerer 15]
- perform greedy search for sequence of rank-1 lattices which reconstruct as much as possible frequencies from $\tilde{I}$
- each used lattice size $\tilde{M} \approx$ number of remaining frequencies
- oversampling factor bounded by $\mathrm{e}$ for hyperbolic cross in performed tests (up to $d = 10$) instead of $\mathcal{O}(N/\log N)$

# Multiple rank-1 Chebyshev lattices

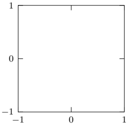
remaining frequencies to reconstruct:




reconstructable freq.:




used rank-1 Chebyshev lattices:



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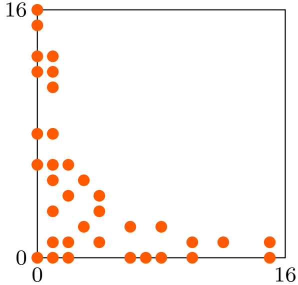
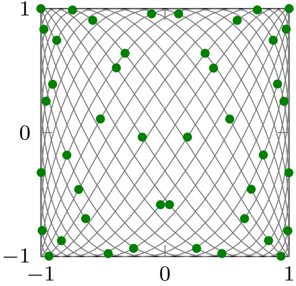
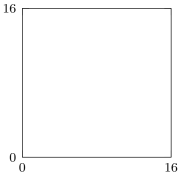


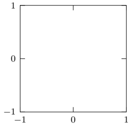

rank-1 Chebyshev lattice:




reconstructable freq.:




used rank-1 Chebyshev lattices:



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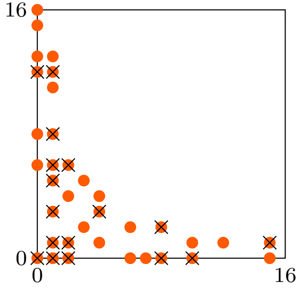
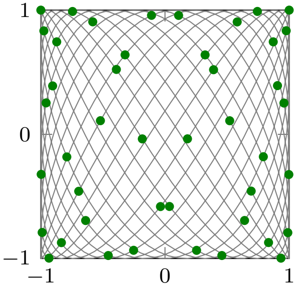
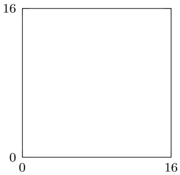


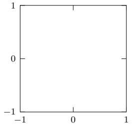

rank-1 Chebyshev lattice:




reconstructable freq.:




used rank-1 Chebyshev lattices:



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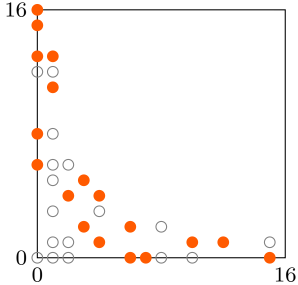
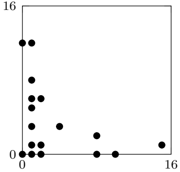
remaining frequencies to reconstruct:




reconstructable freq.:

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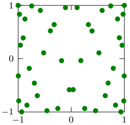
used rank-1 Chebyshev lattices:



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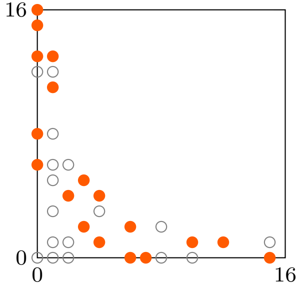


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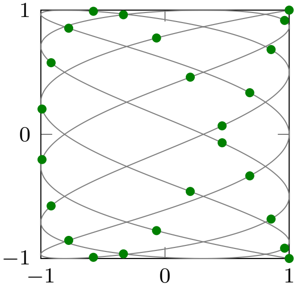
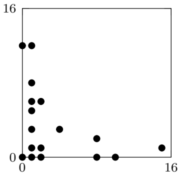



reconstructable freq.:

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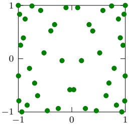
used rank-1 Chebyshev lattices:



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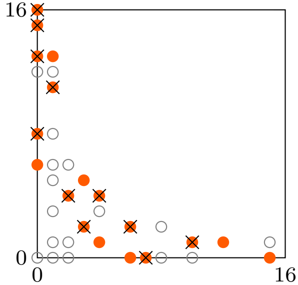
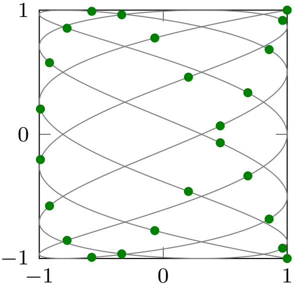
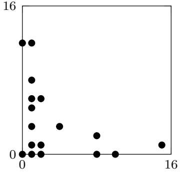
remaining frequencies to reconstruct:




rank-1 Chebyshev lattice:




reconstructable freq.:

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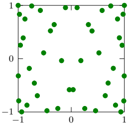

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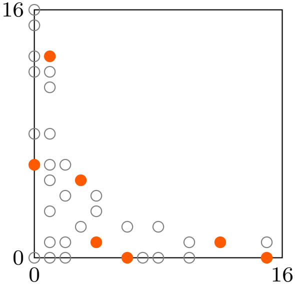
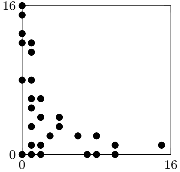
remaining frequencies to reconstruct:

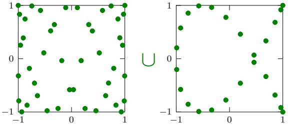



reconstructable freq.:



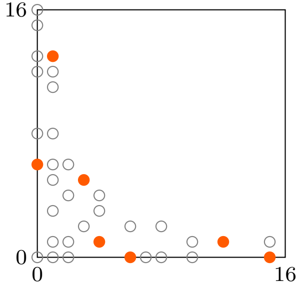

$\longleftarrow$

used rank-1 Chebyshev lattices:

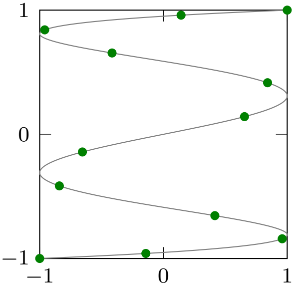


# Multiple rank-1 Chebyshev lattices

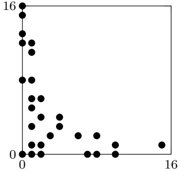
remaining frequencies to reconstruct:




rank-1 Chebyshev lattice:




reconstructable freq.:

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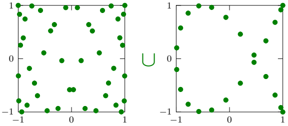
used rank-1 Chebyshev lattices:



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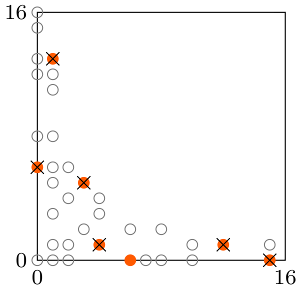
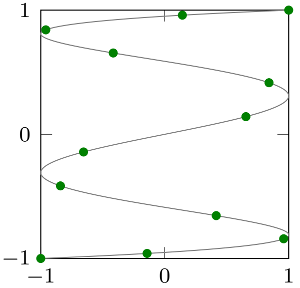
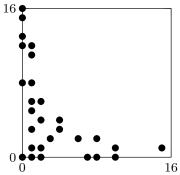
remaining frequencies to reconstruct:




rank-1 Chebyshev lattice:




reconstructable freq.:

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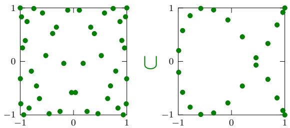
used rank-1 Chebyshev lattices:



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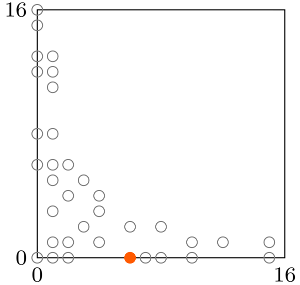
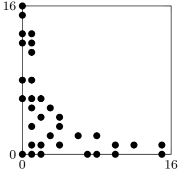
remaining frequencies to reconstruct:

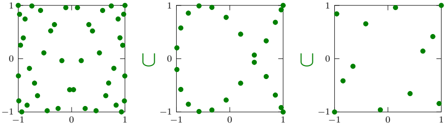



reconstructable freq.:




$\longleftarrow$

used rank-1 Chebyshev lattices:



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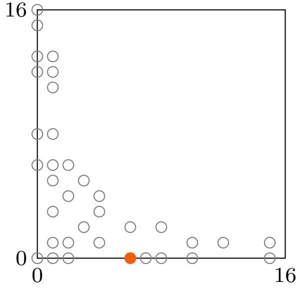
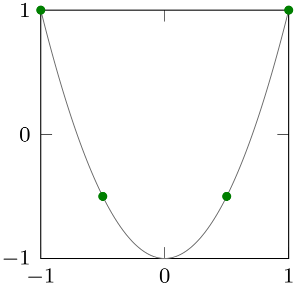
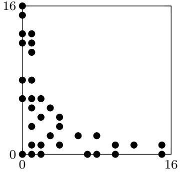
remaining frequencies to reconstruct:




rank-1 Chebyshev lattice:

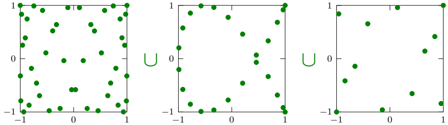



reconstructable freq.:



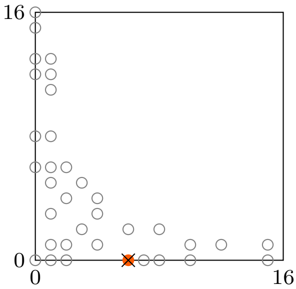

$\longleftarrow$

used rank-1 Chebyshev lattices:

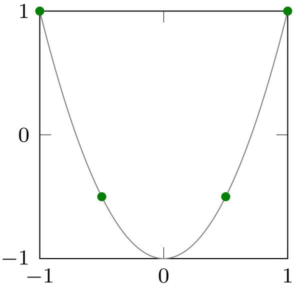


# Multiple rank-1 Chebyshev lattices

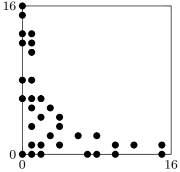
remaining frequencies to reconstruct:




rank-1 Chebyshev lattice:




reconstructable freq.:

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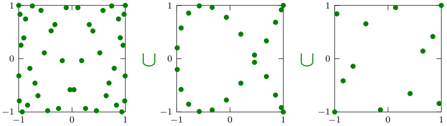
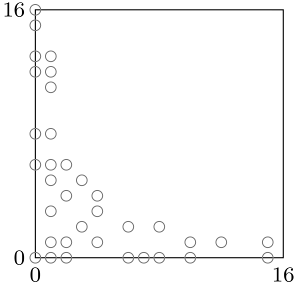
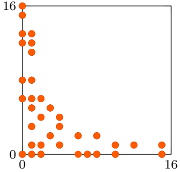
used rank-1 Chebyshev lattices:



# Multiple rank-1 Chebyshev lattices

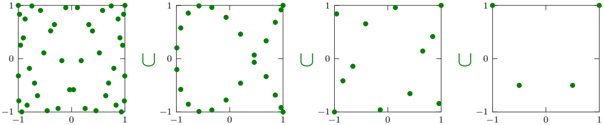



reconstructable freq.:




⟵

used rank-1 Chebyshev lattices:



# Examples of multiple rank-1 Chebyshev lattices

| param. | | card. | direct search 1 CL | | multiple CL | |
|---|---|---|---|---|---|---|
| $d$ | $n$ | $\lvert H_n^d\rvert$ | #samples | $\frac{\#\text{samples}}{\lvert H_n^d\rvert}$ | #samples | $\frac{\#\text{samples}}{\lvert H_n^d\rvert}$ |
| 2 | 256 | 1 979 | 66 051 | 33.38 | 6 104 | 3.08 |
| 2 | 512 | 4 305 | 263 171 | 61.13 | 13 985 | 3.25 |
| 2 | 1024 | 9 311 | 1 050 627 | 112.84 | 31 768 | 3.41 |
| 3 | 256 | 10 303 | 302 884 | 29.40 | 57 094 | 5.54 |
| 3 | 512 | 23 976 | 1 424 614 | 59.42 | 140 618 | 5.86 |
| 3 | 1024 | 55 202 | 4 600 673 | 83.34 | 348 684 | 6.32 |
| 6 | 32 | 26 088 | 1 751 514 | 67.14 | 463 186 | 17.75 |
| 6 | 64 | 76 433 | 8 979 933 | 117.49 | 1 631 406 | 21.34 |
| 6 | 128 | 217 113 | 51 662 222 | 237.95 | 5 406 934 | 25.90 |
| 10 | 2 | 6 144 | 495 452 | 80.64 | 285 946 | 46.54 |
| 10 | 4 | 27 904 | 3 083 989 | 110.52 | 1 732 161 | 62.08 |
| 10 | 8 | 109 824 | 25 099 620 | 228.54 | 8 529 343 | 77.66 |

# Approximation - some references

periodic case:

- weighted Korobov spaces, rank-1 lattices
  [Kuo, Sloan, Woźniakowski 06]
- Sobolev spaces of dominating mixed smoothness
  - sparse grids [Baszenski, Delvos 89] [Hallatschek 92] [Gradinaru 07]
  - rank-1 lattices [Temlyakov 86,...]
- Sobolev spaces of isotropic and dominating mixed smoothness
  - generalized sparse grids [Griebel, Hamaekers 14]
    [Byrenheid, Dũng, Sickel, Ullrich 14]
  - rank-1 lattices [Potts, Kämmerer, V. 15]

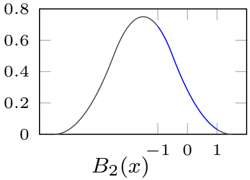
non-periodic case:

- weighted Korobov / Sobolev spaces,
  Chebyshev sparse grids / transformed sparse grids
  [Barthelmann, Novak, Ritter 00] [Shen, Yu 10]
- spaces of half-periodic cosine functions,
  tent-transformed rank-1 lattices
  [Dick, Nuyens, Pillichshammer 14] [Suryanarayana, Nuyens, Cools 15]

# Approximation - non-periodic case example

- dilated B-Spline $B_2(x)$ of order 2, only $x \in [-1,1]$ considered
- weighted $L_{2,\mathrm{W}}([-1,1])$ scalar product $\langle f,g\rangle := \int_{-1}^{1} \frac{f(x)\,g(x)}{\sqrt{1-x^2}}\mathrm{d}x$ and induced norm $\|f\|_{L_{2,\mathrm{W}}([-1,1]^d)} := \sqrt{\langle f,f\rangle}$
- set of $T_k(x) := \cos(k \arccos x)$, $k \in \mathbb{N}_0$, form orthogonal basis

$\Rightarrow$ tensorization: $B_2^d(\boldsymbol{x}) := \prod_{t=1}^{d} B_2(x_t)$, $L_{2,\mathrm{W}}([-1,1]^d)$



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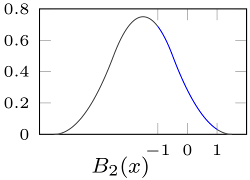


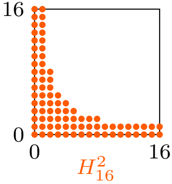

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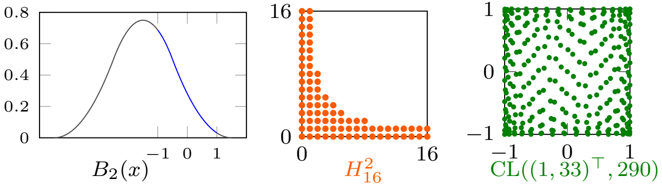
- Determine relative error $\|B_2^d - p\|_{L_{2,\mathrm{W}}([-1,1]^d)} / \|B_2^d\|_{L_{2,\mathrm{W}}([-1,1]^d)}$



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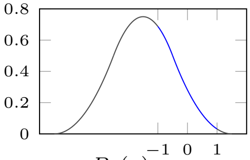
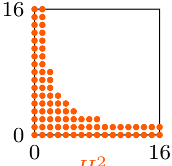
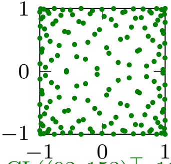


$B_2(x)$

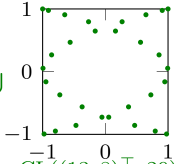


$H_{16}^2$



$\mathrm{CL}((92,153)^\top, 167)$

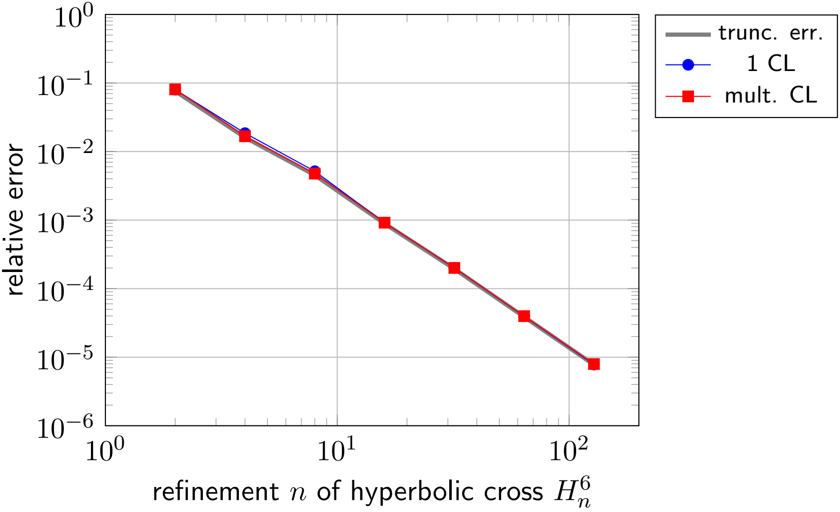
$\cup$



$\mathrm{CL}((13,8)^\top, 29)$

# Approximation - non-periodic case example

$B_2^6$ sampling at (multiple) rank-1 Chebyshev lattices ($d = 6$)



# Approximation - non-periodic case example

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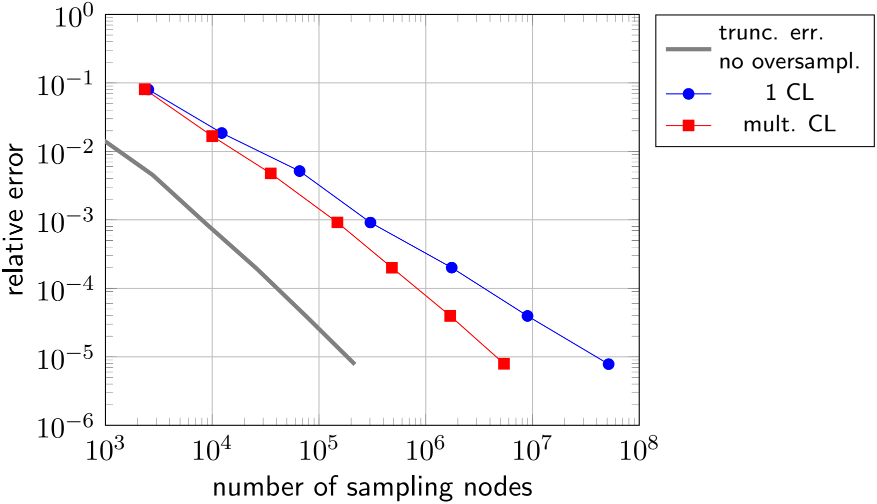

# Summary

| | periodic | non-periodic |
|---|---|---|
| fast evaluation (arbitrary freq.) | rank-1 lattice [Li, Hickernell 03] | rank-1 Chebyshev lattice [Cools, Poppe 11] [Potts, V. 15] |
| fast reconstruction (arbitrary freq.) | rank-1 lattice [Kämmerer, Kunis, Potts 12] [Kämmerer 13] | rank-1 Chebyshev lattice [Potts, V. 15] |
| approximation | sparse grid [Baszenski, Delvos 89] [Hallatschek 92] [Gradinaru 07] [Griebel, Hamaekers 14] [Byrenheid, Dũng, Sickel, Ullrich 14] | Chebyshev sparse grid [Barthelmann, Novak, Ritter 00] [Shen, Yu 10] |
| | rank-1 lattice [Kuo, Sloan, Woźniakowski 06] [Kämmerer, Potts, V. 15] | tent-transf. rank-1 lattice [Dick, Nuyens, Pillichshammer 14] [Suryanarayana, Nuyens, Cools 15] |
| | | rank-1 Chebyshev lattice (example in this talk, work in progress) |

# Next steps

reconstruction of $p\colon [-1,1]^d \to \mathbb{R}$, $p(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$

if $I$ unknown

- adapt methods from periodic case


Potts, D., Volkmer, T.
**Sparse high-dimensional FFT based on rank-1 lattice sampling**.
Appl. Comput. Harm. Anal., accepted, 2015.
(http://www.tu-chemnitz.de/∼tovo)