

Approximation of high-dimensional multivariate periodic functions by trigonometric polynomials based on rank-1 lattice sampling

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joint work with L. Kämmerer and D. Potts

supported by



Introduction

Multivariate trigonometric polynomials

- Fast evaluation at rank-1 lattices

- Fast, exact and stable reconstruction

Approximate reconstruction of functions $f \in \mathcal{H}^\omega(\mathbb{T}^d)$

- by sampling at rank-1 lattice nodes

- Error estimates

- Numerical results

Dimension incremental reconstruction

Summary

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$$p(\mathbf{x}) := \sum_{\mathbf{k} \in I} \hat{p}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}, \quad \hat{p}_{\mathbf{k}} \in \mathbb{C}$$

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where the Fourier coefficients of f are given by

$$\hat{f}_{\mathbf{k}} = \int_{\mathbb{T}^d} f(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} d\mathbf{x}, \quad \mathbf{k} \in \mathbb{Z}^d$$

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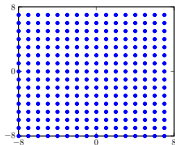
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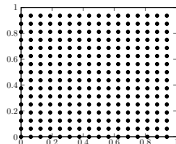
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- e.g., $I = \mathbb{Z}^d \cap [-N, N)^d$ full grid, \mathbf{y}_j equispaced nodes



$$(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in I} \begin{array}{c} \xleftrightarrow{\text{d-dim}} \\ \xleftrightarrow{\text{FFT}} \end{array} (f(\mathbf{y}_j))_{j=0}^{|\mathbf{I}|-1}$$



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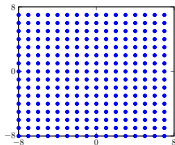
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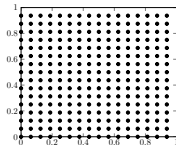
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$$\mathcal{O}(N^d \log N)$$



- problem: $|I| = (2N)^d \Rightarrow$ **curse of dimensionality**

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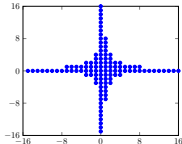
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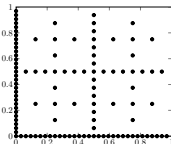
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(Baszanski, Delvos 1989; Hallatschek 1992;
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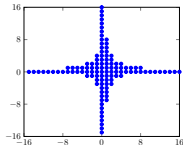
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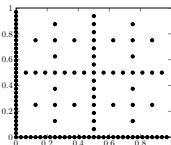
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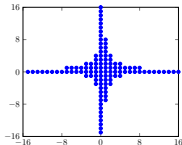
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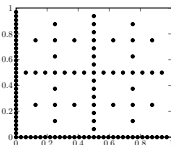
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- problem: **stability** (Kämmerer, Kunis 2011)
- **implementation** of algorithm (HCFFT) is **effortful**

- Hilbert space

$$\mathcal{H}^\omega(\mathbb{T}^d) := \left\{ f \in L^2(\mathbb{T}^d) : \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d} \omega(\mathbf{k})^2 |\hat{f}_{\mathbf{k}}|^2} < \infty \right\}$$

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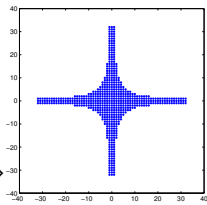
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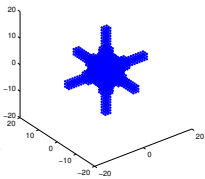


hyperbolic
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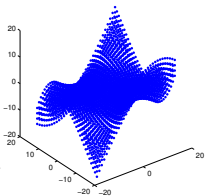
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arbitrary
index set

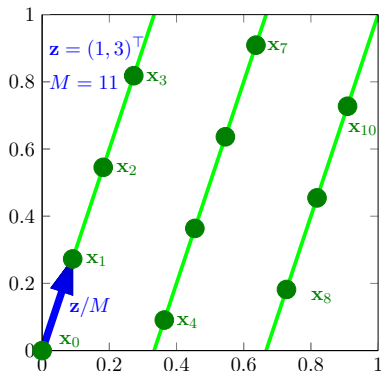
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- large dimension d

Trig. polynomials - Fast evaluation at rank-1 lattices

- rank-1 lattice: $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$

$$\mathbf{x}_j = \frac{j}{M} \mathbf{z} \bmod \mathbf{1}; j = 0, \dots, M-1$$



Korobov 59
Maisonneuve 72
Sloan & Kachoyan 84,87,90
Temlyakov 86
Lyness 89
Sloan & Joe 94
Sloan & Reztsov 01
Li & Hickernell 03

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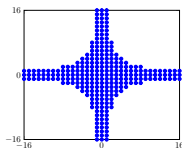
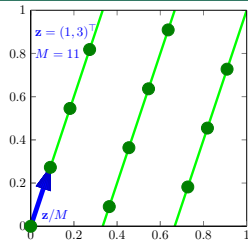
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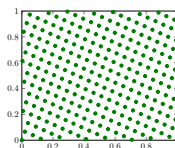
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- reformulation

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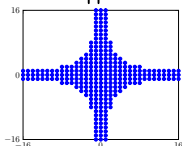
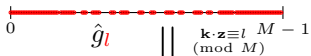
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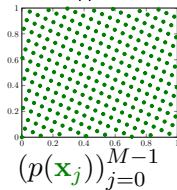
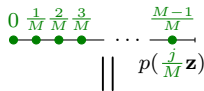
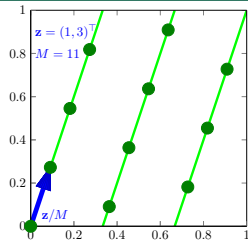
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1-dim
→
FFT

$$\mathcal{O}(M \log M + d |\mathbb{I}_N^d|)$$



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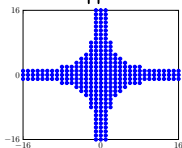
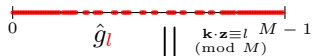
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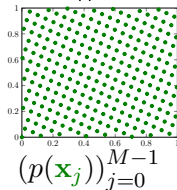
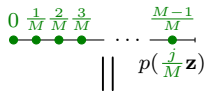
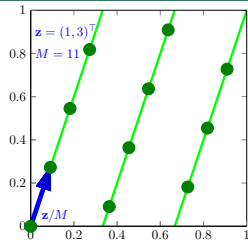


$$(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in \mathbb{I}_N^d}$$

?

←

fast, stable, unique reconstruction

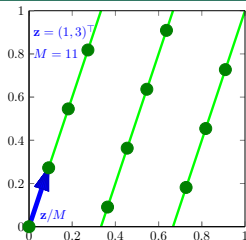


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- reconstruction of the Fourier coefficients of $p(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{I}_N^d} \hat{p}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}$ by applying a lattice rule

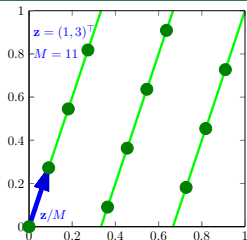


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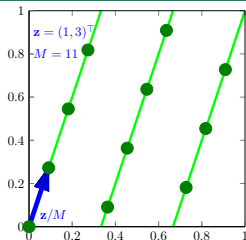
$$\hat{p}_{\mathbf{k}} = \int_{\mathbb{T}^d} p(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} d\mathbf{x} \approx Q(p(\cdot)) e^{-2\pi i \mathbf{k} \cdot (\cdot)} = \underbrace{\frac{1}{M} \sum_{j=0}^{M-1} p(\mathbf{x}_j) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}_j}}_{\hat{\tilde{p}}_{\mathbf{k}}}$$

Trig. polynomials - Fast reconstruction

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$$\mathbf{x}_j = \frac{j}{M} \mathbf{z} \bmod \mathbf{1}; j = 0, \dots, M-1$$

- reconstruction of the Fourier coefficients of $p(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{I}_N^d} \hat{p}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}$ by applying a lattice rule



$$\hat{p}_{\mathbf{k}} = \int_{\mathbb{T}^d} p(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} d\mathbf{x} \approx Q(p(\cdot)) e^{-2\pi i \mathbf{k} \cdot (\cdot)} = \underbrace{\frac{1}{M} \sum_{j=0}^{M-1} p(\mathbf{x}_j) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}_j}}_{\hat{\hat{p}}_{\mathbf{k}}}$$

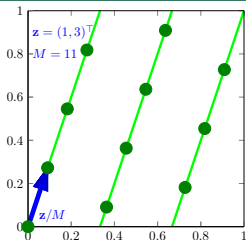
$$\Rightarrow \hat{\hat{p}}_{\mathbf{k}} = \hat{p}_{\mathbf{k}} \Leftrightarrow \mathbf{k}_1 \cdot \mathbf{z} \not\equiv \mathbf{k}_2 \cdot \mathbf{z} \pmod{M} \text{ for all } \mathbf{k}_1 \neq \mathbf{k}_2 \in \mathbb{I}_N^d$$

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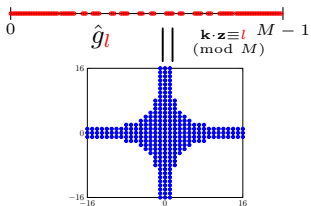
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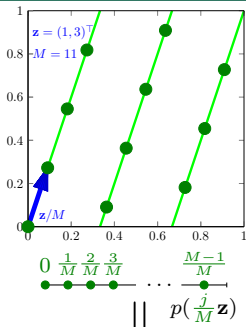
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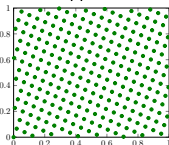
$$(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in \mathbb{I}_N^d}$$

\Rightarrow **Definition:** reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathbb{I}_N^d)$ for \mathbb{I}_N^d , $|\mathbb{I}_N^d| \leq M \leq |\mathbb{I}_N^d|^2$, CBC construction algorithm (Kämmerer 2012)



1-dim
←
iFFT

$$\mathcal{O}(M \log M + d |\mathbb{I}_N^d|)$$



$$(p(\mathbf{x}_j))_{j=0}^{M-1}$$

Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Rank-1 lattice nodes

- reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathbf{I}_N^d)$:
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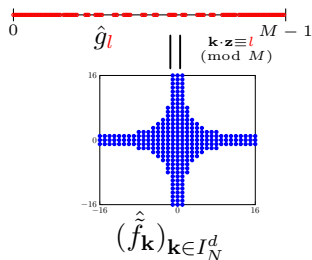
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Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Rank-1 lattice nodes

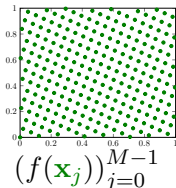
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1-dim
←
iFFT

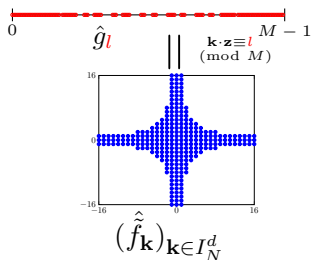
$$\mathcal{O}(M \log M + d|\mathbb{I}_N^d|)$$



Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Rank-1 lattice nodes

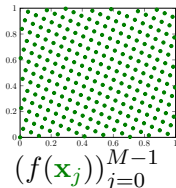
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1-dim
←
iFFT

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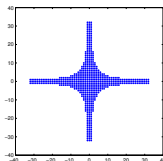
- approximation $\tilde{S}_{\mathbb{I}_N^d} f$ of f by $\tilde{S}_{\mathbb{I}_N^d} f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{I}_N^d} \hat{f}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}$

Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Error estimates

Example: hyperbolic cross index set

$$\mathbb{I}_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d : \prod_{s=1}^d \max(1, |k_s|) \leq N \right\}$$

$$|\mathbb{I}_N^d| \in \mathcal{O}(N \log^{d-1} N)$$



- Hilbert space:

$$\mathcal{H}_{\text{mix}}^\beta(\mathbb{T}^d) := \left\{ f : \|f\|_{\mathcal{H}_{\text{mix}}^\beta(\mathbb{T}^d)} := \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d} \omega_{\text{mix}}^\beta(\mathbf{k})^2 |\hat{f}_{\mathbf{k}}|^2} < \infty \right\}$$

- weights: $\omega_{\text{mix}}^\beta(\mathbf{k}) := \prod_{s=1}^d \max(1, |k_s|)^\beta$, $\beta > 1/2$
- samples: $\mathcal{O}(N^2 \log^{d-2} N)$
- arithmetic complexity: $\mathcal{O}(N^2 \log^{d-1} N)$
- error estimate:

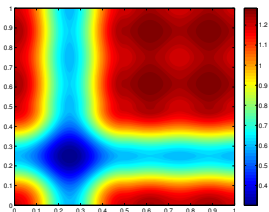
$$\|f - \tilde{S}_{\mathbb{I}_N^d} f\|_{L^2(\mathbb{T}^d)} \lesssim N^{-\beta} \log^{d-1} N \|f\|_{\mathcal{H}_{\text{mix}}^\beta(\mathbb{T}^d)}$$

Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Numerical results

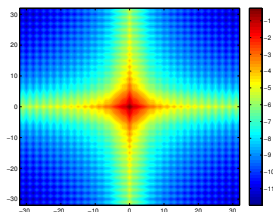
- function $f \in \mathcal{H}_{\text{mix}}^{\frac{5}{2}-\epsilon}(\mathbb{T}^d)$, $\epsilon > 0$,

$$f(\mathbf{x}) := \prod_{s=1}^d \left(4 + \operatorname{sgn} \left(x_s - \frac{1}{2} \right) \sin(2\pi x_s)^2 + \operatorname{sgn} \left(x_s - \frac{1}{2} \right) \sin(2\pi x_s)^3 \right),$$

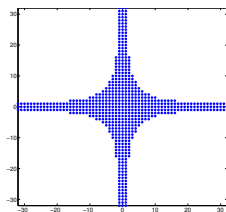
- hyperbolic cross index set $I_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d : \prod_{s=1}^d \max(1, |k_s|) \leq N \right\}$



$f((x_1, x_2)^\top)$



$\log_{10} \left| \hat{f}(k_1, k_2)^\top \right|$



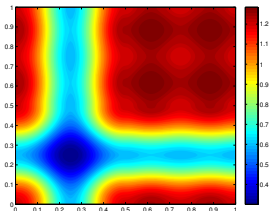
I_N^d

Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Numerical results

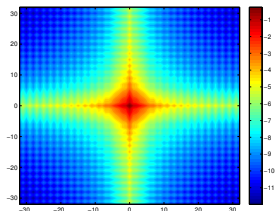
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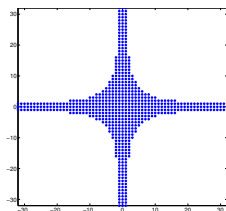
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$\log_{10} \left| \hat{f}_{(k_1, k_2)}^T \right|$



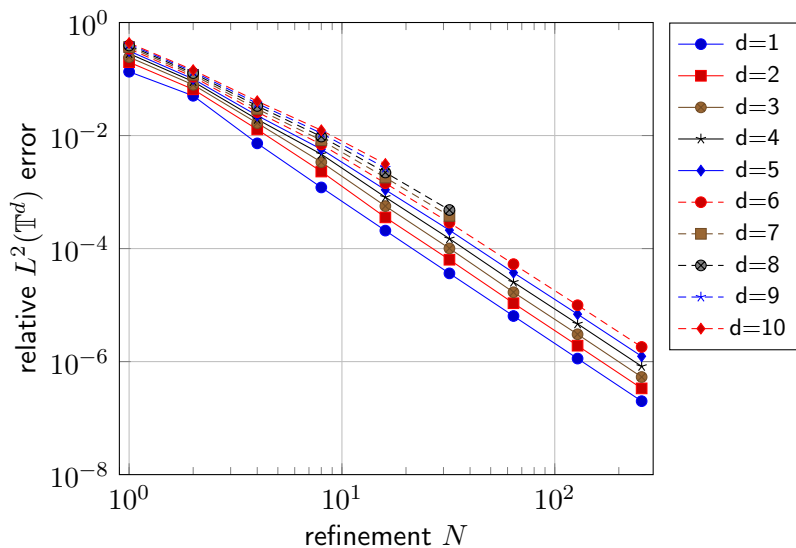
I_N^d

- error estimate: $\epsilon > 0$

$$\|f - \tilde{S}_{I_N^d} f\|_{L^2(\mathbb{T}^d)} \lesssim N^{-\frac{5}{2}+\epsilon} \log^{d-1} N \|f\|_{\mathcal{H}_{\text{mix}}^{\frac{5}{2}-\epsilon}(\mathbb{T}^d)}$$

Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Numerical results

sampling at rank-1 lattice nodes



Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Numerical results

d	N	$ I_N^d $	M	$\frac{M}{ I_N^d }$	rel. $L^2(\mathbb{T}^d)$ error
6	64	1 709 857	31 829 977	18.6	5.3e-05
6	128	5 137 789	192 757 285	37.5	9.9e-06
6	256	14 977 209	1 400 567 254	93.5	1.8e-06
8	8	768 609	6 027 975	7.8	9.6e-03
8	16	2 935 521	49 768 670	17.0	2.2e-03
8	32	10 665 297	359 896 131	33.7	4.8e-04
10	4	2 421 009	30 780 958	12.7	4.1e-02
10	8	10 819 089	194 144 634	17.9	1.2e-02
10	16	45 548 649	2 040 484 044	44.8	3.1e-03

Dimension incremental reconstruction

until now: approximation of function $f(\mathbf{x}) \approx \sum_{\mathbf{k} \in I} \hat{f}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}$

- given/known frequency index set I
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- search for location I of largest Fourier coefficients of f
(and compute $\hat{f}_{\mathbf{k}}, \mathbf{k} \in I$)
- search domain: (possibly) large index set $\Gamma \subset \mathbb{Z}^d$, e.g.,
full grid $\hat{G}_N^d := \{\mathbf{k} \in \mathbb{Z}^d : \|\mathbf{k}\|_{\infty} \leq N\}$, ($|\hat{G}_{64}^{10}| \approx 1.28 \cdot 10^{21}$)

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 - Gilbert, Guha, Indyk, Muthukrishnan, Strauss 2002
 - Iwen, Gilbert, Strauss 2007

Dimension incremental reconstruction

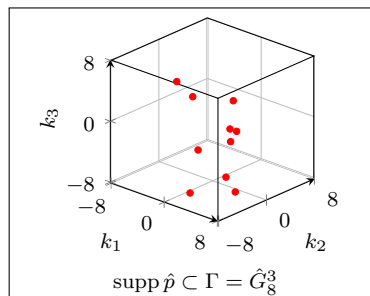
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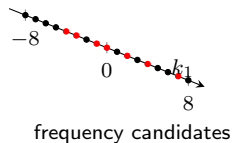
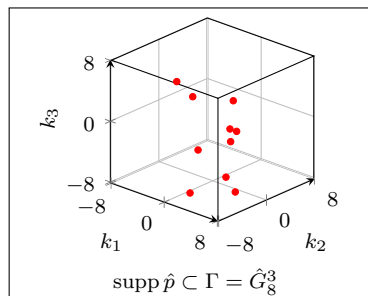
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- our method
 - compute (projected) Fourier coefficients from sampling values and determine frequency locations dimension incremental
 - use reconstructing rank-1 lattices (\Rightarrow 1-dim iFFT)

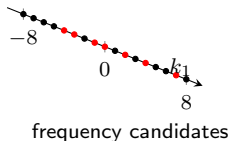
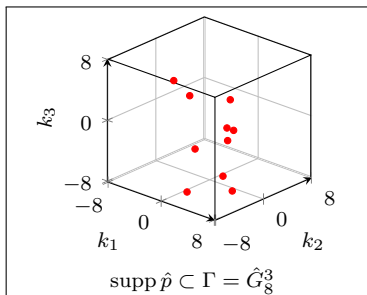
Dimension incremental reconstruction - Method



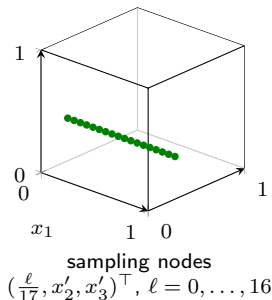
Dimension incremental reconstruction - Method



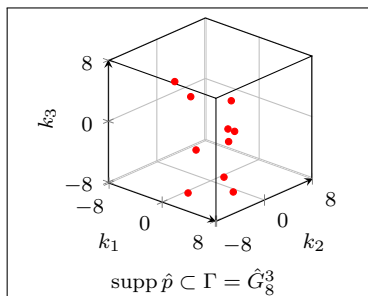
Dimension incremental reconstruction - Method



construct
→
sampling set

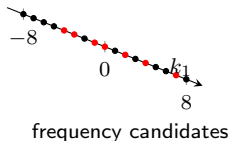


Dimension incremental reconstruction - Method

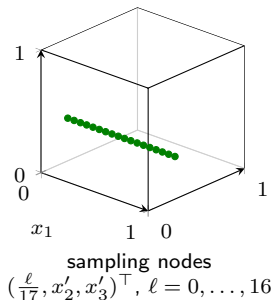


$$\tilde{p}_{k_1} := \frac{1}{17} \sum_{\ell=0}^{16} p \left(\begin{pmatrix} \ell/17 \\ x'_2 \\ x'_3 \end{pmatrix} \right) e^{-2\pi i \frac{\ell k_1}{17}}$$

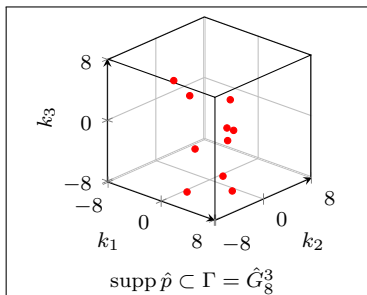
$$k_1 = -8, \dots, 8$$



1-dim
←
iFFT

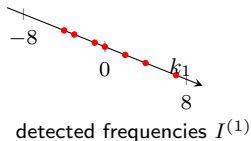


Dimension incremental reconstruction - Method

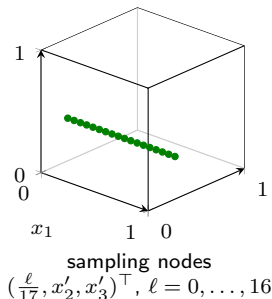


$$\begin{aligned} \tilde{p}_{k_1} &:= \frac{1}{17} \sum_{\ell=0}^{16} p \left(\begin{pmatrix} \ell/17 \\ x'_2 \\ x'_3 \end{pmatrix} \right) e^{-2\pi i \frac{\ell k_1}{17}} \\ &= \sum_{\substack{(h_2, h_3) \in \{-8, \dots, 8\}^2 \\ (k_1, h_2, h_3)^\top \in \text{supp } \hat{p}}} \hat{p} \begin{pmatrix} k_1 \\ h_2 \\ h_3 \end{pmatrix} e^{2\pi i (h_2 x'_2 + h_3 x'_3)}, \end{aligned}$$

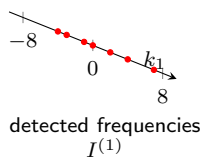
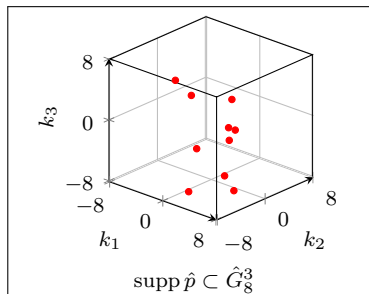
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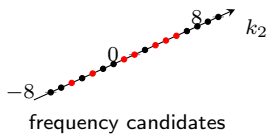
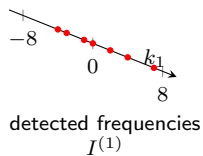
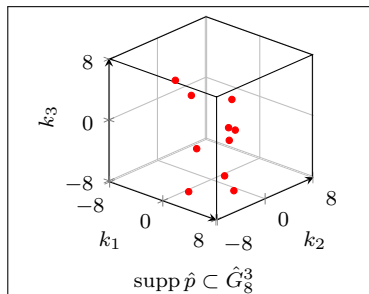
1-dim
←
iFFT



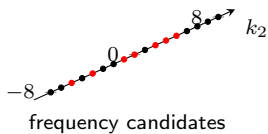
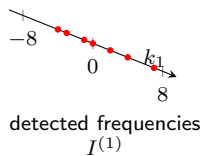
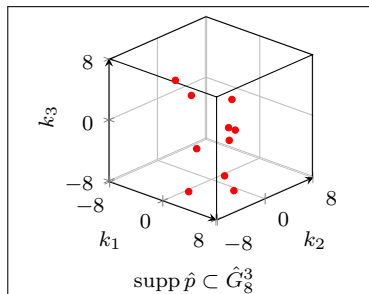
Dimension incremental reconstruction - Method



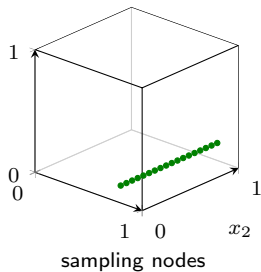
Dimension incremental reconstruction - Method



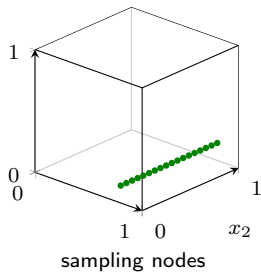
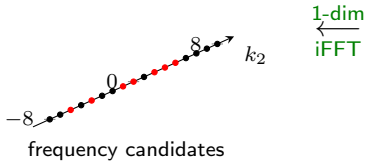
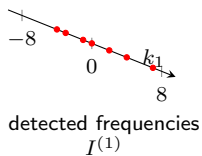
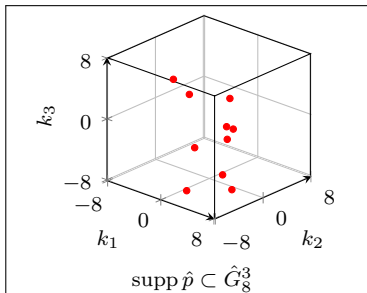
Dimension incremental reconstruction - Method



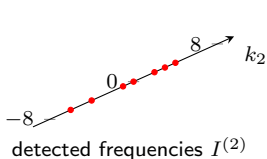
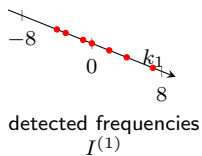
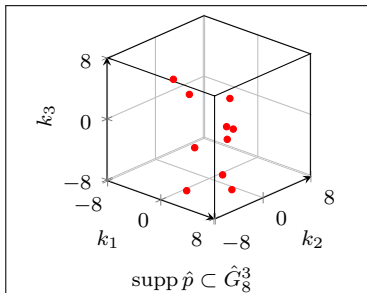
construct
→
sampling set



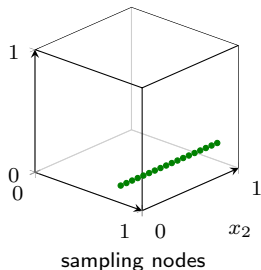
Dimension incremental reconstruction - Method



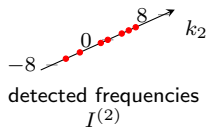
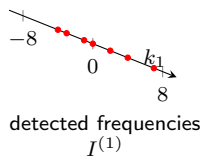
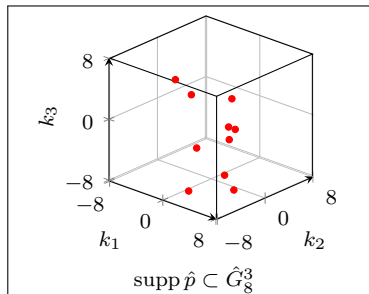
Dimension incremental reconstruction - Method



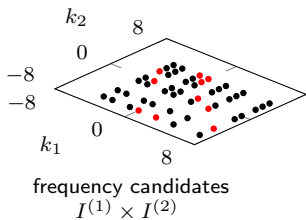
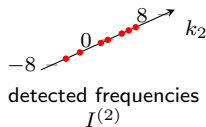
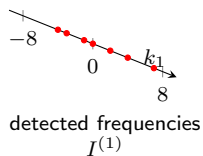
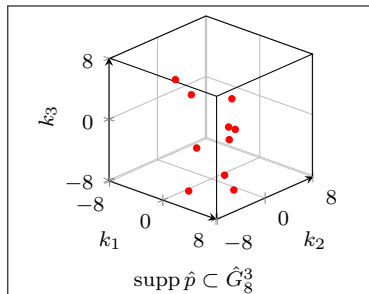
1-dim
←
iFFT



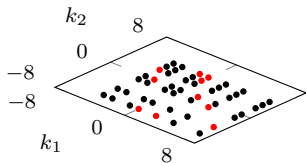
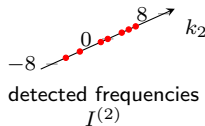
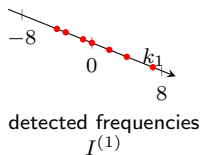
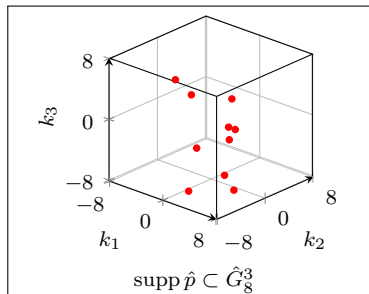
Dimension incremental reconstruction - Method



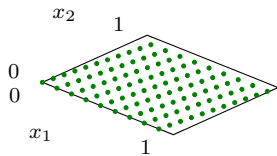
Dimension incremental reconstruction - Method



Dimension incremental reconstruction - Method



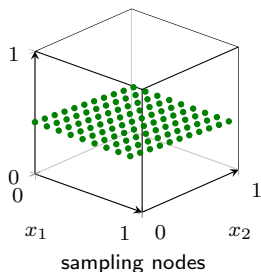
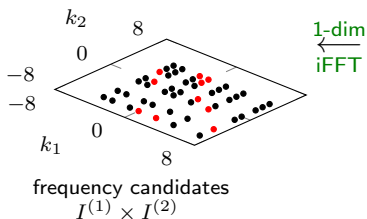
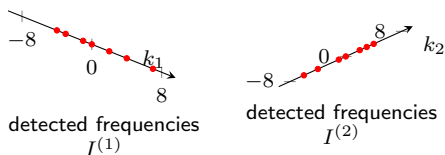
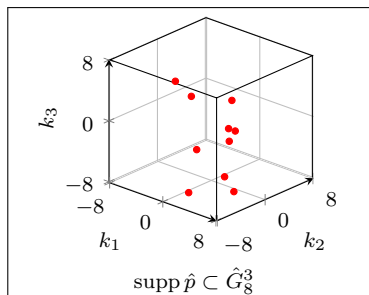
reconstructing
rank-1 lattice



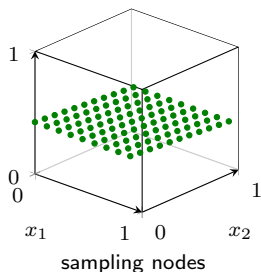
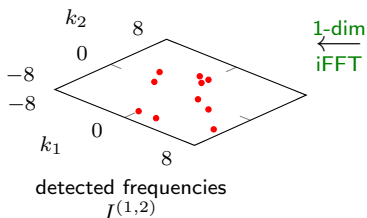
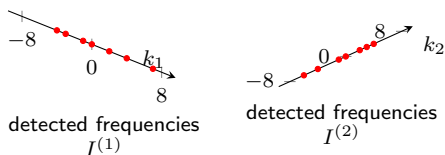
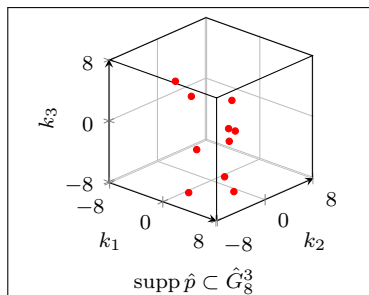
frequency candidates
 $I^{(1)} \times I^{(2)}$

sampling nodes

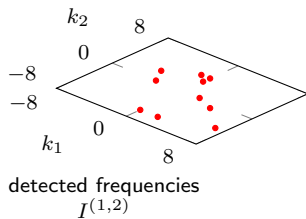
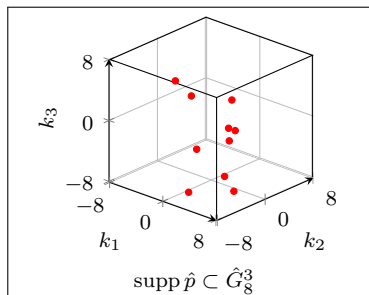
Dimension incremental reconstruction - Method



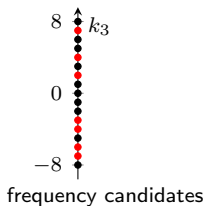
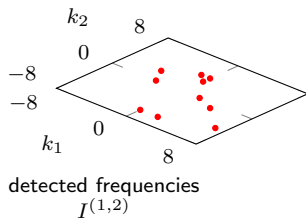
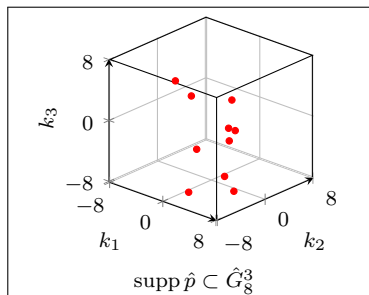
Dimension incremental reconstruction - Method



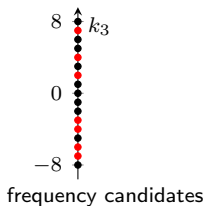
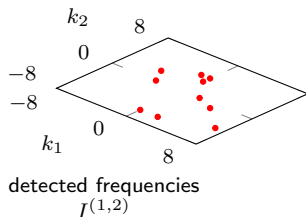
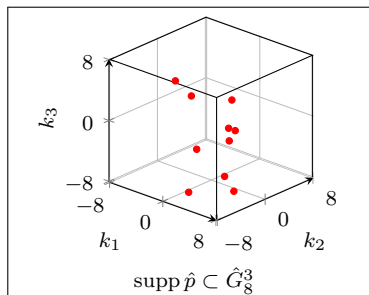
Dimension incremental reconstruction - Method



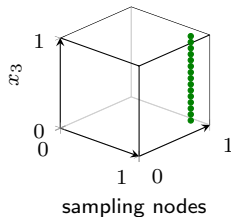
Dimension incremental reconstruction - Method



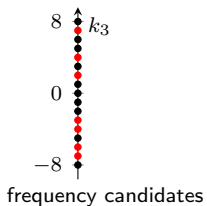
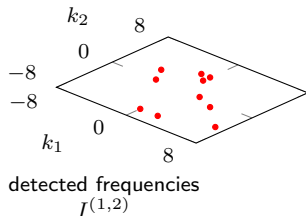
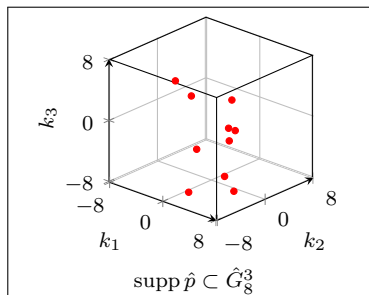
Dimension incremental reconstruction - Method



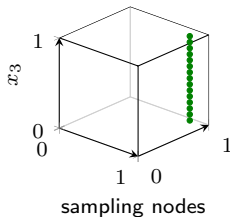
construct
→
sampling set



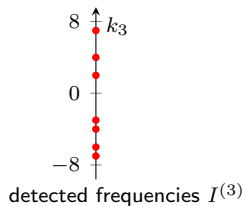
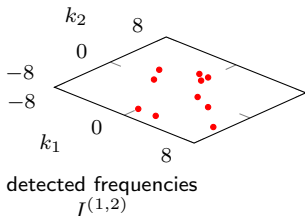
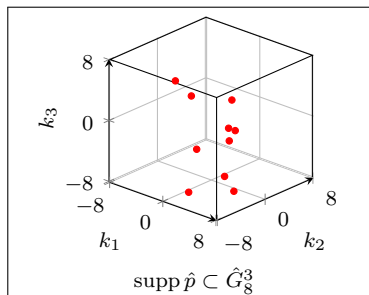
Dimension incremental reconstruction - Method



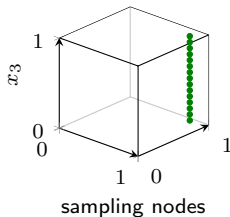
1-dim
←
iFFT



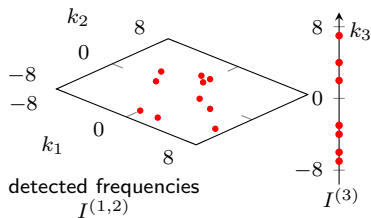
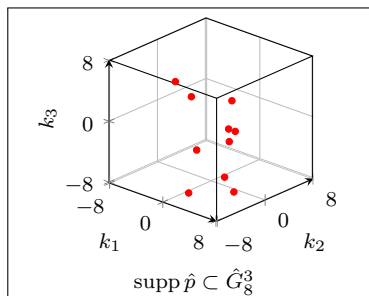
Dimension incremental reconstruction - Method



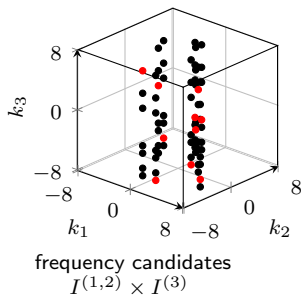
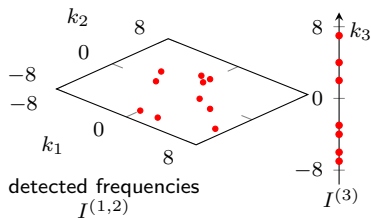
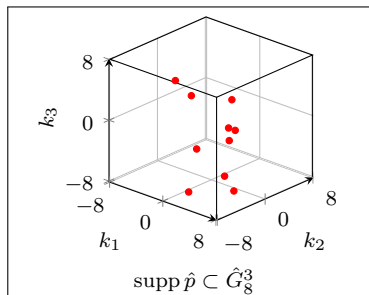
1-dim
←
iFFT



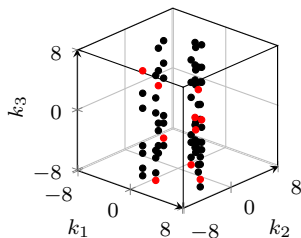
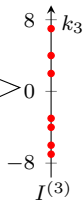
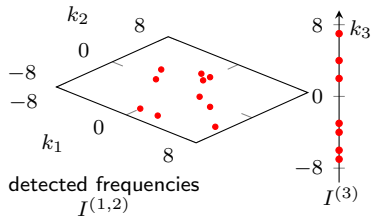
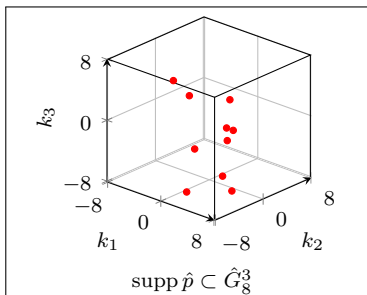
Dimension incremental reconstruction - Method



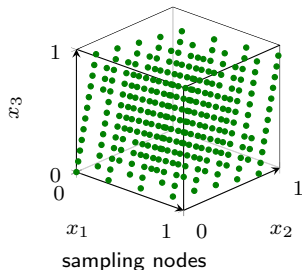
Dimension incremental reconstruction - Method



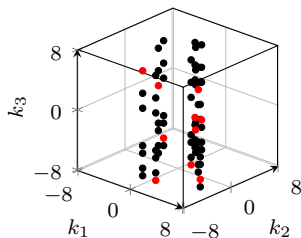
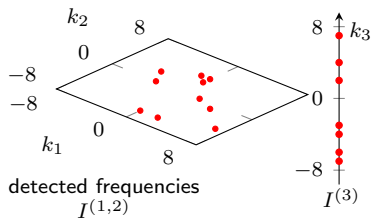
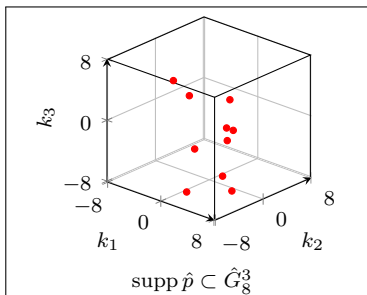
Dimension incremental reconstruction - Method



reconstructing
→
rank-1 lattice

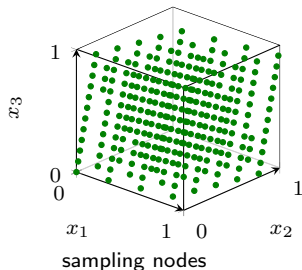


Dimension incremental reconstruction - Method

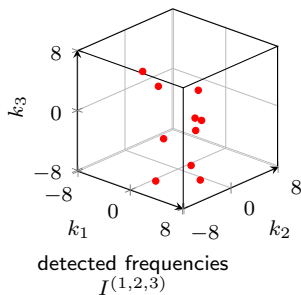
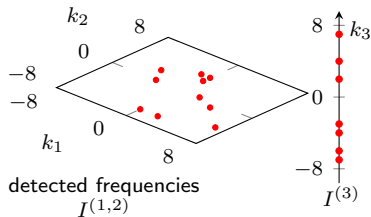
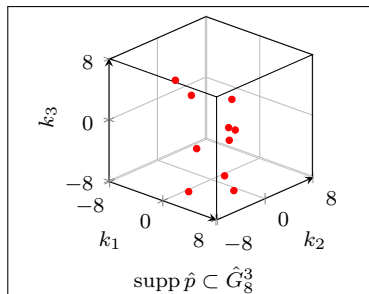


frequency candidates
 $I^{(1,2)} \times I^{(3)}$

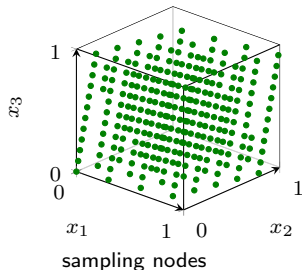
1-dim
←
iFFT



Dimension incremental reconstruction - Method



1-dim
←
iFFT



Dimension incremental reconstruction - Example

- B-spline $N_m(x) := \sum_{k \in \mathbb{Z}} C_m \operatorname{sinc}\left(\frac{\pi}{m}k\right)^m \cos(\pi k) e^{2\pi i k x}$,
 $\|N_m\|_{L^2(\mathbb{T})} = 1$, $|\hat{N}_m(k)| \sim |k|^{-m}$
- $f(\mathbf{x}) := \prod_{t \in \{1,3,8\}} N_2(x_t) + \prod_{t \in \{2,5,6,10\}} N_4(x_t) + \prod_{t \in \{4,7,9\}} N_6(x_t)$

Dimension incremental reconstruction - Example

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- full grid for $N = 64$, $d = 10$: $|\hat{G}_{64}^{10}| = 129^{10} \approx 1.28 \cdot 10^{21}$
- symmetric hyperbolic cross: $|I_{64}^{10}| = 696\,036\,321$
relative $L^2(\mathbb{T}^d)$ -error (best case) 4.1e-04

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relative $L^2(\mathbb{T}^d)$ -error (best case) 4.1e-04
- results for dimension incremental algorithm with $\Gamma = \hat{G}_{64}^{10}$
(tests repeated 10 times):

threshold	#samples	$ I $	rel. L_2 -error
1.0e-02	254 530	491	1.4e-01
1.0e-03	2 789 050	1 121	1.1e-02
1.0e-04	17 836 042	3 013	1.7e-03
1.0e-05	82 222 438	7 163	4.7e-04

Summary

approximate reconstruction of high-dimensional periodic functions $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ by sampling along **rank-1 lattice** nodes

- **perfectly stable**
- computation only uses single **1-dim iFFT** + scalar products
- samples: $\mathcal{O}(|\mathbf{I}_N^d|^2)$, arithm. complexity: $\mathcal{O}(|\mathbf{I}_N^d|^2 \log |\mathbf{I}_N^d|)$
- **oversampling factor** up to $|\mathbf{I}_N^d|$
- **observed** oversampling factor **lower** for realistic problem sizes
- theoretical estimates for approximation error
- numerical tests encourage theoretical results
- dimension incremental reconstruction method
+ numerical results

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Kämmerer, L., Potts, D., Volkmer, T.

Approximation of multivariate functions by trigonometric polynomials based on rank-1 lattice sampling.

DFG-Schwerpunktprogramm 1324, Preprint 145, 2013.

(<http://www.tu-chemnitz.de/~tovo>)