

Fast and exact reconstruction of arbitrary multivariate algebraic polynomials in Chebyshev form

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joint work with L. Kämmerer and D. Potts

supported by

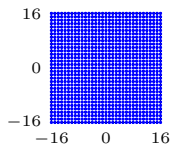


Introduction

Fast evaluation

periodic case $\tilde{p}: \mathbb{T}^d \rightarrow \mathbb{C}, \tilde{p}(\mathbf{x}) := \sum_{\tilde{\mathbf{k}} \in \tilde{I}} \hat{p}_{\tilde{\mathbf{k}}} e^{2\pi i \tilde{\mathbf{k}} \cdot \mathbf{x}}$

non-periodic case $p: [-1, 1]^d \rightarrow \mathbb{R}, p(\mathbf{x}) = \sum_{\mathbf{k} \in I} \hat{p}_{\mathbf{k}} T_{\mathbf{k}}(\mathbf{x})$



Fast and exact reconstruction

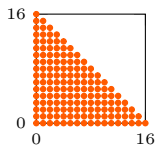
periodic case

non-periodic case

Approximation

non-periodic case

Summary



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Fast and exact reconstruction

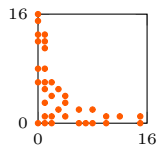
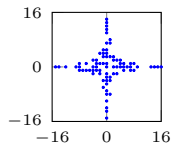
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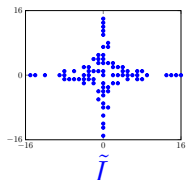


Introduction - periodic case

- d -dimensional torus $\mathbb{T}^d \simeq [0, 1)^d$
- multivariate trigonometric polynomial $\tilde{p}: \mathbb{T}^d \rightarrow \mathbb{C}$ with frequencies supported on $\tilde{\mathbf{I}} \subset \mathbb{Z}^d$, $|\tilde{\mathbf{I}}| < \infty$,

$$\tilde{p}(\mathbf{x}) := \sum_{\tilde{\mathbf{k}} \in \tilde{\mathbf{I}}} \hat{p}_{\tilde{\mathbf{k}}} e^{2\pi i \tilde{\mathbf{k}} \cdot \mathbf{x}}, \quad \hat{p}_{\tilde{\mathbf{k}}} \in \mathbb{C}$$

- fast evaluation $\tilde{p}(\mathbf{x}_j)$, $j = 0, \dots, M - 1$
(e.g. [Li,Hickernell 03])
- fast and exact reconstruction of $\hat{p}_{\tilde{\mathbf{k}}}$, $\tilde{\mathbf{k}} \in \tilde{\mathbf{I}}$,
from samples $\tilde{p}(\mathbf{x}_j)$, $j = 0, \dots, M - 1$
([Kämmerer,Kunis,Potts 12] [Kämmerer 13])

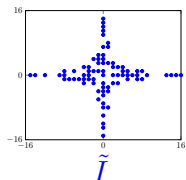


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Aim: transfer results to non-periodic case

Introduction - non-periodic case

- multivariate algebraic polynomial $p : [-1, 1]^d \rightarrow \mathbb{R}$ in Chebyshev form with frequencies supp. on $I \subset \mathbb{N}_0^d$, $|I| < \infty$,

$$p(\mathbf{x}) = \sum_{\mathbf{k} \in I} \hat{p}_{\mathbf{k}} T_{\mathbf{k}}(\mathbf{x}) = \sum_{\mathbf{k} \in I} \hat{p}_{\mathbf{k}} \prod_{t=1}^d T_{k_t}(x_t), \quad \hat{p}_{\mathbf{k}} \in \mathbb{R}, \quad (1)$$

$T_m : [-1, 1] \rightarrow [-1, 1]$, $T_m(x) := \cos(m \arccos x)$, $m \in \mathbb{N}_0$

(T_m is algebraic polynomial of degree m restricted to domain $[-1, 1]$)

$$T_0(x) = 1$$

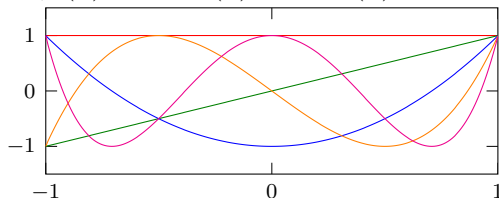
$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x) \quad m \in \mathbb{N}$$



Introduction - non-periodic case

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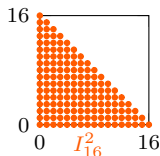
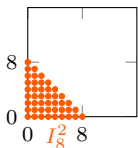
$$p(\mathbf{x}) = \sum_{\mathbf{k} \in I} \hat{p}_{\mathbf{k}} T_{\mathbf{k}}(\mathbf{x}) = \sum_{\mathbf{k} \in I} \hat{p}_{\mathbf{k}} \prod_{t=1}^d T_{k_t}(x_t), \quad \hat{p}_{\mathbf{k}} \in \mathbb{R}, \quad (1)$$

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- e.g. ℓ_1 -ball index set $I = I_n^d := \{\mathbf{k} \in \mathbb{N}_0^d : \|\mathbf{k}\|_1 \leq n\}$, $n \in \mathbb{N}_0$
- \Rightarrow each algebraic polynomial

of total degree $\leq n$

(restricted to the domain $[-1, 1]^d$)
can be written as (1) with $I = I_n^d$



Introduction - non-periodic case

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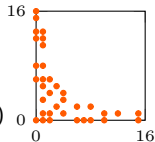
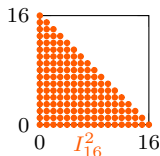
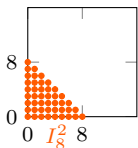
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- Aim:

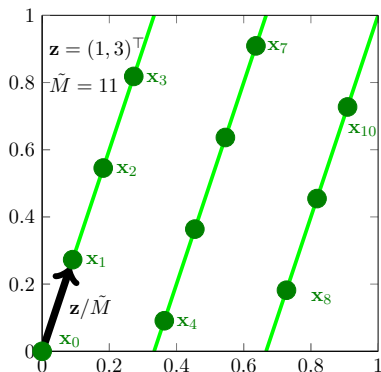
- fast evaluation of high-dimensional arbitrary algebraic polynomials (given in Chebyshev form)
- fast and exact reconstruction from samples
(special case of ℓ_1 -ball index sets $I = I_n^d$ e.g. [Poppe,Cools 12])



Fast evaluation - periodic case

- rank-1 lattice $\text{R1L}(\mathbf{z}, \tilde{M})$: $\mathbf{z} \in \mathbb{N}^d, \tilde{M} \in \mathbb{N}$

$$\mathbf{x}_j = \frac{j}{\tilde{M}} \mathbf{z} \bmod \mathbf{1}; j = 0, \dots, \tilde{M} - 1$$



Korobov 59
Maisonneuve 72
Sloan & Kachoyan 84,87,90
Temlyakov 86
Lyness 89
Sloan & Joe 94
Sloan & Reztsov 01
Li & Hickernell 03

Fast evaluation - periodic case

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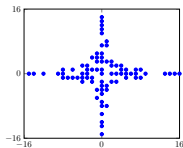
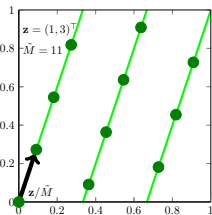
$$\mathbf{x}_j = \frac{j}{\tilde{M}} \mathbf{z} \bmod \mathbf{1}; j = 0, \dots, \tilde{M} - 1$$

- multivariate trigonometric polynomial

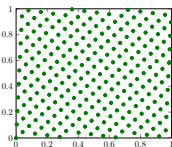
$$\tilde{p}(\mathbf{x}) = \sum_{\tilde{\mathbf{k}} \in \tilde{\mathcal{I}}} \hat{p}_{\tilde{\mathbf{k}}} e^{2\pi i \tilde{\mathbf{k}} \cdot \mathbf{x}}$$

- reformulation

$$\tilde{p}(\mathbf{x}_j) = \sum_{\tilde{\mathbf{k}} \in \tilde{\mathcal{I}}} \hat{p}_{\tilde{\mathbf{k}}} e^{2\pi i \frac{j \tilde{\mathbf{k}} \cdot \mathbf{z}}{\tilde{M}}} = \sum_{l=0}^{\tilde{M}-1} \underbrace{\left(\sum_{\substack{\tilde{\mathbf{k}} \in \tilde{\mathcal{I}} \\ \tilde{\mathbf{k}} \cdot \mathbf{z} \equiv l \pmod{\tilde{M}}} \hat{p}_{\tilde{\mathbf{k}}} \right)}_{\hat{g}_l} e^{2\pi i \frac{j \tilde{\mathbf{k}} \cdot \mathbf{z}}{\tilde{M}}} = \sum_{l=0}^{\tilde{M}-1} \hat{g}_l e^{2\pi i \frac{j l}{\tilde{M}}}$$



$$(\hat{p}_{\tilde{\mathbf{k}}})_{\tilde{\mathbf{k}} \in \tilde{\mathcal{I}}}$$



$$(\tilde{p}(\mathbf{x}_j))_{j=0}^{\tilde{M}-1}$$

Fast evaluation - periodic case

- rank-1 lattice R1L(\mathbf{z}, \tilde{M}): $\mathbf{z} \in \mathbb{N}^d, \tilde{M} \in \mathbb{N}$

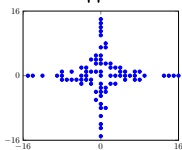
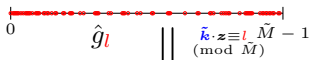
$$\mathbf{x}_j = \frac{j}{\tilde{M}} \mathbf{z} \bmod \mathbf{1}; j = 0, \dots, \tilde{M} - 1$$

- multivariate trigonometric polynomial

$$\tilde{p}(\mathbf{x}) = \sum_{\tilde{\mathbf{k}} \in \tilde{\mathcal{I}}} \hat{p}_{\tilde{\mathbf{k}}} e^{2\pi i \tilde{\mathbf{k}} \cdot \mathbf{x}}$$

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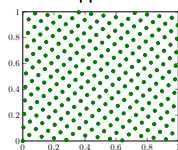
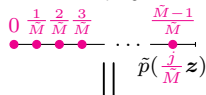
$$\tilde{p}(\mathbf{x}_j) = \sum_{\tilde{\mathbf{k}} \in \tilde{\mathcal{I}}} \hat{p}_{\tilde{\mathbf{k}}} e^{2\pi i \frac{j \tilde{\mathbf{k}} \cdot \mathbf{z}}{\tilde{M}}} = \sum_{l=0}^{\tilde{M}-1} \underbrace{\left(\sum_{\substack{\tilde{\mathbf{k}} \in \tilde{\mathcal{I}} \\ \tilde{\mathbf{k}} \cdot \mathbf{z} \equiv l \pmod{\tilde{M}}} \hat{p}_{\tilde{\mathbf{k}}} \right)}_{\hat{g}_l} e^{2\pi i \frac{j \tilde{\mathbf{k}} \cdot \mathbf{z}}{\tilde{M}}} = \sum_{l=0}^{\tilde{M}-1} \hat{g}_l e^{2\pi i \frac{j l}{\tilde{M}}}$$



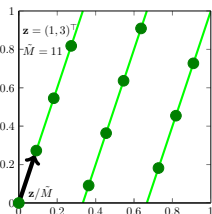
$$(\hat{p}_{\tilde{\mathbf{k}}})_{\tilde{\mathbf{k}} \in \tilde{\mathcal{I}}}$$

1-dim
→
FFT

$$\mathcal{O}(\tilde{M} \log \tilde{M} + d|\tilde{\mathcal{I}}|)$$



$$(\tilde{p}(\mathbf{x}_j))_{j=0}^{\tilde{M}-1}$$



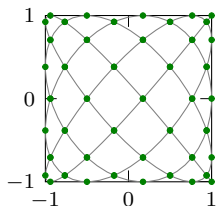
Fast evaluation - non-periodic case

- d -dimensional rank-1 Chebyshev lattice [Cools, Poppe 2011]

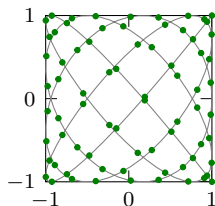
$$\text{CL}(\mathbf{z}, M) := \{\mathbf{x}_j : j = 0, \dots, M\} \subset [-1, 1]^d$$

with nodes $\mathbf{x}_j := \cos\left(\frac{j}{M}\pi\mathbf{z}\right)$, $j = 0, \dots, M$

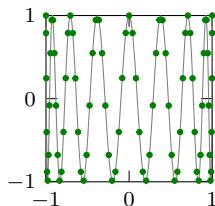
- generating vector $\mathbf{z} \in \mathbb{N}_0^d$
- size parameter $M \in \mathbb{N}_0$



$$\mathbf{z} := (8, 9)^\top, M := 72, \\ |\text{CL}(\mathbf{z}, M)| = 45$$



$$\mathbf{z} := (8, 9)^\top, M := 73, \\ |\text{CL}(\mathbf{z}, M)| = 74$$



$$\mathbf{z} := (1, 16)^\top, M := 76, \\ |\text{CL}(\mathbf{z}, M)| = 77$$

Fast evaluation - non-periodic case

- with Chebyshev nodes $\mathbf{x}_j := \cos\left(\frac{j}{M}\pi\mathbf{z}\right)$, $j = 0, \dots, M$,

$$\begin{aligned} p(\mathbf{x}_j) &= \sum_{\mathbf{k} \in I} \hat{p}_{\mathbf{k}} T_{\mathbf{k}}(\mathbf{x}_j) \\ &= \sum_{l=0}^M \underbrace{\left(\sum_{\substack{\mathbf{k} \in I, \mathbf{s} \in \{-1,1\}^d \\ (\mathbf{s} \odot \mathbf{k}) \cdot \mathbf{z} \text{ emod } M=l}} \frac{\hat{p}_{\mathbf{k}}}{2^d} \right)}_{\hat{g}_l} \cos\left(\frac{jl}{M}\pi\right) = \sum_{l=0}^M \hat{g}_l \cos\left(\frac{jl}{M}\pi\right) \end{aligned}$$

with even-mod relation

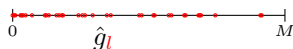
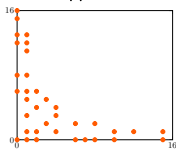
$$l \text{ emod } M := \begin{cases} l \bmod (2M) & \text{for } l \bmod (2M) \leq M, \\ 2M - (l \bmod (2M)) & \text{else} \end{cases}$$

Fast evaluation - non-periodic case

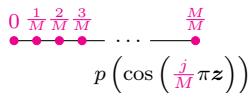
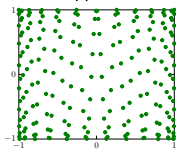
- with Chebyshev nodes $\mathbf{x}_j := \cos\left(\frac{j}{M}\pi\mathbf{z}\right)$, $j = 0, \dots, M$,

$$p(\mathbf{x}_j) = \sum_{\mathbf{k} \in I} \hat{p}_{\mathbf{k}} T_{\mathbf{k}}(\mathbf{x}_j)$$

$$= \sum_{l=0}^M \underbrace{\left(\sum_{\substack{\mathbf{k} \in I, \mathbf{s} \in \{-1,1\}^d \\ (\mathbf{s} \odot \mathbf{k}) \cdot \mathbf{z} \bmod M = l}} \frac{\hat{p}_{\mathbf{k}}}{2^d} \right)}_{\hat{g}_l} \cos\left(\frac{j l}{M} \pi\right) = \sum_{l=0}^M \hat{g}_l \cos\left(\frac{j l}{M} \pi\right)$$


 \hat{g}_l
 \parallel
 $\mathbf{k} \cdot \mathbf{z} \bmod M = l$

 $(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in I}$

1-dim
 \rightarrow
 DCT-I

 $\mathcal{O}(M \log M + d 2^d |I|)$

 $p\left(\cos\left(\frac{j}{M}\pi\mathbf{z}\right)\right)$
 \parallel

 $(p(\mathbf{x}_j))_{j=0}^M$

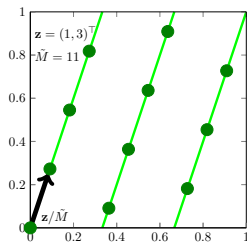
Fast and exact reconstruction - periodic case

- rank-1 lattice R1L(\mathbf{z}, \tilde{M}): $\mathbf{z} \in \mathbb{N}^d, \tilde{M} \in \mathbb{N}$

$$\mathbf{x}_j := \frac{j}{\tilde{M}} \mathbf{z} \bmod \mathbf{1}; j = 0, \dots, \tilde{M} - 1$$

- reconstruction of Fourier coefficients $\hat{p}_{\tilde{\mathbf{k}}}$ of multivariate trigonometric polynomial

$$\tilde{p}(\mathbf{x}) = \sum_{\tilde{\mathbf{k}} \in \tilde{\mathbf{I}}} \hat{p}_{\tilde{\mathbf{k}}} e^{2\pi i \tilde{\mathbf{k}} \cdot \mathbf{x}}$$



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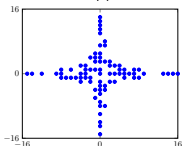
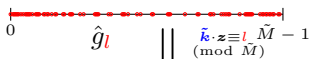
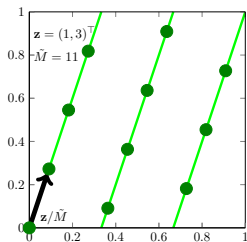
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⇒ **Definition** reconstructing R1L(\mathbf{z}, \tilde{M}) for $\tilde{\mathbf{I}}$:

$$\tilde{\mathbf{k}} \cdot \mathbf{z} \not\equiv \tilde{\mathbf{k}}' \cdot \mathbf{z} \pmod{\tilde{M}} \text{ for all } \tilde{\mathbf{k}}, \tilde{\mathbf{k}}' \in \tilde{\mathbf{I}}, \tilde{\mathbf{k}} \neq \tilde{\mathbf{k}}'$$

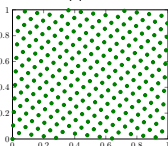
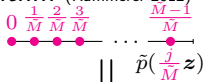
- $|\tilde{\mathbf{I}}| \leq \tilde{M} \leq |\tilde{\mathbf{I}}|^2$, CBC construction algorithm (Kammerer 2012)



$$(\hat{p}_{\tilde{\mathbf{k}}})_{\tilde{\mathbf{k}} \in \tilde{\mathbf{I}}}$$

1-dim
←
iFFT

$$\mathcal{O}(M \log M + d|\tilde{\mathbf{I}}|)$$



$$(\tilde{p}(\mathbf{x}_j))_{j=0}^{\tilde{M}-1}$$

Fast and exact reconstruction - non-periodic case

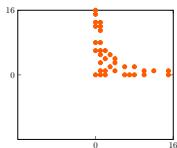
Theorem (Potts, V. 2015)

Let $I \subset \mathbb{N}_0^d$ be an arbitrary index set with $|I| < \infty$. Moreover, let $\text{R1L}(z, \tilde{M}) := \{\mathbf{x}_j := \frac{j}{\tilde{M}}z \bmod \mathbf{1} : j = 0, \dots, \tilde{M} - 1\}$ be a reconstructing rank-1 lattice with even rank-1 lattice size $\tilde{M} \in 2\mathbb{N}$ for the extended symmetric index set $\mathcal{M}(I)$, i.e.,

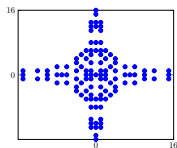
$$\tilde{\mathbf{k}} \cdot z \not\equiv \tilde{\mathbf{k}}' \cdot z \pmod{\tilde{M}} \text{ for all } \tilde{\mathbf{k}}, \tilde{\mathbf{k}}' \in \mathcal{M}(I), \tilde{\mathbf{k}} \neq \tilde{\mathbf{k}}'.$$

Then, the rank-1 Chebyshev lattice

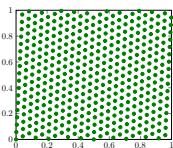
$\text{CL}(z, M = \frac{\tilde{M}}{2}) := \{\mathbf{x}_j := \cos(\frac{2j}{M}\pi z) : j = 0, \dots, \frac{\tilde{M}}{2}\}$ allows the reconstruction of the coefficients $\hat{p}_{\mathbf{k}}$ of $p(\mathbf{x}) = \sum_{\mathbf{k} \in I} \hat{p}_{\mathbf{k}} T_{\mathbf{k}}(\mathbf{x})$.



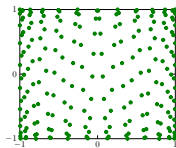
$\mathbf{k} \in I$



$\tilde{\mathbf{k}}, \tilde{\mathbf{k}}' \in \mathcal{M}(I)$



reco. $\text{R1L}(z, \tilde{M})$



reco. $\text{CL}(z, \frac{\tilde{M}}{2})$

Theorem (Potts, V. 2015)

Let $I \subset \mathbb{N}_0^d$ be an arbitrary index set with $|I| < \infty$. Moreover, let $\text{R1L}(\mathbf{z}, \tilde{M}) := \{\mathbf{x}_j := \frac{j}{\tilde{M}}\mathbf{z} \bmod \mathbf{1} : j = 0, \dots, \tilde{M} - 1\}$ be a reconstructing rank-1 lattice with even rank-1 lattice size $\tilde{M} \in 2\mathbb{N}$ for the extended symmetric index set $\mathcal{M}(I)$, i.e.,

$$\tilde{\mathbf{k}} \cdot \mathbf{z} \not\equiv \tilde{\mathbf{k}}' \cdot \mathbf{z} \pmod{\tilde{M}} \text{ for all } \tilde{\mathbf{k}}, \tilde{\mathbf{k}}' \in \mathcal{M}(I), \tilde{\mathbf{k}} \neq \tilde{\mathbf{k}}'.$$

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$\text{CL}(\mathbf{z}, M = \frac{\tilde{M}}{2}) := \{\mathbf{x}_j := \cos(\frac{2j}{\tilde{M}}\pi\mathbf{z}) : j = 0, \dots, \frac{\tilde{M}}{2}\}$ allows the reconstruction of the coefficients $\hat{p}_{\mathbf{k}}$ of $p(\mathbf{x}) = \sum_{\mathbf{k} \in I} \hat{p}_{\mathbf{k}} T_{\mathbf{k}}(\mathbf{x})$.

- under mild assumptions: $\tilde{M} \leq 2^{2d+1}|I|^2$, $M \leq 2^{2d}|I|^2$
[Potts, V. 2014] as a consequence of [Kämmerer 2013]

Fast and exact reconstruction - non-periodic case

- ansatz: compute with DCT-I coefficients

$$\hat{a}_l := \sum_{j=0}^M (\varepsilon_j^M)^2 p(\mathbf{x}_j) \cos\left(\frac{jl}{M}\pi\right), \quad l \in I_M^1, \quad \varepsilon_l^M = \begin{cases} 1/\sqrt{2}, & l = 0, M \\ 1, & \text{else} \end{cases}$$

- If rank-1 Chebyshev lattice allows reconstruction, for $\mathbf{k} \in I$

$$\hat{p}_{\mathbf{k}} = \frac{2(\varepsilon_{\mathbf{k} \cdot \mathbf{z} \bmod M}^M)^2}{M} \hat{a}_{\mathbf{k} \cdot \mathbf{z} \bmod M} \cdot \frac{2^d}{|\{\mathbf{s} \in \{-1, 1\}^d : (\mathbf{s} \odot \mathbf{k}) \cdot \mathbf{z} \bmod M = \mathbf{k} \cdot \mathbf{z} \bmod M\}|}$$

Fast and exact reconstruction - non-periodic case

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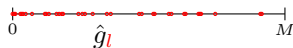
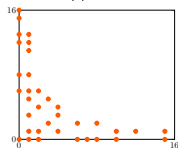
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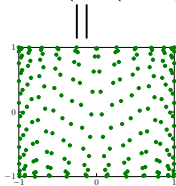
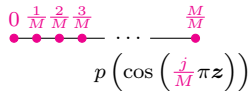
 2^d

$$\cdot \overline{|\{\mathbf{s} \in \{-1, 1\}^d : (\mathbf{s} \odot \mathbf{k}) \cdot \mathbf{z} \bmod M = \mathbf{k} \cdot \mathbf{z} \bmod M\}|}$$


 $\|\mathbf{k} \cdot \mathbf{z} \bmod M = l\|$

 $(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in I}$

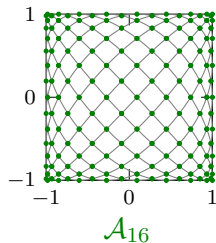
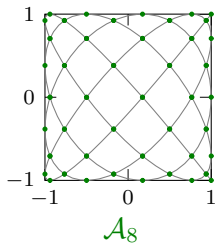
1-dim
←
DCT-I

$$\mathcal{O}(M \log M + d 2^d |I|)$$


 $(p(\mathbf{x}_j))_{j=0}^M$

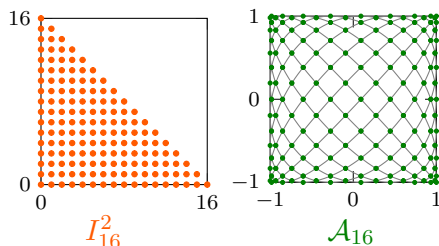
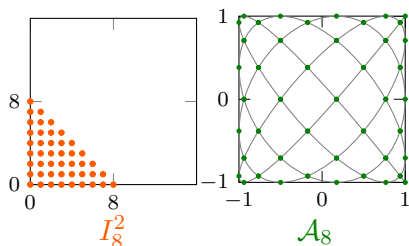
Fast and exact reconstruction - Padua points

- Padua point set (special 2d rank-1 Chebyshev lattice),
e.g. [Bos, Caliari, Marchi, Vianello, Xu 2006]
 $\mathcal{A}_n := \{\mathbf{x}_j := (\cos(j\pi/(n+1)), \cos(j\pi/n))^\top : j = 0, \dots, M\}$
- $\mathcal{A}_n = \text{CL}(\mathbf{z}, M)$, where $\mathbf{z} := (n, n+1)^\top$ and $M := n(n+1)$



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- $\mathcal{A}_n = \text{CL}(\mathbf{z}, M)$, where $\mathbf{z} := (n, n+1)^\top$ and $M := n(n+1)$
- allows exact reconstruction of all coefficients $\hat{p}_{\mathbf{k}}$, $\mathbf{k} \in I_n^2$, of an arbitrary 2d algebraic polynomial of total degree $\leq n$ (restricted to the domain $[-1, 1]^2$)
- $|I_n^2| = |\mathcal{A}_n| = \binom{n+2}{2} = \frac{n^2}{2} + \frac{3}{2}n + 1$, whereas $M = n^2 + n$.



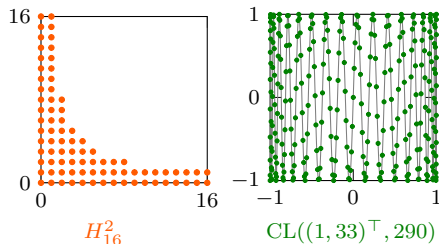
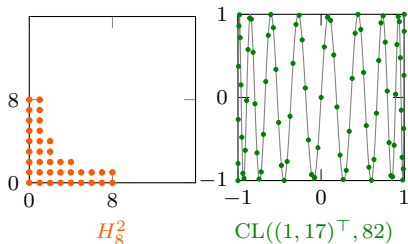
Examples of reconstr. rank-1 Chebyshev lattices

- hyperbolic cross index sets

$$I = H_n^d := \left\{ \mathbf{k} \in \mathbb{N}_0^d : \prod_{t=1}^d \max(1, |k_t|) \leq n \right\}, \quad n, d \in \mathbb{N}$$

- $|H_n^d| = \mathcal{O}(2^d n \log^{d-1} n)$

- search for rank-1 Chebyshev lattices $\text{CL}(\mathbf{z}, M)$ that allow reconstruction of coefficients $\hat{p}_{\mathbf{k}}, \mathbf{k} \in I$, of polynomial $p(\mathbf{x}) = \sum_{\mathbf{k} \in I} \hat{p}_{\mathbf{k}} T_{\mathbf{k}}(\mathbf{x})$



Examples of reconstr. rank-1 Chebyshev lattices

param.		card.	direct search		reco. R1L	
d	n	$ H_n^d $	M	$\frac{M+1}{ H_n^d }$	$\frac{\tilde{M}}{2}$	$\frac{\tilde{M}}{2} + 1$
2	256	1 979	66 050	33.38	66 050	33.38
2	512	4 305	263 170	61.13	263 170	61.13
2	1024	9 311	1 050 626	112.84	1 050 626	112.84
3	256	10 303	302 883	29.40	359 075	34.85
3	512	23 976	1 424 613	59.42	1 424 662	59.42
3	1024	55 202	4 600 672	83.34	5 560 838	100.74
6	16	8 684	303 396	34.94	557 773	64.23
6	32	26 088	1 751 513	67.14	2 867 903	109.93
6	64	76 433	8 979 932	117.49	13 603 339	177.98
10	2	6 144	495 451	80.64	2 157 672	351.18
10	4	27 904	3 083 988	110.52	15 390 479	551.55
10	8	109 824	25 099 619	228.54	88 580 127	806.56

Approximation - non-periodic case

Lemma

Let $f: [-1, 1]^d \rightarrow \mathbb{R}$ be a function with point-wise and absolutely convergent Chebyshev series,

$$f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{N}_0^d} \hat{f}_{\mathbf{k}} T_{\mathbf{k}}(\mathbf{x}).$$

Moreover, let $I \subset \mathbb{N}_0^d$, $|I| < \infty$, and let $\text{CL}(\mathbf{z}, M)$ be a reconstructing rank-1 Chebyshev lattice.

We approximate f by the polynomial

$$p(\mathbf{x}) = \sum_{\mathbf{k} \in I} \hat{p}_{\mathbf{k}} T_{\mathbf{k}}(\mathbf{x}),$$

where the coefficients $\hat{p}_{\mathbf{k}}$, $\mathbf{k} \in I$, are computed by applying a DCT-I on the samples $f(\mathbf{x}_j)$, $\mathbf{x}_j := \cos(j\pi\mathbf{z}/M) \in \text{CL}(\mathbf{z}, M)$.

Then, $\|f - p\|_{L^\infty([-1, 1]^d)} \leq (1 + 2^{d-1}) \sum_{\mathbf{k} \in \mathbb{N}_0^d \setminus I} |\hat{f}_{\mathbf{k}}|.$

Approximation - non-periodic case

Example

Use ℓ_1 -ball index set $I := I_n^d$, $n := 4$, $d := 4$,

and rank-1 Chebyshev lattice $\text{CL}(\mathbf{z}, M)$,

where $\mathbf{z} := (1, 8, 36, 121)^\top$ and $M := 262$.

Consider $f: [-1, 1]^4 \rightarrow \mathbb{R}$,

$f(\mathbf{x}) := \hat{f}_{(166,0,0,0)^\top} T_{(166,0,0,0)^\top}(\mathbf{x})$, $\hat{f}_{(166,0,0,0)^\top} \in \mathbb{R}$.

Compute coefficients $\hat{p}_{\mathbf{k}}$, $\mathbf{k} \in I$, from $f(\mathbf{x}_j)$, $\mathbf{x}_j \in \text{CL}(\mathbf{z}, M)$.

We obtain one non-zero coefficient $\hat{p}_{(1,1,1,1)^\top} = 8 \hat{f}_{(166,0,0,0)^\top}$.

This yields

$$\|f - p\|_{L^\infty([-1, 1]^4)} = (1+8) |\hat{f}_{(166,0,0,0)^\top}| = (1+2^{4-1}) \sum_{\mathbf{k} \in \mathbb{N}_0^4 \setminus I} |\hat{f}_{\mathbf{k}}|.$$

Summary

- fast evaluation of arbitrary high-dimensional multivariate algebraic polynomial $p: [-1, 1]^d \rightarrow \mathbb{R}$ in Chebyshev form, $p(\mathbf{x}) = \sum_{\mathbf{k} \in I} \hat{p}_{\mathbf{k}} T_{\mathbf{k}}(\mathbf{x})$, at the nodes $\mathbf{x}_j := \cos\left(\frac{j}{M}\pi\mathbf{z}\right)$, $j = 0, \dots, M$, of arbitrary rank-1 Chebyshev lattice $\text{CL}(\mathbf{z}, M)$ in $\mathcal{O}(M \log M + d 2^d |I|)$ arithmetic operations
- fast and exact reconstruction of coefficients $\hat{p}_{\mathbf{k}}$, $\mathbf{k} \in I$, from samples $p(\mathbf{x}_j)$ of suitable $\text{CL}(\mathbf{z}, M)$ in $\mathcal{O}(M \log M + d 2^d |I|)$ arithmetic operations
- constructive algorithm for searching suitable $\text{CL}(\mathbf{z}, M)$
- estimates for approximation error



Potts, D., Volkmer, T.

Fast and exact reconstruction of arbitrary multivariate algebraic polynomials in Chebyshev form.

SampTA 2015.

(<http://www.tu-chemnitz.de/~tovo>)