Approximation of high-dimensional multivariate periodic functions by trigonometric polynomials based on rank-1 lattice sampling

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joint work with L. Kämmerer and D. Potts

supported by DFG SPP 1324

Content

Introduction

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$$p(\mathbf{x}) := \sum_{\mathbf{k} \in \mathrm{I}} \hat{p}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}}, \qquad \hat{p}_{\mathbf{k}} \in \mathbb{C}$$

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  where the Fourier coefficients of f are given by

$$\hat{f}_{\mathbf{k}} = \int_{\mathbb{T}^d} f(\mathbf{x}) \mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{x}, \qquad \mathbf{k} \in \mathbb{Z}^d$$

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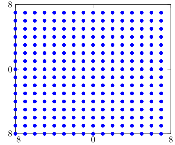
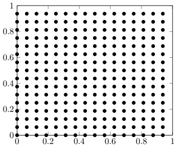
- e.g., $\mathrm{I} = \mathbb{Z}^d \cap [-N, N)^d$ full grid, $\mathbf{y}_j$ equispaced nodes



$$(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in\mathrm{I}} \overset{\text{d-dim}}{\underset{\text{FFT}}{\longleftrightarrow}} (f(\mathbf{y}_j))_{j=0}^{|\mathrm{I}|-1}$$



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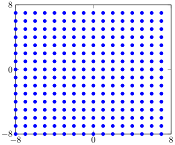

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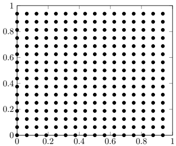
$$\mathcal{O}(N^d \log N)$$



  - problem: $|\mathrm{I}| = (2N)^d \Rightarrow$ curse of dimensionality

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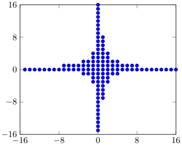

$$(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in \mathrm{I}} \quad \underset{\text{HCFFT}}{\longleftrightarrow} \quad (f(\mathbf{y}_j))_{j=0}^{|\mathrm{I}|-1}$$

(Baszenski, Delvos 1989; Hallatschek 1992; Gradinaru 2007; Griebel, Hamaekers 2013)

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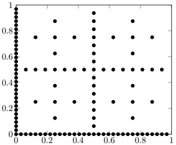

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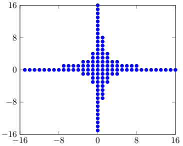

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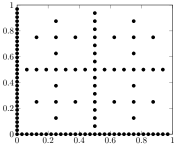

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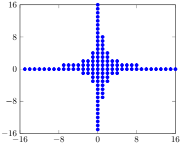

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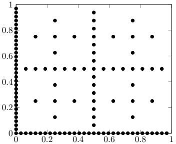

  - problem: stability (Kämmerer, Kunis 2011)
  - implementation of algorithm (HCFFT) is effortful

Introduction - Fct. Spaces $\mathcal{H}^\omega(\mathbb{T}^d)$ and Index Sets I

- Hilbert space

$$\mathcal{H}^\omega(\mathbb{T}^d) := \left\{ f \in L^2(\mathbb{T}^d) : \sqrt{\sum_{\mathbf{k}\in\mathbb{Z}^d} \omega(\mathbf{k})^2 |\hat{f}_{\mathbf{k}}|^2} < \infty \right\}$$

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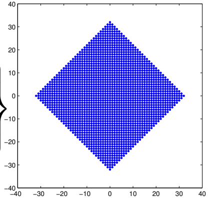

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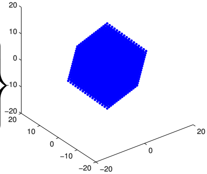

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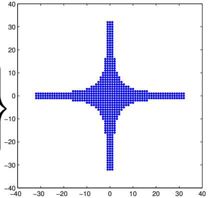

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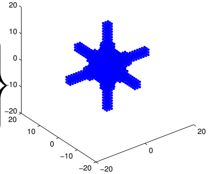

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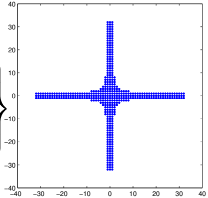

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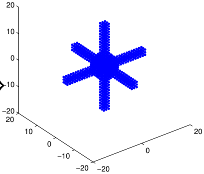

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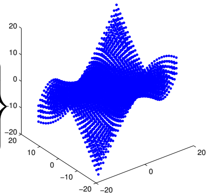

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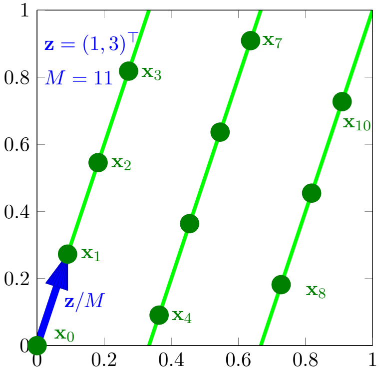
- large dimension d

Trig. polynomials - Fast evaluation at rank-1 lattices

- rank-1 lattice: $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$

$$\mathbf{x}_j = \frac{j}{M}\mathbf{z} \bmod \mathbf{1};\; j = 0, \ldots, M-1$$



Korobov 59
Maisonneuve 72
Sloan & Kachoyan 84,87,90
Temlyakov 86
Lyness 89
Sloan & Joe 94
Sloan & Reztsov 01
Li & Hickernell 03

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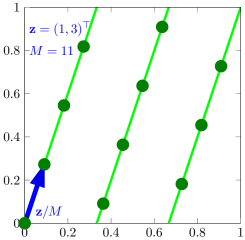

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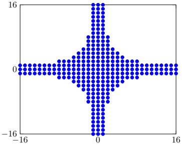
$$\mathbf{x}_j = \frac{j}{M}\mathbf{z} \bmod \mathbf{1};\; j = 0, \dots, M-1$$

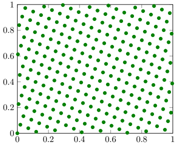
- multivariate trigonometric polynomial
  $p(\mathbf{x}) = \sum_{\mathbf{k}\in \mathrm{I}_N^d} \hat{p}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$
- reformulation

$$p(\mathbf{x}_j) = \sum_{\mathbf{k}\in \mathrm{I}_N^d} \hat{p}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\frac{j\mathbf{k}\cdot\mathbf{z}}{M}} = \sum_{l=0}^{M-1} \underbrace{\left(\sum_{\substack{\mathbf{k}\in \mathrm{I}_N^d \\ \mathbf{k}\cdot\mathbf{z}\equiv l \pmod M}} \hat{p}_{\mathbf{k}} \right)}_{\hat{g}_l} \mathrm{e}^{2\pi\mathrm{i}\frac{j\mathbf{k}\cdot\mathbf{z}}{M}} = \sum_{l=0}^{M-1} \hat{g}_l \mathrm{e}^{2\pi\mathrm{i}\frac{jl}{M}}$$



$(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in I_N^d}$

$$\longrightarrow$$

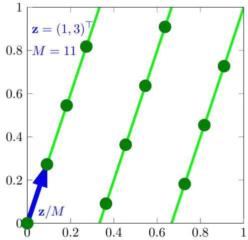


$(p(\mathbf{x}_j))_{j=0}^{M-1}$

Trig. polynomials - Fast evaluation at rank-1 lattices

- rank-1 lattice: $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$

$$\mathbf{x}_j = \frac{j}{M}\mathbf{z} \bmod \mathbf{1};\; j = 0, \ldots, M-1$$



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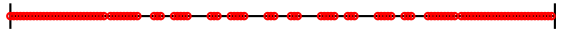
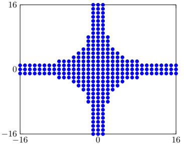


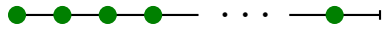
$\Big|\Big|$ $\mathbf{k}\cdot\mathbf{z} \equiv l \pmod{M}$



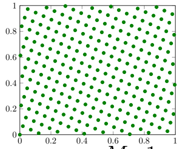
$(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in I_N^d}$

$\xrightarrow[\text{FFT}]{\text{1-dim}}$

$\mathcal{O}(M \log M + d|\mathrm{I}_N^d|)$



$\Big|\Big|$ $p(\frac{j}{M}\mathbf{z})$



$(p(\mathbf{x}_j))_{j=0}^{M-1}$

Trig. polynomials - Fast evaluation at rank-1 lattices

- rank-1 lattice: $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$

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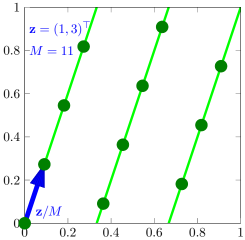
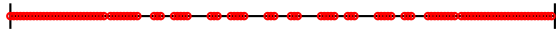


- reformulation

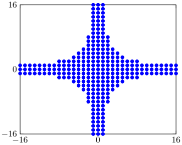
$$p(\mathbf{x}_j) = \sum_{\mathbf{k}\in \mathrm{I}_N^d} \hat{p}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \frac{j\mathbf{k}\cdot\mathbf{z}}{M}} = \sum_{l=0}^{M-1} \underbrace{\left(\sum_{\substack{\mathbf{k}\in \mathrm{I}_N^d \\ \mathbf{k}\cdot\mathbf{z} \equiv l \pmod{M}}} \hat{p}_{\mathbf{k}} \right)}_{\hat{g}_l} \mathrm{e}^{2\pi \mathrm{i} \frac{j\mathbf{k}\cdot\mathbf{z}}{M}} = \sum_{l=0}^{M-1} \hat{g}_l \mathrm{e}^{2\pi \mathrm{i} \frac{jl}{M}}$$



$\hat{g}_l$ $\Vert$ $\mathbf{k}\cdot\mathbf{z} \equiv l \pmod{M}$

$0 \quad \frac{1}{M} \quad \frac{2}{M} \quad \frac{3}{M} \quad \dots \quad \frac{M-1}{M}$

$\Vert$ $p(\frac{j}{M}\mathbf{z})$

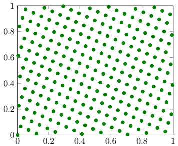


$(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in I_N^d}$

?

$\longleftarrow$

fast, stable, unique reconstruction



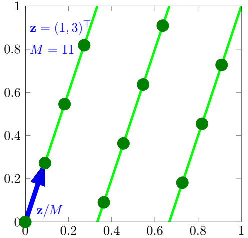
$(p(\mathbf{x}_j))_{j=0}^{M-1}$

Trig. polynomials - Fast reconstruction

- rank-1 lattice: $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$

$$\mathbf{x}_j = \frac{j}{M}\mathbf{z} \bmod \mathbf{1};\ j = 0, \ldots, M-1$$

- reconstruction of the Fourier coefficients of $p(\mathbf{x}) = \sum_{\mathbf{k}\in \mathrm{I}_N^d} \hat{p}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{x}}$ by applying a lattice rule



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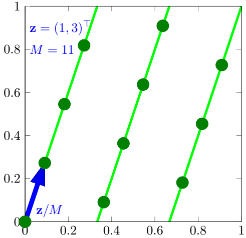

$$\hat{p}_{\mathbf{k}} = \int_{\mathbb{T}^d} p(\mathbf{x}) \mathrm{e}^{-2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}} \mathrm{d}\mathbf{x} \approx \mathrm{Q}(p(\cdot)\mathrm{e}^{-2\pi \mathrm{i} \mathbf{k} \cdot (\cdot)}) = \underbrace{\frac{1}{M} \sum_{j=0}^{M-1} p(\mathbf{x}_j) \mathrm{e}^{-2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}_j}}_{\hat{\tilde{p}}_{\mathbf{k}}}$$

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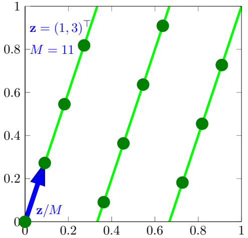

- reconstruction of the Fourier coefficients of $p(\mathbf{x}) = \sum_{\mathbf{k}\in I_N^d} \hat{p}_{\mathbf{k}} \mathrm{e}^{2\pi i \mathbf{k}\cdot\mathbf{x}}$ by applying a lattice rule

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$$\Rightarrow\ \hat{\tilde{p}}_{\mathbf{k}} = \hat{p}_{\mathbf{k}} \Leftrightarrow \mathbf{k}_1\cdot\mathbf{z} \not\equiv \mathbf{k}_2\cdot\mathbf{z} \pmod{M} \text{ for all } \mathbf{k}_1 \neq \mathbf{k}_2 \in I_N^d$$

Trig. polynomials - Fast reconstruction

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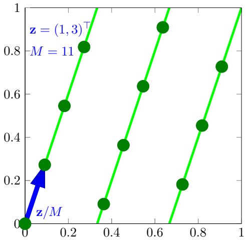

$$\hat{p}_{\mathbf{k}} = \int_{\mathbb{T}^d} p(\mathbf{x})\mathrm{e}^{-2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{x}}\mathrm{d}\mathbf{x} \approx \mathrm{Q}(p(\cdot)\mathrm{e}^{-2\pi \mathrm{i}\mathbf{k}\cdot(\cdot)}) = \underbrace{\frac{1}{M}\sum_{j=0}^{M-1} p(\mathbf{x}_j)\mathrm{e}^{-2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{x}_j}}_{\hat{\tilde{p}}_{\mathbf{k}}}$$

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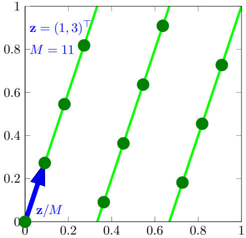
$\Rightarrow$ Definition: reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathrm{I}_N^d)$ for I_N^d,

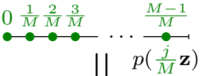
Trig. polynomials - Fast reconstruction

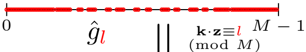
- rank-1 lattice: $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$

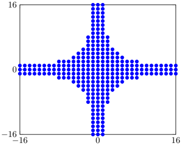
$$\mathbf{x}_j = \frac{j}{M}\mathbf{z} \bmod \mathbf{1};\ j = 0, \ldots, M-1$$

- reconstruction of the Fourier coefficients of $p(\mathbf{x}) = \sum_{\mathbf{k} \in \mathrm{I}_N^d} \hat{p}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}}$ by applying a lattice rule





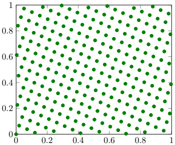




$(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in I_N^d}$

$\xleftarrow[\text{iFFT}]{\text{1-dim}}$

$\mathcal{O}(M \log M + d|\mathrm{I}_N^d|)$



$(p(\mathbf{x}_j))_{j=0}^{M-1}$

$\Rightarrow$ Definition: reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathrm{I}_N^d)$ for I_N^d, $|\mathrm{I}_N^d| \leq M \leq |\mathrm{I}_N^d|^2$, CBC construction algorithm (Kämmerer 2012)

Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Rank-1 lattice nodes

- reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathrm{I}_N^d)$:
  $\mathbf{x}_j = \frac{j}{M}\mathbf{z} \bmod \mathbf{1};\ j = 0, \ldots, M-1; \quad |\mathrm{I}_N^d| \leq M \leq |\mathrm{I}_N^d|^2$
- approximate reconstruction of the Fourier coefficients of $f \in \mathcal{H}^\omega(\mathbb{T}^d) \cap \mathcal{C}(\mathbb{T}^d)$ by applying a lattice rule

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$$\hat{f}_{\mathbf{k}} = \int_{\mathbb{T}^d} f(\mathbf{x}) \mathrm{e}^{-2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}} \mathrm{d}\mathbf{x} \approx \mathrm{Q}(f(\cdot)\mathrm{e}^{-2\pi \mathrm{i} \mathbf{k} \cdot (\cdot)}) = \underbrace{\frac{1}{M} \sum_{j=0}^{M-1} f(\mathbf{x}_j) \mathrm{e}^{-2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}_j}}_{\hat{\hat{f}}_{\mathbf{k}}}$$

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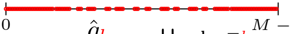
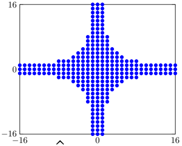


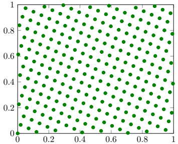
$\hat{g}_l$ $\quad ||$ $\quad \mathbf{k}\cdot\mathbf{z} \equiv l \pmod{M}$



$(\hat{\tilde{f}}_{\mathbf{k}})_{\mathbf{k}\in I_N^d}$

$\xleftarrow[\text{iFFT}]{\text{1-dim}}$

$\mathcal{O}(M \log M + d|\mathrm{I}_N^d|)$

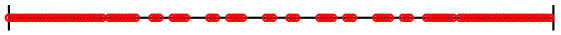


$(f(\mathbf{x}_j))_{j=0}^{M-1}$

Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Rank-1 lattice nodes

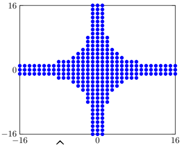
- reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathrm{I}_N^d)$:
  $\mathbf{x}_j = \frac{j}{M}\mathbf{z} \bmod \mathbf{1};\ j = 0, \ldots, M-1; \quad |\mathrm{I}_N^d| \le M \le |\mathrm{I}_N^d|^2$
- approximate reconstruction of the Fourier coefficients of $f \in \mathcal{H}^\omega(\mathbb{T}^d) \cap \mathcal{C}(\mathbb{T}^d)$ by applying a lattice rule

$$\hat{f}_{\mathbf{k}} = \int_{\mathbb{T}^d} f(\mathbf{x}) \mathrm{e}^{-2\pi \mathrm{i} \mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{x} \approx \mathrm{Q}(f(\cdot)\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot(\cdot)}) = \underbrace{\frac{1}{M}\sum_{j=0}^{M-1} f(\mathbf{x}_j)\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}_j}}_{\hat{\tilde{f}}_{\mathbf{k}}}$$



$0 \qquad \hat{g}_l \qquad M-1$

$\Vert \quad \mathbf{k}\cdot\mathbf{z} \equiv l \pmod{M}$



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$\xleftarrow[\text{iFFT}]{\text{1-dim}}$

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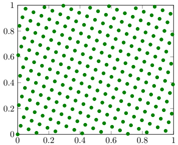

$(f(\mathbf{x}_j))_{j=0}^{M-1}$

- approximation $\tilde{S}_{\mathrm{I}_N^d} f$ of f by $\tilde{S}_{\mathrm{I}_N^d} f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathrm{I}_N^d} \hat{\tilde{f}}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$

Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Error estimates

$$\underbrace{\|f - \tilde{S}_{\mathrm{I}_N^d} f | L^2(\mathbb{T}^d)\|}_{\text{approximation error}} \leq \underbrace{\|f - S_{\mathrm{I}_N^d} f | L^2(\mathbb{T}^d)\|}_{\text{truncation error}} + \underbrace{\|S_{\mathrm{I}_N^d} f - \tilde{S}_{\mathrm{I}_N^d} f | L^2(\mathbb{T}^d)\|}_{\text{aliasing error}}$$

Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Error estimates

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$$\|f - S_{\mathrm{I}_N^d} f | L^2(\mathbb{T}^d)\| \quad \leq \quad \frac{1}{N} \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d} \omega(\mathbf{k})^2 \left| \hat{f}_{\mathbf{k}} \right|^2} = \frac{1}{N} \, \|f | \mathcal{H}^\omega(\mathbb{T}^d)\|$$

Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Error estimates

$$\underbrace{\|f - \tilde{S}_{\mathrm{I}_N^d} f | L^2(\mathbb{T}^d)\|}_{\text{approximation error}} \leq \underbrace{\|f - S_{\mathrm{I}_N^d} f | L^2(\mathbb{T}^d)\|}_{\text{truncation error}} + \underbrace{\|S_{\mathrm{I}_N^d} f - \tilde{S}_{\mathrm{I}_N^d} f | L^2(\mathbb{T}^d)\|}_{\text{aliasing error}}$$

$$\|f - S_{\mathrm{I}_N^d} f | L^2(\mathbb{T}^d)\| \leq \frac{1}{N} \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d} \omega(\mathbf{k})^2 \left|\hat{f}_{\mathbf{k}}\right|^2} = \frac{1}{N} \|f | \mathcal{H}^\omega(\mathbb{T}^d)\|$$

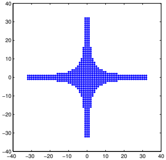
$$\|S_{\mathrm{I}_N^d} f - \tilde{S}_{\mathrm{I}_N^d} f | L^2(\mathbb{T}^d)\| \leq \frac{1}{N} \left(1 + 2\zeta(2\lambda)\right)^{\frac{d}{2}} \cdot \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d} \omega(\mathbf{k})^2 \left|\hat{f}_{\mathbf{k}}\right|^2 \prod_{s=1}^{d} \max(1, |k_s|)^{2\lambda}}$$

$(\lambda > 1/2)$

Reconstruction of $f \in \mathcal{H}^{\omega}(\mathbb{T}^d)$ - Error estimates

Example: hyperbolic cross index set: $|\mathrm{I}_N^d| \in \mathcal{O}\left(N \log^{d-1} N\right)$

$$\mathrm{I}_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d \colon \prod_{s=1}^{d} \max(1, |k_s|) \leq N \right\}$$



weights: $\omega(\mathbf{k}) := \omega^{0,\beta}(\mathbf{k}) := \prod_{s=1}^{d} \max(1, |k_s|)^{\beta}$

Hilbert space:

$$\mathcal{H}^{0,\beta}(\mathbb{T}^d) := \left\{ f \colon \|f|\mathcal{H}^{0,\beta}(\mathbb{T}^d)\| := \sqrt{\sum_{\mathbf{k}\in\mathbb{Z}^d} \omega^{0,\beta}(\mathbf{k})^2 |\hat{f}_{\mathbf{k}}|^2} < \infty \right\}$$

samples: $\mathcal{O}\left(N^2 \log^{d-2} N\right)$

arithmetic complexity: $\mathcal{O}\left(N^2 \log^{d-1} N\right)$

error estimate: $\beta > t \geq 0$, $\lambda > 1/2$

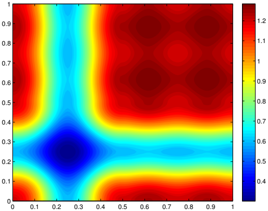
$$\|f - \tilde{S}_{\mathrm{I}_N^d} f|\mathcal{H}^{0,t}(\mathbb{T}^d)\| \leq (1 + 2\zeta(2\lambda))^{\frac{d}{2}} N^{t-\beta} \|f|\mathcal{H}^{0,\beta+\lambda}(\mathbb{T}^d)\|$$

Reconstruction of $f \in \mathcal{H}^{\omega}(\mathbb{T}^d)$ - Numerical results

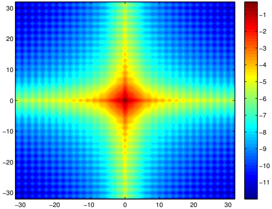
- function $f \in \mathcal{H}^{0,\frac{5}{2}-\epsilon}(\mathbb{T}^d)$, $\epsilon > 0$,

$$f(\mathbf{x}) := \prod_{s=1}^{d} \left(4 + \operatorname{sgn}\left(x_s - \frac{1}{2}\right) \sin\left(2\pi x_s\right)^2 + \operatorname{sgn}\left(x_s - \frac{1}{2}\right) \sin\left(2\pi x_s\right)^3 \right),$$

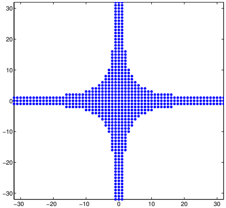
- hyperbolic cross index set $\mathrm{I}_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d \colon \prod_{s=1}^{d} \max(1, |k_s|) \le N \right\}$



$f\left((x_1, x_2)^\top\right)$



$\log_{10} \left| \hat{f}_{(k_1,k_2)^\top} \right|$



I_N^d

Reconstruction of $f \in \mathcal{H}^{\omega}(\mathbb{T}^d)$ - Numerical results

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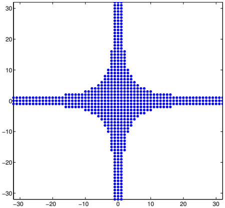
- error estimate: $\tilde{\epsilon} > 0$, $\lambda > 1/2$

$$\|f - \tilde{S}_{\mathrm{I}_N^d} f | L^2(\mathbb{T}^d)\| \lesssim N^{-2+\tilde{\epsilon}} \|f | \mathcal{H}^{0,2-\tilde{\epsilon}+\lambda}(\mathbb{T}^d)\|$$



Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Numerical results

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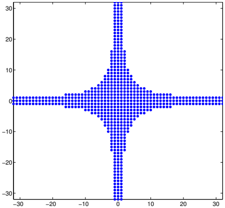
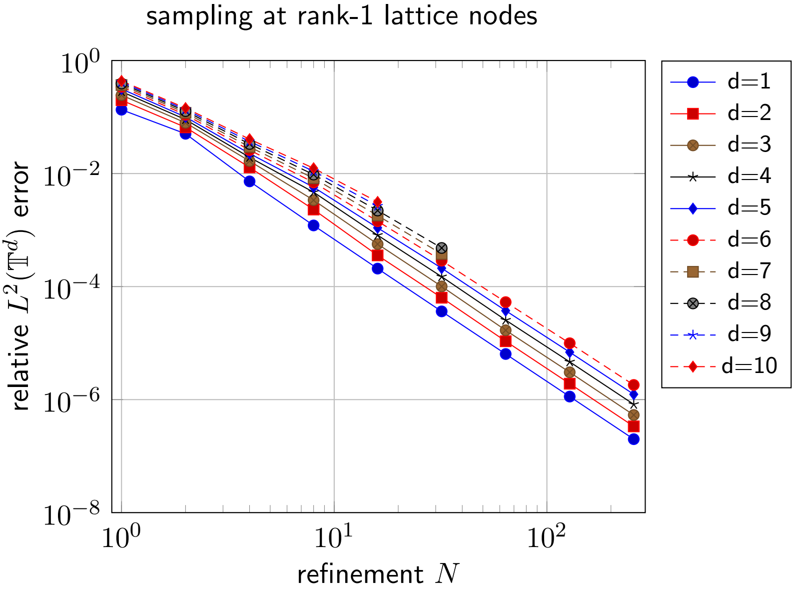


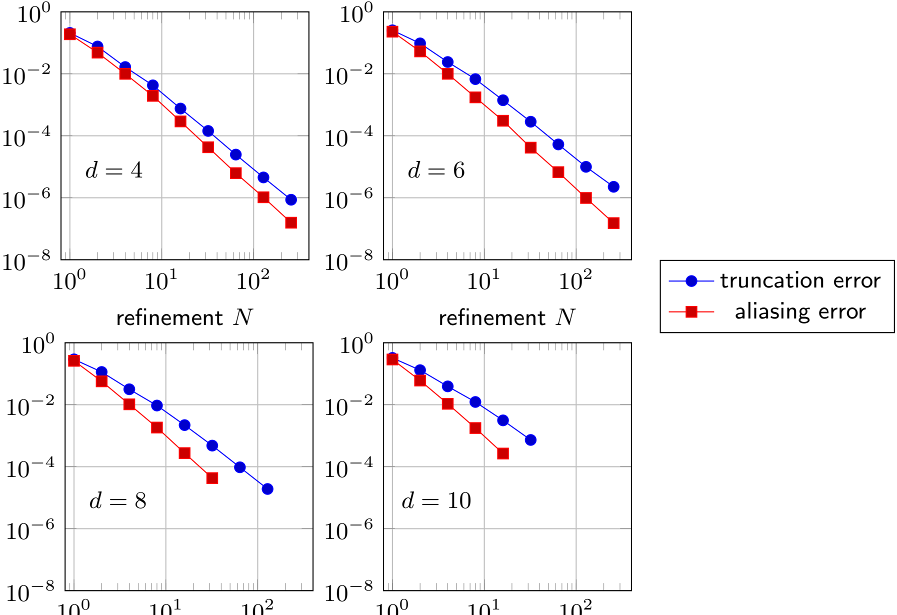
- we compute the relative $L^2(\mathbb{T}^d) = \mathcal{H}^{0,0}(\mathbb{T}^d)$
  - i.e., $\|f - \tilde{S}_{\mathrm{I}_N^d} f | L^2(\mathbb{T}^d)\| / \|f | L^2(\mathbb{T}^d)\|$
  - corresponds to the above error estimate with $r = t = 0$ up to a "constant" since

$$\frac{\|f - \tilde{S}_{\mathrm{I}_N^d} f | \mathcal{H}^{0,0,\gamma}(\mathbb{T}^d)\|}{\|f | \mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)\|} = \underbrace{\frac{\|f | L^2(\mathbb{T}^d)\|}{\|f | \mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)\|}}_{\leq 1} \frac{\|f - \tilde{S}_{\mathrm{I}_N^d} f | L^2(\mathbb{T}^d)\|}{\|f | L^2(\mathbb{T}^d)\|}$$

Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Numerical results



Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Numerical results



Reconstruction of $f \in \mathcal{H}^{\omega}(\mathbb{T}^d)$ - Numerical results

d	N	$\lvert I_N^d \rvert$	M	$\frac{M}{\lvert I_N^d \rvert}$	rel. $L^2(\mathbb{T}^d)$ error
6	64	1 709 857	31 829 977	18.6	5.3e-05
6	128	5 137 789	192 757 285	37.5	9.9e-06
6	256	14 977 209	1 400 567 254	93.5	1.8e-06
8	8	768 609	6 027 975	7.8	9.6e-03
8	16	2 935 521	49 768 670	17.0	2.2e-03
8	32	10 665 297	359 896 131	33.7	4.8e-04
10	4	2 421 009	30 780 958	12.7	4.1e-02
10	8	10 819 089	194 144 634	17.9	1.2e-02
10	16	45 548 649	2 040 484 044	44.8	3.1e-03

Dimension incremental reconstruction

until now: approximation of function $f(\mathbf{x}) \approx \sum_{\mathbf{k} \in \mathrm{I}} \hat{\tilde{f}}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}}$

- given/known frequency index set I
- compute $\hat{\tilde{f}}_{\mathbf{k}}$ from samples along reconstructing rank-1 lattice

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next:

- search for location I of largest Fourier coefficients of f
  (and compute $\hat{\tilde{\hat{f}}}_{\mathbf{k}}$, $\mathbf{k} \in I$)
- search domain: (possibly) large index set $\Gamma \subset \mathbb{Z}^d$, e.g.,
  full grid $\hat{G}_N^d := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \|\boldsymbol{k}\|_\infty \leq N\}$, ($|\hat{G}_{64}^{10}| \approx 1.28 \cdot 10^{21}$)

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  - Gilbert, Guha, Indyk, Muthukrishnan, Strauss 2002
  - Iwen, Gilbert, Strauss 2007

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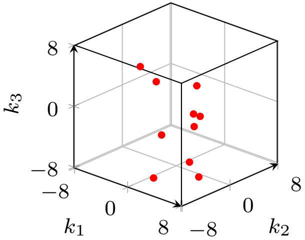
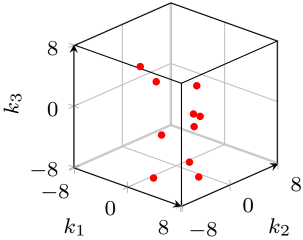
- several existing methods from compressed sensing, e.g.
  - Gilbert, Guha, Indyk, Muthukrishnan, Strauss 2002
  - Iwen, Gilbert, Strauss 2007
- our method
  - compute (projected) Fourier coefficients from sampling values and determine frequency locations dimension incremental
  - use reconstructing rank-1 lattices ($\Rightarrow$ 1-dim iFFT)

Dimension incremental reconstruction - Method



$\operatorname{supp} \hat{p} \subset \Gamma = \hat{G}_8^3$

Dimension incremental reconstruction - Method



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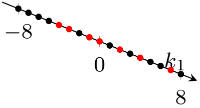
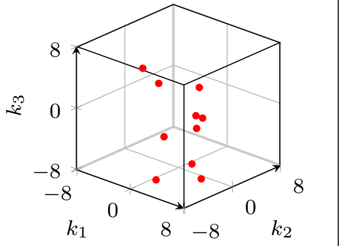


frequency candidates

Dimension incremental reconstruction - Method



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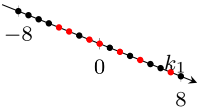
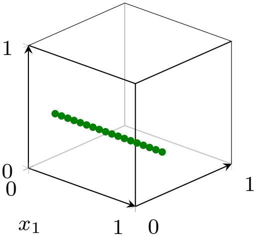


frequency candidates

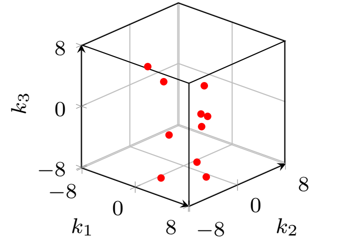
construct $\longrightarrow$ sampling set



sampling nodes

$(\frac{\ell}{17}, x_2', x_3')^\top$, $\ell = 0, \ldots, 16$

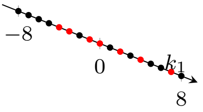
Dimension incremental reconstruction - Method



$\operatorname{supp}\hat{p} \subset \Gamma = \hat{G}_8^3$

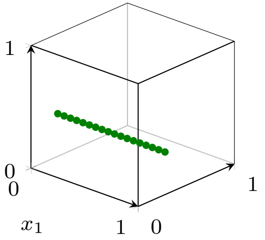
$$\tilde{\hat{p}}_{k_1} := \frac{1}{17} \sum_{\ell=0}^{16} p\left(\begin{pmatrix} \ell/17 \\ x_2' \\ x_3' \end{pmatrix}\right) \mathrm{e}^{-2\pi\mathrm{i}\frac{\ell k_1}{17}}$$

$k_1 = -8, \ldots, 8$



frequency candidates

1-dim ⟵ iFFT



sampling nodes

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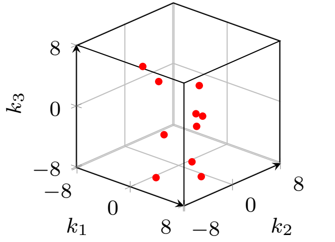
Dimension incremental reconstruction - Method



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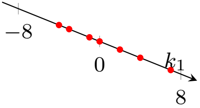
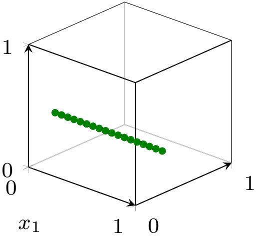
$$= \sum_{\substack{(h_2,h_3)\in\{-8,\ldots,8\}^2 \\ (k_1,h_2,h_3)^\top\in\operatorname{supp}\hat{p}}} \hat{p}_{\begin{pmatrix} k_1 \\ h_2 \\ h_3 \end{pmatrix}} \mathrm{e}^{2\pi\mathrm{i}(h_2 x_2' + h_3 x_3')},$$

$k_1 = -8, \ldots, 8$



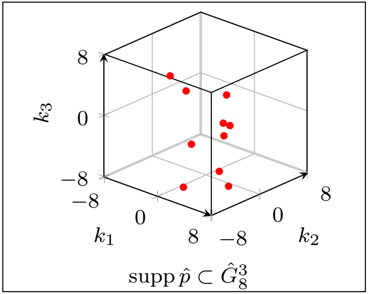
detected frequencies $I^{(1)}$

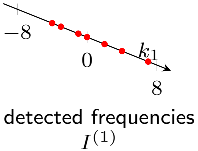
1-dim
$\longleftarrow$
iFFT



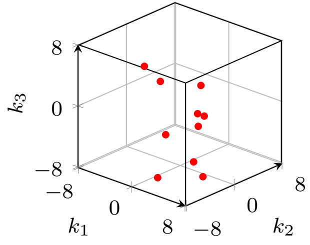
sampling nodes
$(\frac{\ell}{17}, x_2', x_3')^\top$, $\ell = 0, \ldots, 16$

Dimension incremental reconstruction - Method

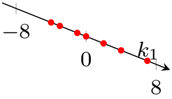




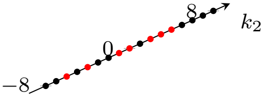
Dimension incremental reconstruction - Method



$\operatorname{supp} \hat{p} \subset \hat{G}_8^3$

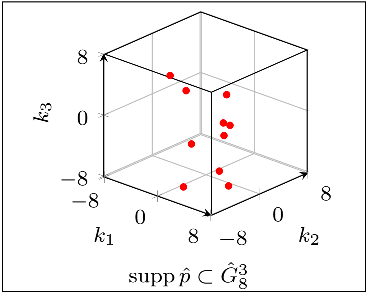


detected frequencies
$I^{(1)}$

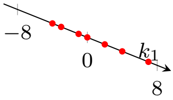


frequency candidates

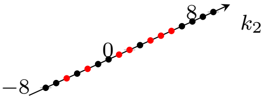
Dimension incremental reconstruction - Method



$\operatorname{supp}\hat{p} \subset \hat{G}_8^3$

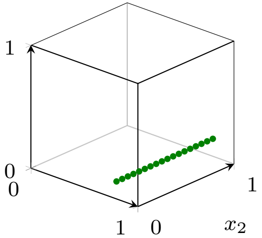


detected frequencies
$I^{(1)}$



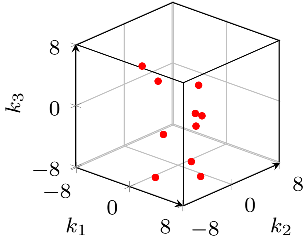
frequency candidates

reconstructing $\longrightarrow$ rank-1 lattice

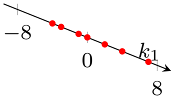


sampling nodes

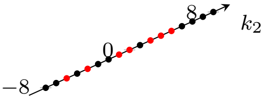
Dimension incremental reconstruction - Method



$\operatorname{supp} \hat{p} \subset \hat{G}_8^3$

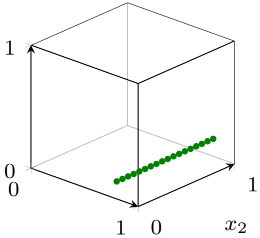


detected frequencies
$I^{(1)}$



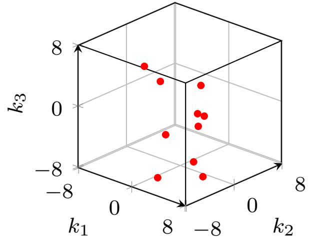
frequency candidates

1-dim ⟵ iFFT

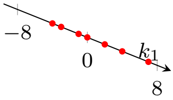


sampling nodes

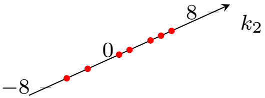
Dimension incremental reconstruction - Method



$\operatorname{supp} \hat{p} \subset \hat{G}_8^3$

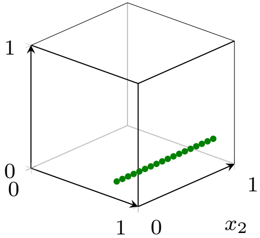


detected frequencies $I^{(1)}$



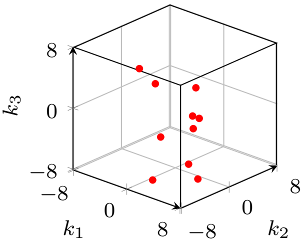
detected frequencies $I^{(2)}$

1-dim ← iFFT

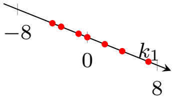


sampling nodes

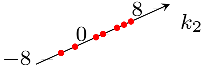
Dimension incremental reconstruction - Method



$\operatorname{supp} \hat{p} \subset \hat{G}_8^3$

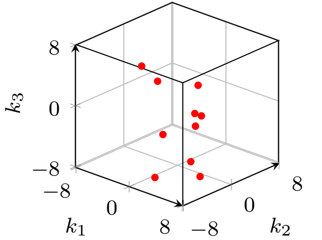


detected frequencies $I^{(1)}$

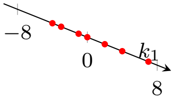


detected frequencies $I^{(2)}$

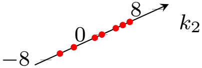
Dimension incremental reconstruction - Method



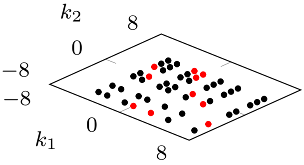
$\operatorname{supp} \hat{p} \subset \hat{G}_8^3$



detected frequencies
$I^{(1)}$

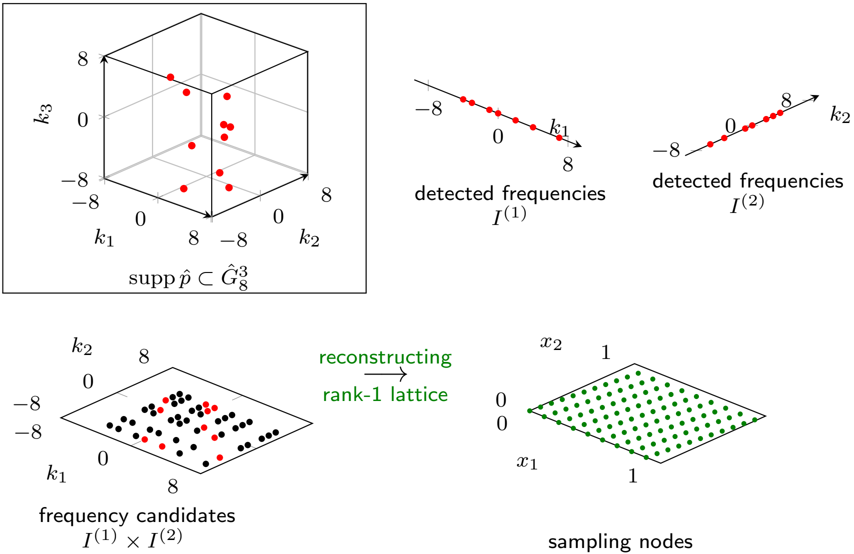


detected frequencies
$I^{(2)}$

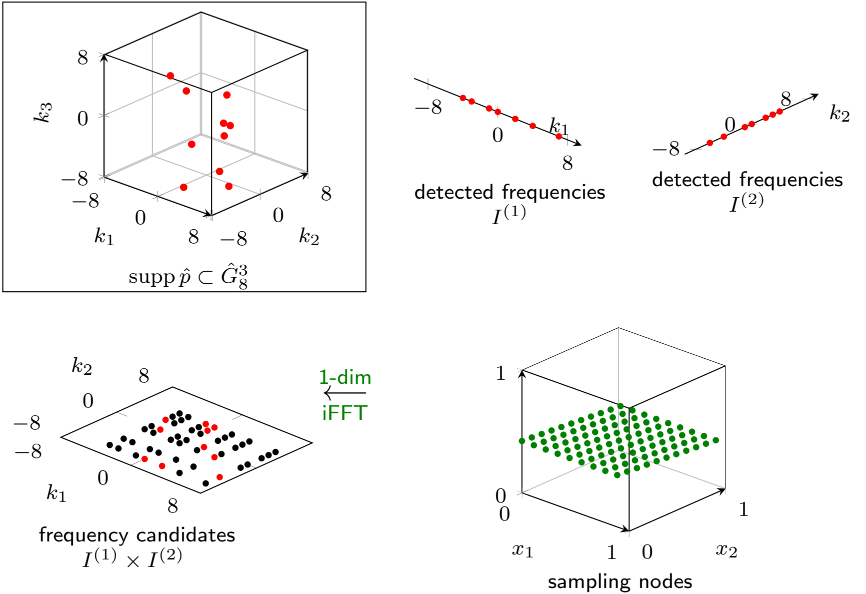


frequency candidates
$I^{(1)} \times I^{(2)}$

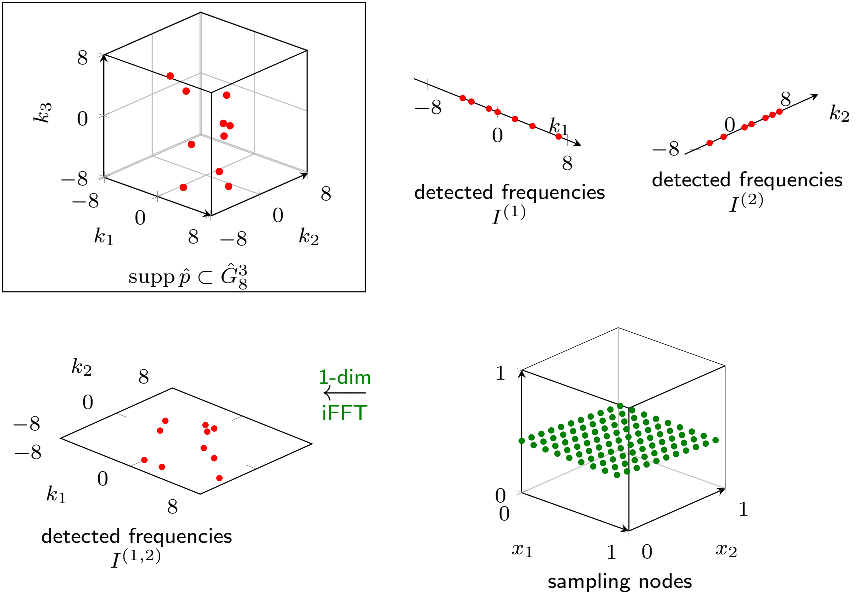
Dimension incremental reconstruction - Method

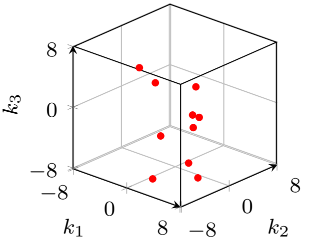


Dimension incremental reconstruction - Method

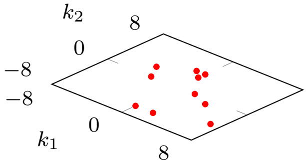


Dimension incremental reconstruction - Method



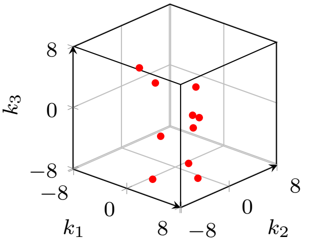


$\operatorname{supp}\hat{p} \subset \hat{G}_8^3$

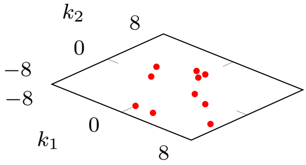


detected frequencies
$I^{(1,2)}$

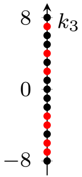
Dimension incremental reconstruction - Method



$\operatorname{supp} \hat{p} \subset \hat{G}_8^3$

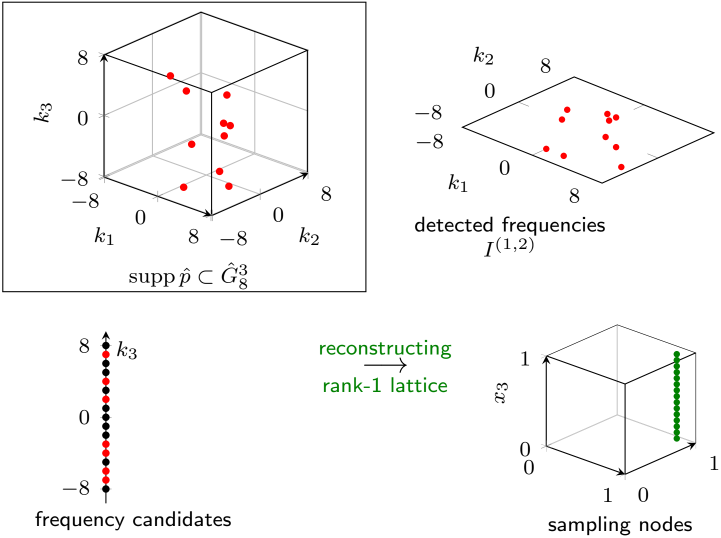


detected frequencies $I^{(1,2)}$

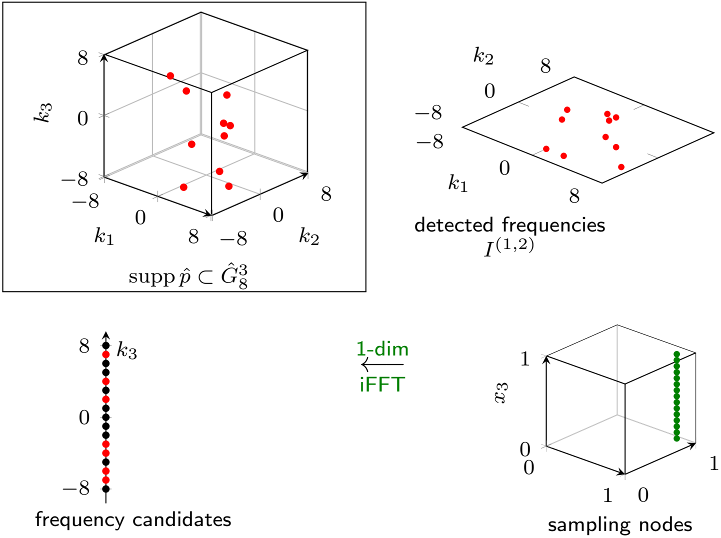


frequency candidates

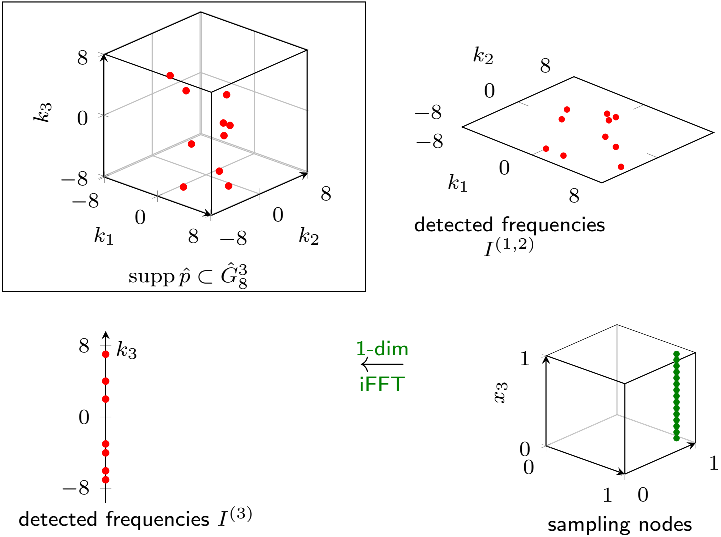
Dimension incremental reconstruction - Method



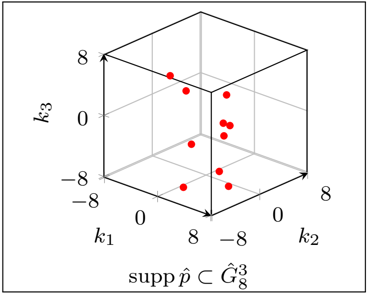
Dimension incremental reconstruction - Method

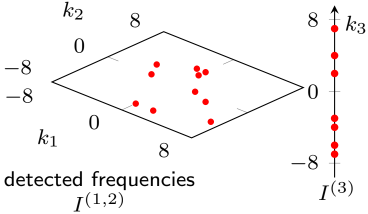


Dimension incremental reconstruction - Method

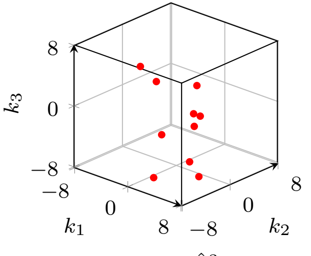


Dimension incremental reconstruction - Method

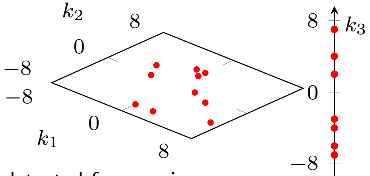




Dimension incremental reconstruction - Method

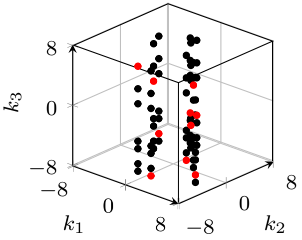


$\operatorname{supp}\hat{p} \subset \hat{G}_8^3$



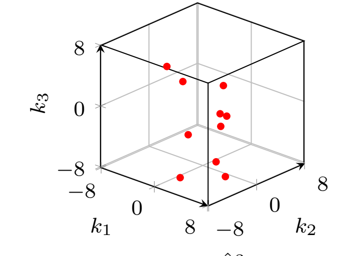
detected frequencies
$I^{(1,2)}$

$I^{(3)}$



frequency candidates
$I^{(1,2)} \times I^{(3)}$

Dimension incremental reconstruction - Method



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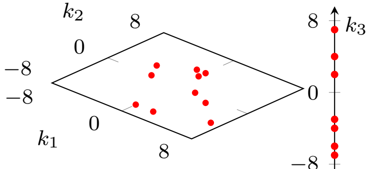

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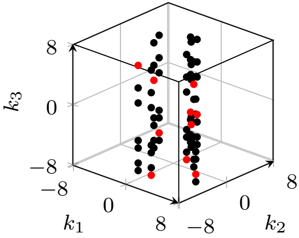
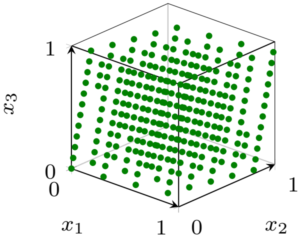


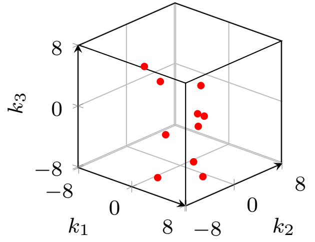
frequency candidates
$I^{(1,2)} \times I^{(3)}$

reconstructing $\longrightarrow$ rank-1 lattice

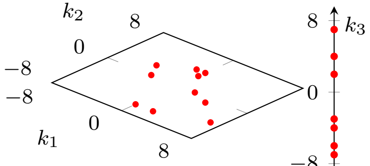


sampling nodes

Dimension incremental reconstruction - Method

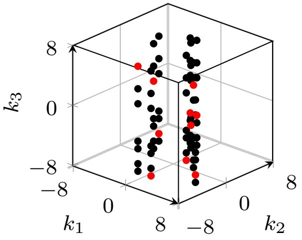


$\operatorname{supp} \hat{p} \subset \hat{G}_8^3$



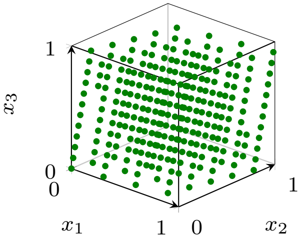
detected frequencies
$I^{(1,2)}$

$I^{(3)}$



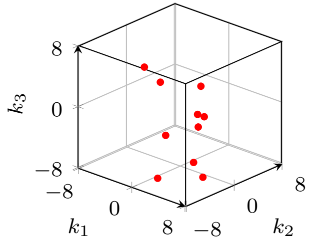
frequency candidates
$I^{(1,2)} \times I^{(3)}$

1-dim
$\longleftarrow$
iFFT



sampling nodes

Dimension incremental reconstruction - Method



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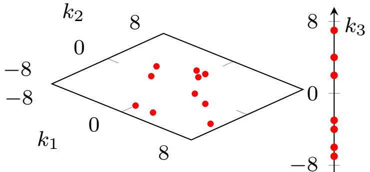
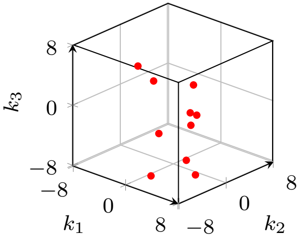


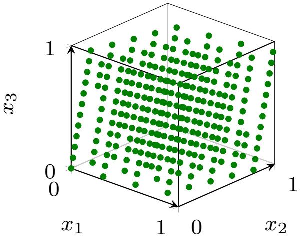
detected frequencies
$I^{(1,2)}$

$I^{(3)}$



detected frequencies
$I^{(1,2,3)}$

1-dim
$\longleftarrow$
iFFT



sampling nodes

Dimension incremental reconstruction - Example

- B-spline $N_m(x) := \sum_{k\in\mathbb{Z}} C_m \operatorname{sinc}\left(\frac{\pi}{m}k\right)^m \cos(\pi k)\, \mathrm{e}^{2\pi \mathrm{i} k x}$, $\|N_m|L^2(\mathbb{T})\| = 1$, $|\hat{N}_m(k)| \sim |k|^{-m}$
- $f(\boldsymbol{x}) := \prod_{t\in\{1,3,8\}} N_2(x_t) + \prod_{t\in\{2,5,6,10\}} N_4(x_t) + \prod_{t\in\{4,7,9\}} N_6(x_t)$

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- symmetric hyperbolic cross: $|I_{64}^{10}| = 696\,036\,321$

Dimension incremental reconstruction - Example

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- full grid for $N = 64$, $d = 10$: $|\hat{G}_{64}^{10}| = 129^{10} \approx 1.28 \cdot 10^{21}$
- symmetric hyperbolic cross: $|I_{64}^{10}| = 696\,036\,321$
- results for dimension incremental algorithm with $\Gamma = \hat{G}_{64}^{10}$ (tests repeated 10 times):

threshold	$\|I\|$	max cand	max M	#samples	rel. L_2-error
1.0e-02	491	3 885	21 970	254 530	1.4e-01
1.0e-03	1 121	27 521	217 494	2 789 050	1.1e-02
1.0e-04	3 013	123 195	903 906	17 836 042	1.7e-03
1.0e-05	7 163	256 065	7 820 238	82 222 438	4.7e-04

Summary

approximate reconstruction of high-dimensional periodic functions $f \in \mathcal{H}^{\omega}(\mathbb{T}^d)$ by sampling along rank-1 lattice nodes

- perfectly stable
- computation only uses single 1-dim iFFT + scalar products
- oversampling factor up to $|\mathrm{I}_N^d|$
- arithmetic complexity $\mathcal{O}\left(|\mathrm{I}_N^d|^2 \log |\mathrm{I}_N^d|\right)$
- observed oversampling factor lower for realistic problem sizes
- theoretical estimates for approximation error
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Kämmerer, L., Potts, D., Volkmer, T.
Approximation of multivariate periodic functions by trigonometric polynomials based on sampling along rank-1 lattice with generating vector of Korobov form.
DFG-Schwerpunktprogramm 1324, Preprint 159, 2014.
(http://www.tu-chemnitz.de/∼tovo)