Fast and stable reconstruction of high-dimensional polynomials based on samples at rank-1 lattice nodes

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joint work with L. Kämmerer and D. Potts

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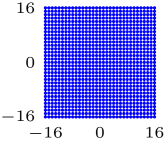
Content

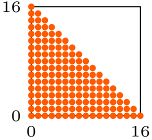
Introduction



Fast evaluation

periodic case $\tilde{p}\colon \mathbb{T}^d \to \mathbb{C}$, $\tilde{p}(\boldsymbol{x}) := \sum_{\tilde{\boldsymbol{k}}\in\tilde{\mathrm{I}}} \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}} \mathrm{e}^{2\pi \mathrm{i}\tilde{\boldsymbol{k}}\cdot\boldsymbol{x}}$

non-periodic case $p\colon [-1,1]^d \to \mathbb{R}$, $p(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$



Fast, exact and stable reconstruction

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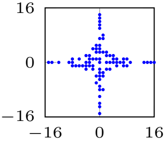
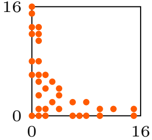


non-periodic case $p\colon [-1,1]^d \to \mathbb{R}$, $p(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x})$



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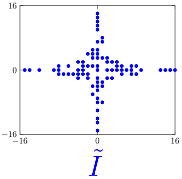
Summary

Introduction - periodic case

- d-dimensional torus $\mathbb{T}^d \simeq [0,1)^d$
- multivariate trigonometric polynomial $\tilde{p}\colon \mathbb{T}^d \to \mathbb{C}$ with frequencies supported on $\tilde{\mathrm{I}} \subset \mathbb{Z}^d$, $|\tilde{\mathrm{I}}| < \infty$,

$$\tilde{p}(\boldsymbol{x}) := \sum_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}} \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}} \mathrm{e}^{2\pi \mathrm{i} \tilde{\boldsymbol{k}} \cdot \boldsymbol{x}}, \quad \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}} \in \mathbb{C}$$

  - fast evaluation
    (e.g. [Li,Hickernell 03])
  - fast, exact and stable reconstruction
    ([Kämmerer,Kunis,Potts 12] [Kämmerer 13])



Introduction - periodic case

- d-dimensional torus $\mathbb{T}^d \simeq [0,1)^d$
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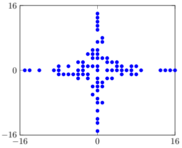
  - fast evaluation (e.g. [Li,Hickernell 03])
  - fast, exact and stable reconstruction ([Kämmerer,Kunis,Potts 12] [Kämmerer 13])



$\tilde{I}$

Aim: transfer results to non-periodic case

Introduction - non-periodic case

- multivariate algebraic polynomial $p : [-1,1]^d \to \mathbb{R}$ in Chebyshev form with frequencies supp. on $I \subset \mathbb{N}_0^d$, $|I| < \infty$,

$$p(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \prod_{t=1}^{d} T_{k_t}(x_t), \quad \hat{p}_{\boldsymbol{k}} \in \mathbb{R}, \qquad (1)$$

$T_m : [-1,1] \to [-1,1]$, $T_m(x) := \cos(m \arccos x)$, $m \in \mathbb{N}_0$

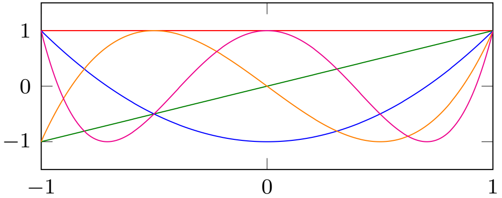
(T_m is algebraic polynomial of degree m restricted to domain $[-1,1]$)

$T_0(x) = 1$ $\quad T_1(x) = x$

$T_2(x) = 2x^2 - 1$ $\quad T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 + 1$

$T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x) \quad m \in \mathbb{N}$



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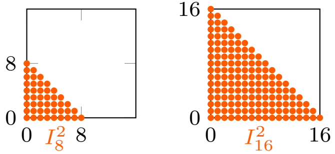
- e.g. ℓ_1-ball index set $I = I_n^d := \{\boldsymbol{k} \in \mathbb{N}_0^d \colon \|\boldsymbol{k}\|_1 \le n\}$, $n \in \mathbb{N}_0$

$\Rightarrow$ each algebraic polynomial of total degree $\le n$ (restricted to the domain $[-1,1]^d$) can be written as (1) with $I = I_n^d$



Introduction - non-periodic case

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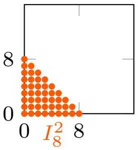
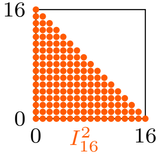
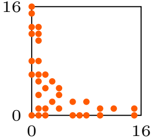




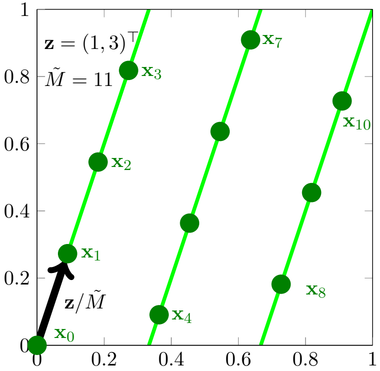
- Aim:
  - fast evaluation of high-dimensional arbitrary algebraic polynomials (given in Chebyshev form)
  - fast, exact and stable reconstruction from samples (special case of ℓ_1-ball index sets $I = I_n^d$ e.g. [Poppe,Cools 12])



Fast evaluation - periodic case

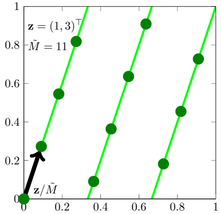
- rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, \tilde{M})$: $\boldsymbol{z} \in \mathbb{N}^d, \tilde{M} \in \mathbb{N}$

$$\boldsymbol{x}_j = \frac{j}{\tilde{M}} \boldsymbol{z} \bmod \mathbf{1};\ j = 0, \ldots, \tilde{M} - 1$$



Korobov 59
Maisonneuve 72
Sloan & Kachoyan 84,87,90
Temlyakov 86
Lyness 89
Sloan & Joe 94
Sloan & Reztsov 01
Li & Hickernell 03

Fast evaluation - periodic case



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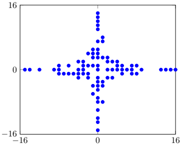
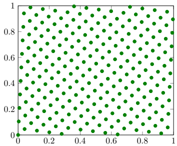
- multivariate trigonometric polynomial
  $\tilde{p}(\boldsymbol{x}) = \sum_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}} \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}} \mathrm{e}^{2\pi \mathrm{i} \tilde{\boldsymbol{k}} \cdot \boldsymbol{x}}$
- reformulation

$$\tilde{p}(\boldsymbol{x}_j) = \sum_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}} \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}} \mathrm{e}^{2\pi \mathrm{i} \frac{j \tilde{\boldsymbol{k}} \cdot \boldsymbol{z}}{\tilde{M}}} = \sum_{l=0}^{\tilde{M}-1} \underbrace{\left(\sum_{\substack{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}} \\ \tilde{\boldsymbol{k}} \cdot \boldsymbol{z} \equiv l \pmod{\tilde{M}}}} \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}} \right)}_{\hat{g}_l} \mathrm{e}^{2\pi \mathrm{i} \frac{j \tilde{\boldsymbol{k}} \cdot \boldsymbol{z}}{\tilde{M}}} = \sum_{l=0}^{\tilde{M}-1} \hat{g}_l \mathrm{e}^{2\pi \mathrm{i} \frac{j l}{\tilde{M}}}$$



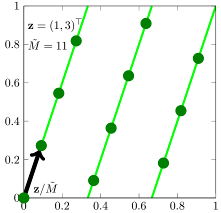
$(\hat{\tilde{p}}_{\tilde{\boldsymbol{k}}})_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}}$

$\longrightarrow$



$(\tilde{p}(\boldsymbol{x}_j))_{j=0}^{\tilde{M}-1}$

Fast evaluation - periodic case



- rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, \tilde{M})$: $\boldsymbol{z} \in \mathbb{N}^d, \tilde{M} \in \mathbb{N}$

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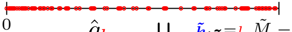
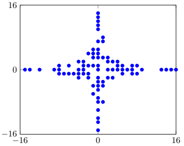
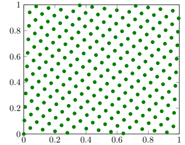




$(\hat{\tilde{p}}_{\tilde{\boldsymbol{k}}})_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}}$

$\xrightarrow[\text{FFT}]{\text{1-dim}}$

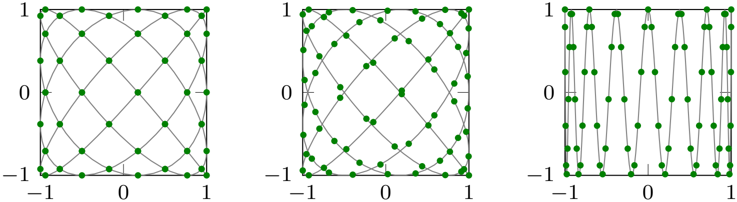
$\mathcal{O}(\tilde{M} \log \tilde{M} + d|\tilde{\mathrm{I}}|)$



$(\tilde{p}(\boldsymbol{x}_j))_{j=0}^{\tilde{M}-1}$

Fast evaluation - non-periodic case

- d-dimensional rank-1 Chebyshev lattice [Cools, Poppe 2011]
  $\mathrm{CL}(\boldsymbol{z}, M) := \{\boldsymbol{x}_j \colon j = 0, \ldots, M\} \subset [-1,1]^d$
  with nodes $\boldsymbol{x}_j := \cos\left(\frac{j}{M}\pi\boldsymbol{z}\right)$, $j = 0, \ldots, M$
  - generating vector $\boldsymbol{z} \in \mathbb{N}_0^d$
  - size parameter $M \in \mathbb{N}_0$



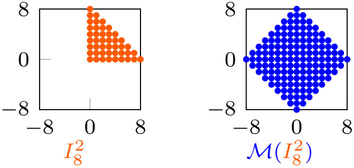
$\boldsymbol{z} := (8,9)^\top$, $M := 72$, $|\mathrm{CL}(\boldsymbol{z}, M)| = 45$

$\boldsymbol{z} := (8,9)^\top$, $M := 73$, $|\mathrm{CL}(\boldsymbol{z}, M)| = 74$

$\boldsymbol{z} := (1,16)^\top$, $M := 76$, $|\mathrm{CL}(\boldsymbol{z}, M)| = 77$

Fast evaluation - non-periodic case

- define extended symmetric index set
  $\mathcal{M}(I) := \{\tilde{\boldsymbol{k}} \in \mathbb{Z}^d : (|\tilde{k}_1|, \ldots, |\tilde{k}_d|)^\top \in I\}$ for $I \subset \mathbb{N}_0^d$



- for $M \in \mathbb{N}_0$ and $l \in \mathbb{Z}$, define even-mod relation

$$l \operatorname{emod} M := \begin{cases} l \bmod (2M) & \text{for } l \bmod (2M) \leq M, \\ 2M - (l \bmod (2M)) & \text{else} \end{cases}$$

Fast evaluation - non-periodic case

- with Chebyshev nodes $\boldsymbol{x}_j := \cos\left(\frac{j}{M}\pi\boldsymbol{z}\right)$, $j = 0, \dots, M$,

$$p(\boldsymbol{x}_j) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x}_j)$$

$$= \sum_{l=0}^{M} \underbrace{\left(\sum_{\substack{\boldsymbol{k}\in I,\ \boldsymbol{s}\in\{-1,1\}^d \\ (\boldsymbol{s}\odot\boldsymbol{k})\cdot\boldsymbol{z} \,\mathrm{emod}\, M = l}} \frac{\hat{p}_{\boldsymbol{k}}}{2^d} \right)}_{\hat{g}_l} \cos\left(\frac{jl}{M}\pi\right) = \sum_{l=0}^{M} \hat{g}_l \cos\left(\frac{jl}{M}\pi\right)$$

Fast evaluation - non-periodic case

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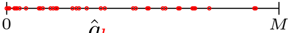
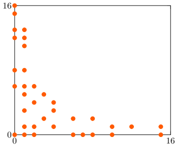


$\hat{g}_l$

$\Big\|$ $\boldsymbol{k}\cdot\boldsymbol{z} \,\mathrm{emod}\, M = l$



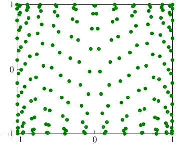
$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I}$

$$\xrightarrow[\text{DCT-I}]{\text{1-dim}}$$

$$\mathcal{O}(M \log M + d\, 2^d |I|)$$

$p\left(\cos\left(\frac{j}{M}\pi \boldsymbol{z}\right)\right)$

$\Big\|$



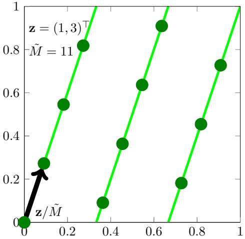
$(p(\boldsymbol{x}_j))_{j=0}^{M}$

Fast, exact, stable reconstruction - periodic case

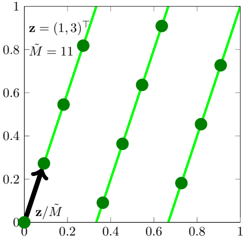
- rank-1 lattice $\mathrm{R1L}(\boldsymbol{z}, \tilde{M})$: $\boldsymbol{z} \in \mathbb{N}^d, \tilde{M} \in \mathbb{N}$

$$\boldsymbol{x}_j := \frac{j}{\tilde{M}}\boldsymbol{z} \text{ mod } \mathbf{1};\ j = 0, \ldots, \tilde{M} - 1$$

- reconstruction of Fourier coefficients $\hat{\tilde{p}}_{\tilde{\boldsymbol{k}}}$ of multivariate trigonometric polynomial $\tilde{p}(\boldsymbol{x}) = \sum_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}} \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}} \mathrm{e}^{2\pi\mathrm{i}\tilde{\boldsymbol{k}}\cdot\boldsymbol{x}}$



Fast, exact, stable reconstruction - periodic case



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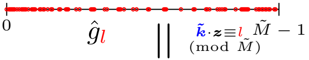
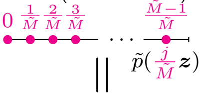
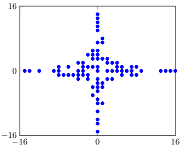
$\Rightarrow$ Definition reconstructing $\mathrm{R1L}(\boldsymbol{z}, \tilde{M})$ for $\tilde{\mathrm{I}}$:
$\tilde{\boldsymbol{k}} \cdot \boldsymbol{z} \not\equiv \tilde{\boldsymbol{k}}' \cdot \boldsymbol{z} \pmod{\tilde{M}}$ for all $\tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}}' \in \tilde{\mathrm{I}}$, $\tilde{\boldsymbol{k}} \neq \tilde{\boldsymbol{k}}'$

- $|\tilde{\mathrm{I}}| \leq \tilde{M} \leq |\tilde{\mathrm{I}}|^2$, CBC construction algorithm (Kämmerer 2012)



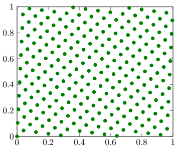




$(\hat{\tilde{p}}_{\tilde{\boldsymbol{k}}})_{\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}}$

1-dim $\longleftarrow$ iFFT

$\mathcal{O}(M \log M + d|\tilde{\mathrm{I}}|)$



$(\tilde{p}(\boldsymbol{x}_j))_{j=0}^{M-1}$

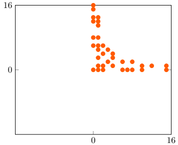
Fast, exact, stable reconstruction - non-periodic case

Theorem (Potts, V. 2015)

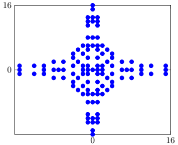
Let $I \subset \mathbb{N}_0^d$ *be an arbitrary index set with* $|I| < \infty$. *Moreover, let* $\mathrm{R1L}(\boldsymbol{z}, \tilde{M}) := \{\boldsymbol{x}_j := \frac{j}{\tilde{M}}\boldsymbol{z} \bmod \boldsymbol{1} \colon j = 0, \ldots, \tilde{M}-1\}$ *be a reconstructing rank-1 lattice with even rank-1 lattice size* $\tilde{M} \in 2\mathbb{N}$ *for the extended symmetric index set* $\mathcal{M}(I)$, *i.e.,*

$$\tilde{\boldsymbol{k}} \cdot \boldsymbol{z} \not\equiv \tilde{\boldsymbol{k}}' \cdot \boldsymbol{z} \pmod{\tilde{M}} \text{ for all } \tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}}' \in \mathcal{M}(I),\ \tilde{\boldsymbol{k}} \neq \tilde{\boldsymbol{k}}'.$$

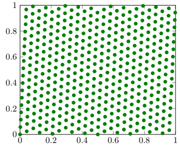
Then, the rank-1 *Chebyshev lattice* $\mathrm{CL}(\boldsymbol{z}, M = \frac{\tilde{M}}{2}) := \{\boldsymbol{x}_j := \cos(\frac{2j}{\tilde{M}}\pi\boldsymbol{z}) \colon j = 0, \ldots, \frac{\tilde{M}}{2}\}$ *allows the reconstruction of the coefficients* $\hat{p}_{\boldsymbol{k}}$ *of* $p(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}} T_{\boldsymbol{k}}(\boldsymbol{x})$.



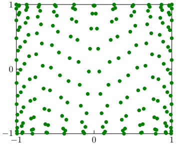
$\boldsymbol{k} \in I$



$\tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}}' \in \mathcal{M}(I)$



reco. $\mathrm{R1L}(\boldsymbol{z}, \tilde{M})$



reco. $\mathrm{CL}(\boldsymbol{z}, \frac{\tilde{M}}{2})$

Fast, exact, stable reconstruction - non-periodic case

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Then, the rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, M = \frac{\tilde{M}}{2}) := \{\boldsymbol{x}_j := \cos(\frac{2j}{\tilde{M}}\pi\boldsymbol{z}) \colon j = 0, \ldots, \frac{\tilde{M}}{2}\}$ *allows the reconstruction of the coefficients* $\hat{p}_{\boldsymbol{k}}$ *of* $p(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}} T_{\boldsymbol{k}}(\boldsymbol{x})$.

- under mild assumptions: $\tilde{M} \leq 2^{2d+1}|I|^2$, $M \leq 2^{2d}|I|^2$

  [Potts, V. 2014] as a consequence of [Kämmerer 2013]

Fast, exact, stable reconstruction - non-periodic case

- ansatz: compute with DCT-I coefficients
$$\hat{a}_l := \sum_{j=0}^{M} (\varepsilon_j^M)^2 \, p(\boldsymbol{x}_j) \, \cos\left(\frac{jl}{M}\pi\right), \; l \in I_M^1, \; \varepsilon_l^M = \begin{cases} 1/\sqrt{2}, & l = 0, M \\ 1, & \text{else} \end{cases}$$
- If rank-1 Chebyshev lattice allows reconstruction, for $\boldsymbol{k} \in I$
$$\hat{p}_{\boldsymbol{k}} = \frac{2(\varepsilon^M_{\boldsymbol{k}\cdot\boldsymbol{z} \,\mathrm{emod}\, M})^2}{M} \; \hat{a}_{\boldsymbol{k}\cdot\boldsymbol{z} \,\mathrm{emod}\, M} \cdot \frac{2^d}{|\{\boldsymbol{s} \in \{-1,1\}^d \colon (\boldsymbol{s} \odot \boldsymbol{k}) \cdot \boldsymbol{z} \,\mathrm{emod}\, M = \boldsymbol{k} \cdot \boldsymbol{z} \,\mathrm{emod}\, M\}|}$$

Fast, exact, stable reconstruction - non-periodic case

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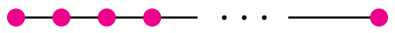
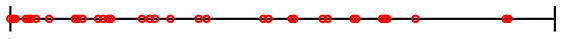
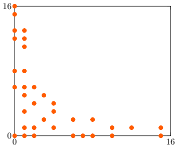


$p\left(\cos\left(\frac{j}{M}\pi \boldsymbol{z}\right)\right)$



$\hat{g}_l$

$\| \; \boldsymbol{k}\cdot\boldsymbol{z} \operatorname{emod} M = l$

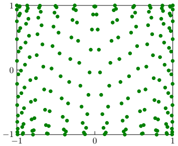


$(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I}$

1-dim
$\longleftarrow$
DCT-I

$\mathcal{O}(M \log M + d\, 2^d |I|)$

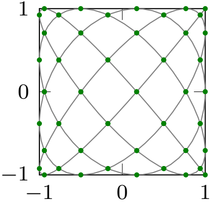
$\|$



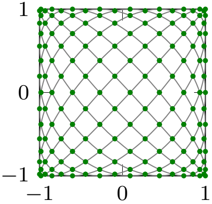
$(p(\boldsymbol{x}_j))_{j=0}^{M}$

Fast, exact, stable reconstruction - Padua points

- Padua point set (special 2d rank-1 Chebyshev lattice), e.g. [Bos, Caliari, Marchi, Vianello, Xu 2006]
  $\mathcal{A}_n := \{\boldsymbol{x}_j := (\cos(j\pi/(n+1)), \cos(j\pi/n))^\top : j = 0, \ldots, M\}$
- $\mathcal{A}_n = \mathrm{CL}(\boldsymbol{z}, M)$, where $\boldsymbol{z} := (n, n+1)^\top$ and $M := n\,(n+1)$



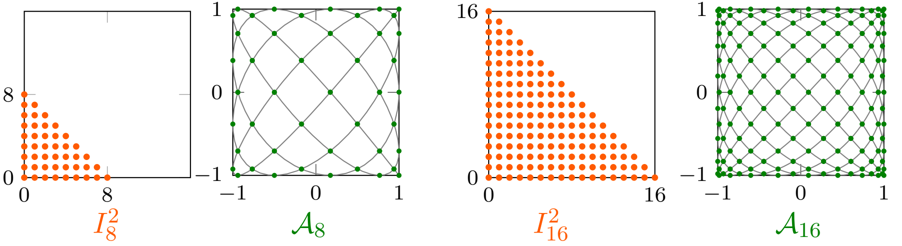
$\mathcal{A}_8$



$\mathcal{A}_{16}$

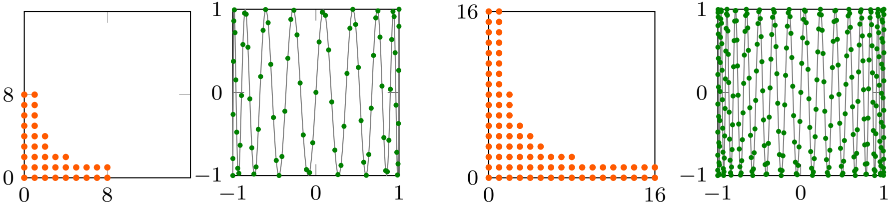
Fast, exact, stable reconstruction - Padua points

- Padua point set (special 2d rank-1 Chebyshev lattice),
  e.g. [Bos, Caliari, Marchi, Vianello, Xu 2006]
  $\mathcal{A}_n := \{\boldsymbol{x}_j := (\cos(j\pi/(n+1)), \cos(j\pi/n))^\top : j = 0, \ldots, M\}$
- $\mathcal{A}_n = \mathrm{CL}(\boldsymbol{z}, M)$, where $\boldsymbol{z} := (n, n+1)^\top$ and $M := n\,(n+1)$
- allows exact reconstruction of all coefficients $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I_n^2$, of an arbitrary 2d algebraic polynomial of total degree $\leq n$ (restricted to the domain $[-1,1]^2$)
- $|I_n^2| = |\mathcal{A}_n| = \binom{n+2}{2} = \frac{n^2}{2} + \frac{3}{2}n + 1$, whereas $M = n^2 + n$.



Examples of reconstr. rank-1 Chebyshev lattices

- hyperbolic cross index sets
  $I = H_n^d := \left\{ \boldsymbol{k} \in \mathbb{N}_0^d \colon \prod_{t=1}^d \max(1, |k_t|) \le n \right\}, \, n, d \in \mathbb{N}$
- $|H_n^d| = \mathcal{O}(2^d \, n \log^{d-1} n)$
- search for rank-1 Chebyshev lattices $\mathrm{CL}(\boldsymbol{z}, M)$
  that allow reconstruction of coefficients $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$,
  of polynomial $p(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}} \, T_{\boldsymbol{k}}(\boldsymbol{x})$



H_8^2 | $\mathrm{CL}((1, 17)^\top, 82)$ | H_{16}^2 | $\mathrm{CL}((1, 33)^\top, 290)$

Examples of reconstr. rank-1 Chebyshev lattices

param.		card.	direct search		reco. R1L	
d	n	$\lvert H_n^d \rvert$	M	$\frac{M+1}{\lvert H_n^d \rvert}$	$\frac{\tilde{M}}{2}$	$\frac{\frac{\tilde{M}}{2}+1}{\lvert H_n^d \rvert}$
2	256	1 979	66 050	33.38	66 050	33.38
2	512	4 305	263 170	61.13	263 170	61.13
2	1024	9 311	1 050 626	112.84	1 050 626	112.84
3	256	10 303	302 883	29.40	359 075	34.85
3	512	23 976	1 424 613	59.42	1 424 662	59.42
3	1024	55 202	4 600 672	83.34	5 560 838	100.74
6	16	8 684	303 396	34.94	557 773	64.23
6	32	26 088	1 751 513	67.14	2 867 903	109.93
6	64	76 433	8 979 932	117.49	13 603 339	177.98
9	2	2 816	132 708	47.13	473 013	167.97
9	4	12 032	781 974	64.99	3 449 019	286.65
9	8	45 056	6 329 397	140.48	16 125 059	357.89

Approximation - periodic case

Lemma

Let $\tilde{f}\colon \mathbb{T}^d \to \mathbb{C}$ *be a function with point-wise and absolutely convergent Fourier series,* $\tilde{f}(\boldsymbol{x}) = \sum_{\tilde{\boldsymbol{k}}\in\mathbb{Z}^d} \hat{\tilde{f}}_{\tilde{\boldsymbol{k}}}\, \mathrm{e}^{2\pi\mathrm{i}\tilde{\boldsymbol{k}}\cdot\boldsymbol{x}}$, $\hat{\tilde{f}}_{\tilde{\boldsymbol{k}}} \in \mathbb{C}$ *for* $\tilde{\boldsymbol{k}} \in \mathbb{Z}^d$, *and* $\sum_{\tilde{\boldsymbol{k}}\in\mathbb{Z}^d} |\hat{\tilde{f}}_{\tilde{\boldsymbol{k}}}| < \infty$.
Moreover, let $\tilde{\mathrm{I}} \subset \mathbb{Z}^d$, $|\tilde{\mathrm{I}}| < \infty$, *and*
let $\mathrm{R1L}(\boldsymbol{z}, \tilde{M})$ *be a reconstructing rank-1 lattice for* $\tilde{\mathrm{I}}$.
We approximate $\tilde{f}$ *by the trigonometric polynomial* $\tilde{p}(\boldsymbol{x}) = \sum_{\tilde{\boldsymbol{k}}\in I} \hat{\tilde{p}}_{\tilde{\boldsymbol{k}}}\, \mathrm{e}^{2\pi\mathrm{i}\tilde{\boldsymbol{k}}\cdot\boldsymbol{x}}$, *where the Fourier coefficients* $\hat{\tilde{p}}_{\tilde{\boldsymbol{k}}}$, $\tilde{\boldsymbol{k}} \in \tilde{\mathrm{I}}$, *are computed as explained on slide 9 by applying an 1d iFFT on the samples* $\tilde{f}(\boldsymbol{x}_j)$, $\boldsymbol{x}_j := \frac{j}{\tilde{M}}\boldsymbol{z} \bmod \boldsymbol{1}$, $j = 0, \ldots, \tilde{M}-1$.
Then, the approximation error is bounded by

$$\|\tilde{f} - \tilde{p} | L^\infty(\mathbb{T}^d)\| \leq 2 \sum_{\tilde{\boldsymbol{k}}\in\mathbb{Z}^d\setminus\tilde{\mathrm{I}}} |\hat{\tilde{f}}_{\tilde{\boldsymbol{k}}}|.$$

more estimates available for $\tilde{f} \in \mathcal{A}^\omega(\mathbb{T}^d)$ and $\mathcal{H}^{\alpha,\beta}_{\mathrm{mix}}(\mathbb{T}^d)$

[Temlyakov 86,...] [Kuo,Sloan,Woźniakowski 06] [Potts,Kämmerer,V. 14]

Approximation - non-periodic case

Lemma

Let $f\colon [-1,1]^d \to \mathbb{R}$ *be a function with point-wise and absolutely convergent Chebyshev series,*
$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \hat{f}_{\boldsymbol{k}} T_{\boldsymbol{k}}(\boldsymbol{x})$ *for* $\boldsymbol{x} \in [-1,1]^d$,
$\hat{f}_{\boldsymbol{k}} \in \mathbb{R}$ *for* $\boldsymbol{k} \in \mathbb{N}_0^d$, *and* $\sum_{\boldsymbol{k}\in\mathbb{N}_0^d} |\hat{f}_{\boldsymbol{k}}| < \infty$.
Moreover, let $I \subset \mathbb{N}_0^d$, $|I| < \infty$, *and let* $\mathrm{CL}(\boldsymbol{z}, M)$ *be a rank-1 Chebyshev lattice which allows the reconstruction.*
We approximate f *by the polynomial* $p(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} T_{\boldsymbol{k}}(\boldsymbol{x})$, *where the coefficients* $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, *are computed as explained on slide 11 by applying a DCT-I on the samples* $f(\boldsymbol{x}_j)$,
$\boldsymbol{x}_j := \cos(j\pi\boldsymbol{z}/M) \in \mathrm{CL}(\boldsymbol{z}, M)$, $j = 0, \ldots, M$.
Then, the approximation error is bounded by

$$\|f - p|L^\infty([-1,1]^d)\| \le (1 + 2^{d-1}) \sum_{\boldsymbol{k}\in\mathbb{N}_0^d\setminus I} |\hat{f}_{\boldsymbol{k}}|.$$

Approximation - non-periodic case

Example

Use ℓ_1-ball index set $I := I_n^d$, $n := 4$, $d := 4$,
and rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, M)$,
where $\boldsymbol{z} := (1, 8, 36, 121)^\top$ and $M := 262$.
Consider $f\colon [-1,1]^4 \to \mathbb{R}$,
$f(\boldsymbol{x}) := \hat{f}_{(166,0,0,0)^\top} T_{(166,0,0,0)^\top}(\boldsymbol{x})$, $\hat{f}_{(166,0,0,0)^\top} \in \mathbb{R}$.
Compute coefficients $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, from $f(\boldsymbol{x}_j)$, $\boldsymbol{x}_j \in \mathrm{CL}(\boldsymbol{z}, M)$.
We obtain one non-zero coefficient $\hat{p}_{(1,1,1,1)^\top} = 8\,\hat{f}_{(166,0,0,0)^\top}$.
This yields

$$\|f - p | L^\infty([-1,1]^4)\| = (1+8)\,|\hat{f}_{(166,0,0,0)^\top}| = (1+2^{4-1}) \sum_{\boldsymbol{k} \in \mathbb{N}_0^4 \setminus I} |\hat{f}_{\boldsymbol{k}}|.$$

Summary

- fast evaluation of arbitrary high-dimensional multivariate algebraic polynomial $p\colon [-1,1]^d \to \mathbb{R}$ in Chebyshev form, $p(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} T_{\boldsymbol{k}}(\boldsymbol{x})$,
  at the nodes $\boldsymbol{x}_j := \cos\left(\frac{j}{M}\pi\boldsymbol{z}\right)$, $j = 0, \ldots, M$,
  of arbitrary rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, M)$
  in $\mathcal{O}(M \log M + d\,2^d |I|)$ arithmetic operations
- fast, exact and stable reconstruction of coefficients $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, from samples $p(\boldsymbol{x}_j)$ of suitable $\mathrm{CL}(\boldsymbol{z}, M)$
  in $\mathcal{O}(M \log M + d\,2^d |I|)$ arithmetic operations
- constructive algorithm for searching suitable $\mathrm{CL}(\boldsymbol{z}, M)$
- estimates for approximation error

Potts, D., Volkmer, T.
Fast, exact and stable reconstruction of multivariate algebraic polynomials in Chebyshev form.
Preprint 2015-03, Faculty of Mathematics, TU Chemnitz.
(http://www.tu-chemnitz.de/∼tovo)