

# Approximation hochdimensionaler multivariater periodischer Funktionen durch trigonometrische Polynome

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joint work with L. Kämmerer and D. Potts

supported by



## Introduction

## Multivariate trigonometric polynomials

- Fast evaluation at rank-1 lattices

- Fast, exact and stable reconstruction

## Approximate reconstruction of functions $f \in \mathcal{H}^\omega(\mathbb{T}^d)$

- by sampling at rank-1 lattice nodes

- Error estimates

- Numerical results

- Improved error estimates

## Summary

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where the Fourier coefficients of  $f$  are given by

$$\hat{f}_{\mathbf{k}} = \int_{\mathbb{T}^d} f(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} d\mathbf{x}, \quad \mathbf{k} \in \mathbb{Z}^d$$

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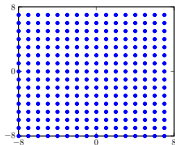
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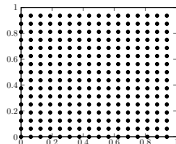
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- e.g.,  $I = \mathbb{Z}^d \cap [-N, N)^d$  full grid,  $\mathbf{y}_j$  equispaced nodes



$$(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in I} \begin{array}{c} \xleftrightarrow{\text{d-dim}} \\ \xleftrightarrow{\text{FFT}} \end{array} (f(\mathbf{y}_j))_{j=0}^{|\mathbf{I}|-1}$$



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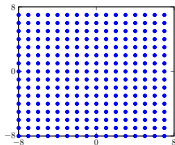
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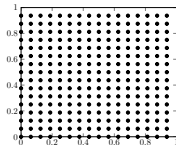
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$$\mathcal{O}(N^d \log N)$$



- problem:  $|I| = (2N)^d \Rightarrow$  **curse of dimensionality**

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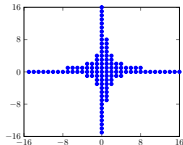
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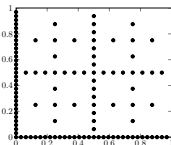
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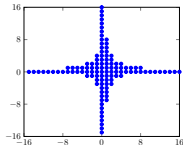
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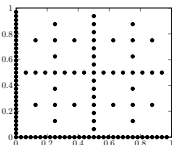
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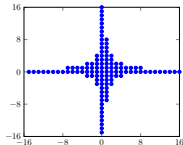
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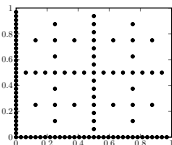
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- **implementation** of algorithm (HCFFT) is **effortful**

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- Hilbert space

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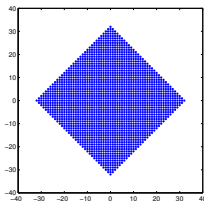
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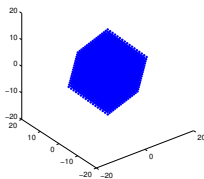
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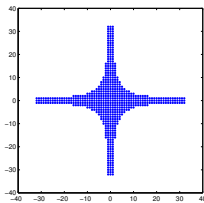
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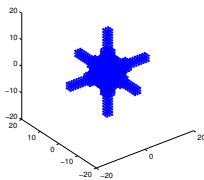
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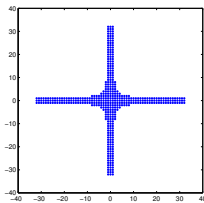
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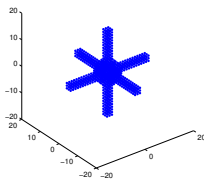
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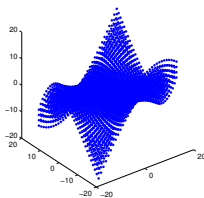
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arbitrary  
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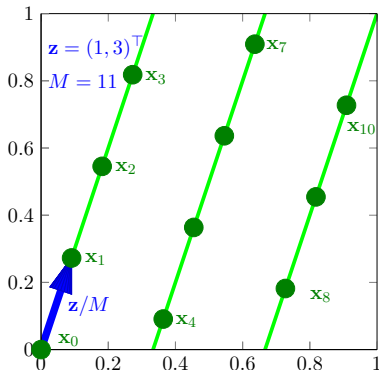
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- large dimension  $d$

# Trig. polynomials - Fast evaluation at rank-1 lattices

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$$\mathbf{x}_j = \frac{j}{M} \mathbf{z} \bmod \mathbf{1}; j = 0, \dots, M-1$$



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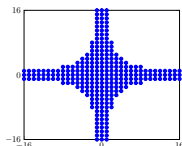
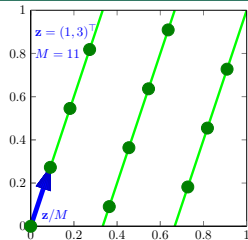
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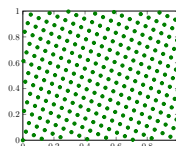
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- reformulation

$$p(\mathbf{x}_j) = \sum_{\mathbf{k} \in I_N^d} \hat{p}_{\mathbf{k}} e^{2\pi i \frac{j \mathbf{k} \cdot \mathbf{z}}{M}} = \sum_{l=0}^{M-1} \underbrace{\left( \sum_{\substack{\mathbf{k} \in I_N^d \\ \mathbf{k} \cdot \mathbf{z} \equiv l \pmod{M}}} \hat{p}_{\mathbf{k}} \right)}_{\hat{g}_l} e^{2\pi i \frac{j \mathbf{k} \cdot \mathbf{z}}{M}} = \sum_{l=0}^{M-1} \hat{g}_l e^{2\pi i \frac{j l}{M}}$$



$$(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in I_N^d}$$



$$(p(\mathbf{x}_j))_{j=0}^{M-1}$$

# Trig. polynomials - Fast evaluation at rank-1 lattices

- rank-1 lattice:  $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$

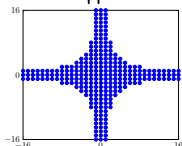
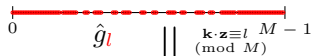
$$\mathbf{x}_j = \frac{j}{M} \mathbf{z} \bmod \mathbf{1}; j = 0, \dots, M-1$$

- multivariate trigonometric polynomial

$$p(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{I}_N^d} \hat{p}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}$$

- reformulation

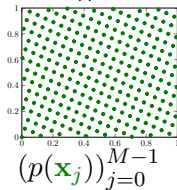
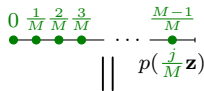
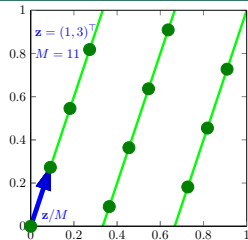
$$p(\mathbf{x}_j) = \sum_{\mathbf{k} \in \mathbb{I}_N^d} \hat{p}_{\mathbf{k}} e^{2\pi i \frac{j \mathbf{k} \cdot \mathbf{z}}{M}} = \sum_{l=0}^{M-1} \underbrace{\left( \sum_{\substack{\mathbf{k} \in \mathbb{I}_N^d \\ \mathbf{k} \cdot \mathbf{z} \equiv l \pmod{M}}} \hat{p}_{\mathbf{k}} \right)}_{\hat{g}_l} e^{2\pi i \frac{j \mathbf{k} \cdot \mathbf{z}}{M}} = \sum_{l=0}^{M-1} \hat{g}_l e^{2\pi i \frac{j l}{M}}$$



$$(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in \mathbb{I}_N^d}$$

1-dim  
→  
FFT

$$\mathcal{O}(M \log M + d |\mathbb{I}_N^d|)$$



# Trig. polynomials - Fast evaluation at rank-1 lattices

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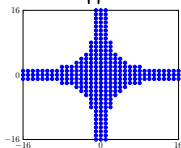
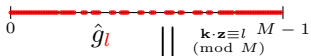
$$\mathbf{x}_j = \frac{j}{M} \mathbf{z} \bmod \mathbf{1}; j = 0, \dots, M-1$$

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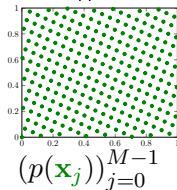
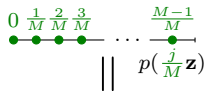
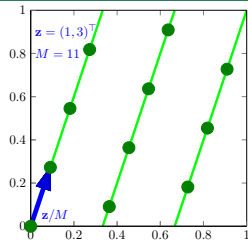


$$(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in \mathbb{I}_N^d}$$

?

←

fast, stable, unique reconstruction



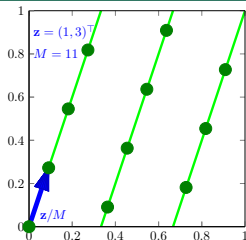


# Trig. polynomials - Fast reconstruction

- rank-1 lattice:  $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$

$$\mathbf{x}_j = \frac{j}{M} \mathbf{z} \bmod \mathbf{1}; j = 0, \dots, M - 1$$

- reconstruction of the Fourier coefficients of  $p(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{I}_N^d} \hat{p}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}$  by applying a lattice rule

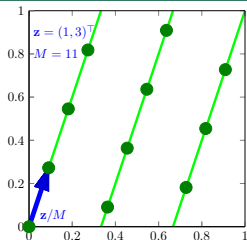


# Trig. polynomials - Fast reconstruction

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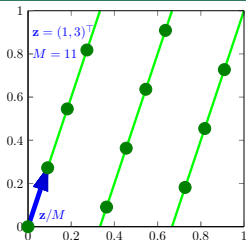
$$\hat{p}_{\mathbf{k}} = \int_{\mathbb{T}^d} p(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} d\mathbf{x} \approx Q(p(\cdot)) e^{-2\pi i \mathbf{k} \cdot (\cdot)} = \underbrace{\frac{1}{M} \sum_{j=0}^{M-1} p(\mathbf{x}_j) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}_j}}_{\hat{\tilde{p}}_{\mathbf{k}}}$$

# Trig. polynomials - Fast reconstruction

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$$\hat{p}_{\mathbf{k}} = \int_{\mathbb{T}^d} p(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} d\mathbf{x} \approx Q(p(\cdot)) e^{-2\pi i \mathbf{k} \cdot (\cdot)} = \underbrace{\frac{1}{M} \sum_{j=0}^{M-1} p(\mathbf{x}_j) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}_j}}_{\hat{\hat{p}}_{\mathbf{k}}}$$

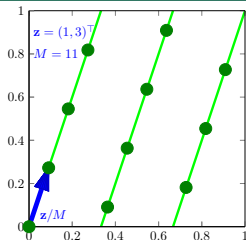
$$\Rightarrow \hat{\hat{p}}_{\mathbf{k}} = \hat{p}_{\mathbf{k}} \Leftrightarrow \mathbf{k}_1 \cdot \mathbf{z} \not\equiv \mathbf{k}_2 \cdot \mathbf{z} \pmod{M} \text{ for all } \mathbf{k}_1 \neq \mathbf{k}_2 \in \mathbb{I}_N^d$$

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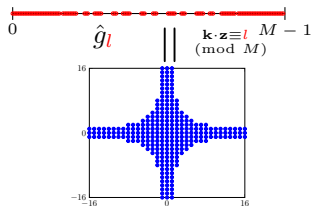
$\Rightarrow$  **Definition:** reconstructing rank-1 lattice  $\Lambda(\mathbf{z}, M, \mathbb{I}_N^d)$  for  $\mathbb{I}_N^d$ ,

# Trig. polynomials - Fast reconstruction

- rank-1 lattice:  $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$

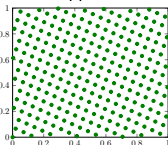
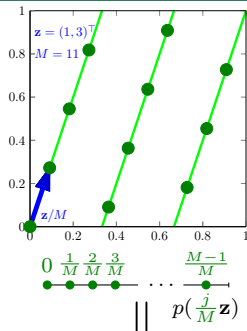
$$\mathbf{x}_j = \frac{j}{M} \mathbf{z} \bmod \mathbf{1}; j = 0, \dots, M-1$$

- reconstruction of the Fourier coefficients of  $p(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{I}_N^d} \hat{p}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}$  by applying a lattice rule



$$(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in \mathbb{I}_N^d}$$

⇒ **Definition:** reconstructing rank-1 lattice  $\Lambda(\mathbf{z}, M, \mathbb{I}_N^d)$  for  $\mathbb{I}_N^d$ ,  $|\mathbb{I}_N^d| \leq M \leq |\mathbb{I}_N^d|^2$ , CBC construction algorithm (Kämmerer 2012)



$$(p(\mathbf{x}_j))_{j=0}^{M-1}$$

1-dim  
←  
iFFT

$$\mathcal{O}(M \log M + d|\mathbb{I}_N^d|)$$

# Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Rank-1 lattice nodes

- reconstructing rank-1 lattice  $\Lambda(\mathbf{z}, M, \mathbf{I}_N^d)$ :  
 $\mathbf{x}_j = \frac{j}{M} \mathbf{z} \bmod \mathbf{1}; j = 0, \dots, M - 1; \quad |\mathbf{I}_N^d| \leq M \leq |\mathbf{I}_N^d|^2$
- approximate reconstruction of the Fourier coefficients of  $f \in \mathcal{H}^\omega(\mathbb{T}^d)$  applying a lattice rule

# Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Rank-1 lattice nodes

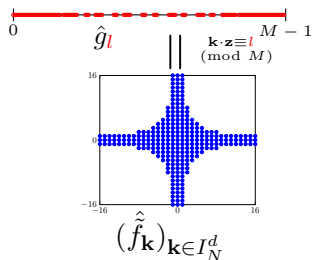
- reconstructing rank-1 lattice  $\Lambda(\mathbf{z}, M, \mathbf{I}_N^d)$ :  
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$$\hat{f}_{\mathbf{k}} = \int_{\mathbb{T}^d} f(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} d\mathbf{x} \approx Q(f(\cdot) e^{-2\pi i \mathbf{k} \cdot (\cdot)}) = \underbrace{\frac{1}{M} \sum_{j=0}^{M-1} f(\mathbf{x}_j) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}_j}}_{\hat{\hat{f}}_{\mathbf{k}}}$$

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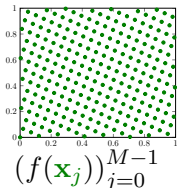
- reconstructing rank-1 lattice  $\Lambda(\mathbf{z}, M, \mathbb{I}_N^d)$ :  
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1-dim  
←  
iFFT

$$\mathcal{O}(M \log M + d|\mathbb{I}_N^d|)$$

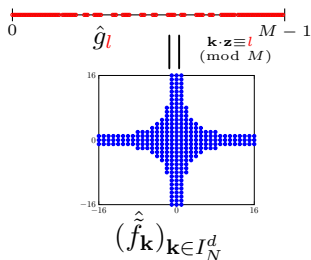




# Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Rank-1 lattice nodes

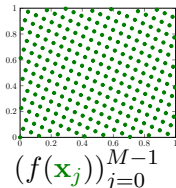
- reconstructing rank-1 lattice  $\Lambda(\mathbf{z}, M, \mathbb{I}_N^d)$ :  
 $\mathbf{x}_j = \frac{j}{M} \mathbf{z} \bmod \mathbf{1}$ ;  $j = 0, \dots, M-1$ ;  $|\mathbb{I}_N^d| \leq M \leq |\mathbb{I}_N^d|^2$
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1-dim  
←  
iFFT

$$\mathcal{O}(M \log M + d|\mathbb{I}_N^d|)$$



- approximation  $\tilde{S}_{\mathbb{I}_N^d} f$  of  $f$  by  $\tilde{S}_{\mathbb{I}_N^d} f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{I}_N^d} \hat{f}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}$

# Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Error estimates

$$\underbrace{\|f - \tilde{S}_{I_N^d} f\|_{L^2(\mathbb{T}^d)}}_{\text{approximation error}} \leq \underbrace{\|f - S_{I_N^d} f\|_{L^2(\mathbb{T}^d)}}_{\text{truncation error}} + \underbrace{\|S_{I_N^d} f - \tilde{S}_{I_N^d} f\|_{L^2(\mathbb{T}^d)}}_{\text{aliasing error}}$$

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$$\|f - S_{I_N^d} f\|_{L^2(\mathbb{T}^d)} = \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d \setminus I_N^d} |\hat{f}_{\mathbf{k}}|^2}$$

# Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Error estimates

$$\underbrace{\|f - \tilde{S}_{I_N^d} f\|_{L^2(\mathbb{T}^d)}}_{\text{approximation error}} \leq \underbrace{\|f - S_{I_N^d} f\|_{L^2(\mathbb{T}^d)}}_{\text{truncation error}} + \underbrace{\|S_{I_N^d} f - \tilde{S}_{I_N^d} f\|_{L^2(\mathbb{T}^d)}}_{\text{aliasing error}}$$

$$\begin{aligned} \|f - S_{I_N^d} f\|_{L^2(\mathbb{T}^d)} &= \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d \setminus I_N^d} |\hat{f}_{\mathbf{k}}|^2} \\ &= \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d \setminus I_N^d} \frac{\omega(\mathbf{k})^2}{\omega(\mathbf{k})^2} |\hat{f}_{\mathbf{k}}|^2} \end{aligned}$$

# Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Error estimates

$$\underbrace{\|f - \tilde{S}_{I_N^d} f\|_{L^2(\mathbb{T}^d)}}_{\text{approximation error}} \leq \underbrace{\|f - S_{I_N^d} f\|_{L^2(\mathbb{T}^d)}}_{\text{truncation error}} + \underbrace{\|S_{I_N^d} f - \tilde{S}_{I_N^d} f\|_{L^2(\mathbb{T}^d)}}_{\text{aliasing error}}$$

$$\begin{aligned} \|f - S_{I_N^d} f\|_{L^2(\mathbb{T}^d)} &= \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d \setminus I_N^d} |\hat{f}_{\mathbf{k}}|^2} \\ &= \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d \setminus I_N^d} \frac{\omega(\mathbf{k})^2}{\omega(\mathbf{k})^2} |\hat{f}_{\mathbf{k}}|^2} \\ &\leq \max_{\mathbf{k} \in \mathbb{Z}^d \setminus I_N^d} \frac{1}{\omega(\mathbf{k})} \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d \setminus I_N^d} \omega(\mathbf{k})^2 |\hat{f}_{\mathbf{k}}|^2} \\ &\leq \max_{\mathbf{k} \in \mathbb{Z}^d \setminus I_N^d} \frac{1}{\omega(\mathbf{k})} \|f\|_{\mathcal{H}^\omega(\mathbb{T}^d)} \\ &\leq \frac{1}{N} \|f\|_{\mathcal{H}^\omega(\mathbb{T}^d)} \end{aligned}$$

# Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Error estimates

$$\|S_{I_N^d} f - \tilde{S}_{I_N^d} f\|_{L^2(\mathbb{T}^d)} = \sqrt{\sum_{\mathbf{k} \in I_N^d} \left| \sum_{\substack{\mathbf{h} \in \mathbb{Z}^d \setminus \{\mathbf{0}\} \\ \mathbf{h} \cdot \mathbf{z} \equiv 0 \pmod{M}}} \hat{f}_{\mathbf{k}+\mathbf{h}} \right|^2}$$

# Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Error estimates

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# Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Error estimates

$$\begin{aligned} \|S_{I_N^d} f - \tilde{S}_{I_N^d} f\|_{L^2(\mathbb{T}^d)} &= \sqrt{\sum_{\mathbf{k} \in I_N^d} \left| \sum_{\substack{\mathbf{h} \in \mathbb{Z}^d \setminus \{\mathbf{0}\} \\ \mathbf{h} \cdot \mathbf{z} \equiv 0 \pmod{M}}} \hat{f}_{\mathbf{k}+\mathbf{h}} \right|^2} \\ &\leq \sum_{\mathbf{k} \in I_N^d} \left| \sum_{\substack{\mathbf{h} \in \mathbb{Z}^d \setminus \{\mathbf{0}\} \\ \mathbf{h} \cdot \mathbf{z} \equiv 0 \pmod{M}}} \hat{f}_{\mathbf{k}+\mathbf{h}} \right| \leq \sum_{\mathbf{k} \in I_N^d} \sum_{\substack{\mathbf{h} \in \mathbb{Z}^d \setminus \{\mathbf{0}\} \\ \mathbf{h} \cdot \mathbf{z} \equiv 0 \pmod{M}}} |\hat{f}_{\mathbf{k}+\mathbf{h}}| \\ &\leq \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus I_N^d} |\hat{f}_{\mathbf{k}}| \leq \max_{\mathbf{k} \in \mathbb{Z}^d \setminus I_N^d} \frac{1}{\omega(\mathbf{k})} \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus I_N^d} \omega(\mathbf{k}) |\hat{f}_{\mathbf{k}}| \end{aligned}$$

# Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Error estimates

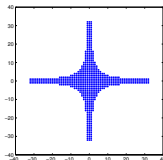
$$\begin{aligned}
 \|S_{I_N^d} f - \tilde{S}_{I_N^d} f\|_{L^2(\mathbb{T}^d)} &= \sqrt{\sum_{\mathbf{k} \in I_N^d} \left| \sum_{\substack{\mathbf{h} \in \mathbb{Z}^d \setminus \{\mathbf{0}\} \\ \mathbf{h} \cdot \mathbf{z} \equiv 0 \pmod{M}}} \hat{f}_{\mathbf{k}+\mathbf{h}} \right|^2} \\
 &\leq \sum_{\mathbf{k} \in I_N^d} \left| \sum_{\substack{\mathbf{h} \in \mathbb{Z}^d \setminus \{\mathbf{0}\} \\ \mathbf{h} \cdot \mathbf{z} \equiv 0 \pmod{M}}} \hat{f}_{\mathbf{k}+\mathbf{h}} \right| \leq \sum_{\mathbf{k} \in I_N^d} \sum_{\substack{\mathbf{h} \in \mathbb{Z}^d \setminus \{\mathbf{0}\} \\ \mathbf{h} \cdot \mathbf{z} \equiv 0 \pmod{M}}} |\hat{f}_{\mathbf{k}+\mathbf{h}}| \\
 &\leq \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus I_N^d} |\hat{f}_{\mathbf{k}}| \leq \max_{\mathbf{k} \in \mathbb{Z}^d \setminus I_N^d} \frac{1}{\omega(\mathbf{k})} \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus I_N^d} \omega(\mathbf{k}) |\hat{f}_{\mathbf{k}}| \\
 &\leq \frac{1}{N} (1 + 2\zeta(2\lambda))^{\frac{d}{2}} \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d} \omega(\mathbf{k})^2 |\hat{f}_{\mathbf{k}}|^2 \prod_{s=1}^d \max(1, |k_s|)^{2\lambda}}
 \end{aligned}$$

# Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Error estimates

Example: hyperbolic cross

index set:  $|I_N^d| \in \mathcal{O}(N \log^{d-1} N)$

$$I_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d : \prod_{s=1}^d \max(1, |k_s|) \leq N \right\}$$



weights:  $\omega(\mathbf{k}) := \omega^{0,\beta}(\mathbf{k}) := \prod_{s=1}^d \max(1, |k_s|)^\beta$

Hilbert space:

$$\mathcal{H}^{0,\beta}(\mathbb{T}^d) := \left\{ f : \|f\|_{\mathcal{H}^{0,\beta}(\mathbb{T}^d)} := \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d} \omega^{0,\beta}(\mathbf{k})^2 |\hat{f}_{\mathbf{k}}|^2} < \infty \right\}$$

arithmetic complexity:  $\mathcal{O}(N^2 \log^{d-1} N)$

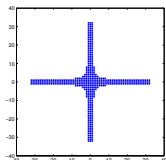
error estimate:  $\beta > t \geq 0$ ,  $\lambda > 1/2$

$$\|f - \tilde{S}_{I_N^d} f\|_{\mathcal{H}^{0,t}(\mathbb{T}^d)} \leq (1 + 2\zeta(2\lambda))^{\frac{d}{2}} N^{t-\beta} \|f\|_{\mathcal{H}^{0,\beta+\lambda}(\mathbb{T}^d)}$$

# Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Error estimates

Example: energy-norm based hyperbolic cross

index set: for  $0 < \alpha - r < \beta - t$ ,  $|\mathbb{I}_N^d| \in \mathcal{O}(N)$



$$\mathbb{I}_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d : \max(1, \|\mathbf{k}\|_1)^{\frac{\alpha-r}{\alpha-r+\beta-t}} \prod_{s=1}^d \max(1, |k_s|)^{\frac{\beta-t}{\alpha-r+\beta-t}} \leq N \right\}$$

weights:  $\omega^{\alpha,\beta}(\mathbf{k}) := \max(1, \|\mathbf{k}\|_1)^\alpha \prod_{s=1}^d \max(1, |k_s|)^\beta$

Hilbert space:

$$\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d) := \left\{ f : \|f\|_{\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)} := \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d} \omega^{\alpha,\beta}(\mathbf{k})^2 |\hat{f}_{\mathbf{k}}|^2} < \infty \right\}$$

arithmetic complexity:  $\mathcal{O}(N^2 \log N)$

error estimate:  $\beta > t \geq 0$ ,  $\lambda > 1/2$

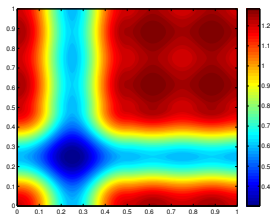
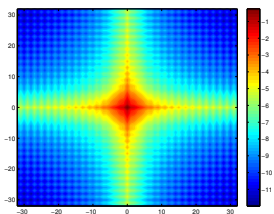
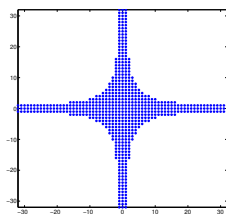
$$\|f - \tilde{S}_{\mathbb{I}_N^d} f\|_{\mathcal{H}^{r,t}(\mathbb{T}^d)} \leq (1 + 2\zeta(2\lambda))^{\frac{d}{2}} N^{r-\alpha+t-\beta} \|f\|_{\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)}$$

# Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Numerical results

- function  $f \in \mathcal{H}^{0, \frac{5}{2} - \epsilon}(\mathbb{T}^d)$ ,  $\epsilon > 0$ ,

$$f(\mathbf{x}) := \prod_{s=1}^d \left( 4 + \operatorname{sgn} \left( x_s - \frac{1}{2} \right) \sin(2\pi x_s)^2 + \operatorname{sgn} \left( x_s - \frac{1}{2} \right) \sin(2\pi x_s)^3 \right),$$

- hyperbolic cross index set  $I_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d : \prod_{s=1}^d \max(1, |k_s|) \leq N \right\}$

 $f((x_1, x_2)^\top)$  $\log_{10} \left| \hat{f}_{(k_1, k_2)^\top} \right|$  $I_N^d$

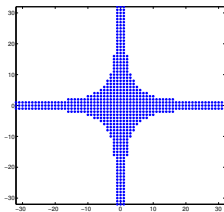
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$$\|f - \tilde{S}_{I_N^d} f\|_{L^2(\mathbb{T}^d)} \lesssim N^{-2+\tilde{\epsilon}} \|f\|_{\mathcal{H}^{0, 2-\tilde{\epsilon}+\lambda}(\mathbb{T}^d)}$$



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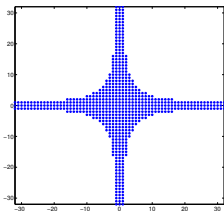
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- we compute the relative  $L^2(\mathbb{T}^d) = \mathcal{H}^{0,0}(\mathbb{T}^d)$

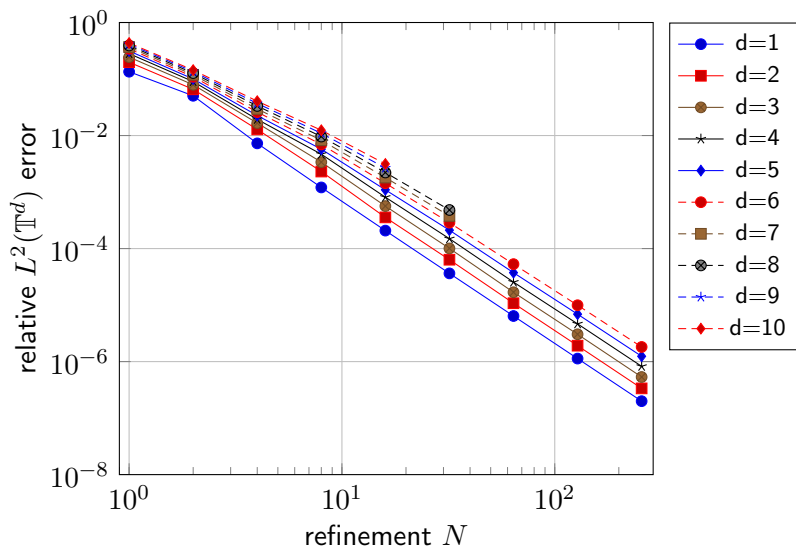
- i.e.,  $\|f - \tilde{S}_{I_N^d} f\|_{L^2(\mathbb{T}^d)} / \|f\|_{L^2(\mathbb{T}^d)}$
- corresponds to the above error estimate with  $r = t = 0$  up to a “constant” since

$$\frac{\|f - \tilde{S}_{I_N^d} f\|_{\mathcal{H}^{0,0,\gamma}(\mathbb{T}^d)}}{\|f\|_{\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)}} = \underbrace{\frac{\|f\|_{L^2(\mathbb{T}^d)}}{\|f\|_{\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)}}}_{\leq 1} \frac{\|f - \tilde{S}_{I_N^d} f\|_{L^2(\mathbb{T}^d)}}{\|f\|_{L^2(\mathbb{T}^d)}}$$



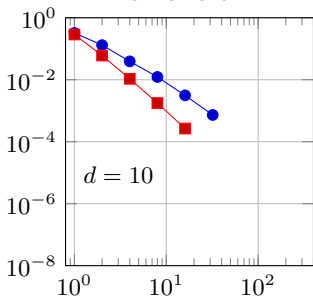
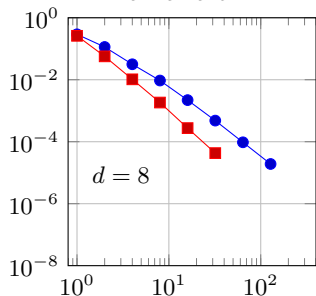
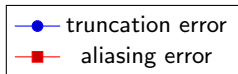
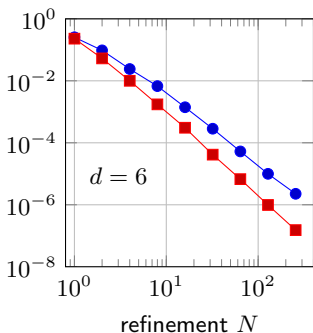
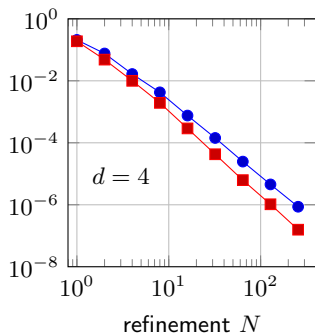
# Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Numerical results

sampling at rank-1 lattice nodes





# Reconstruction of $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ - Numerical results



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$d$	$N$	$ I_N^d $	$M$	$\frac{M}{ I_N^d }$	rel. $L^2(\mathbb{T}^d)$ error
6	64	1 709 857	31 829 977	18.6	5.3e-05
6	128	5 137 789	192 757 285	37.5	9.9e-06
6	256	14 977 209	1 400 567 254	93.5	1.8e-06
8	8	768 609	6 027 975	7.8	9.6e-03
8	16	2 935 521	49 768 670	17.0	2.2e-03
8	32	10 665 297	359 896 131	33.7	4.8e-04
10	4	2 421 009	30 780 958	12.7	4.1e-02
10	8	10 819 089	194 144 634	17.9	1.2e-02
10	16	45 548 649	2 040 484 044	44.8	3.1e-03

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idea from Temlyakov 1986:

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$$\|S_{I_N^{d, \alpha, \beta}} f - \tilde{S}_{I_N^{d, \alpha, \beta}} f\|_{L^2(\mathbb{T}^d)} \leq C(d, \alpha, \beta) N^{-(\alpha+\beta)} \|f\|_{\mathcal{H}^{\alpha, \beta}(\mathbb{T}^d)}$$

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drawback: **non-constructive proof**, i.e., for sufficiently large prime  $M$ ,

$|I_N^{d, \alpha, \beta}| \lesssim M \lesssim |I_N^{d, \alpha, \beta}|^2$ , there exists  $\mathbf{z} = (1, a, a^2, \dots, a^{d-1})^\top$  such that

above error estimates are valid

# Summary

approximate reconstruction of (multivariate) functions  $f \in \mathcal{H}^\omega(\mathbb{T}^d)$  by sampling along rank-1 lattice nodes

- perfectly stable
- computation only uses single one-dimensional FFT + SP
- oversampling factor up to  $|I_N^d|$
- arithmetic complexity  $\mathcal{O}(|I_N^d|^2 \log |I_N^d|)$
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Kämmerer, L., Potts, D., Volkmer, T.

**Approximation of multivariate periodic functions by trigonometric polynomials based on sampling along rank-1 lattice with generating vector of Korobov form.**

DFG-Schwerpunktprogramm 1324, Preprint 159, 2014.

(<http://www.tu-chemnitz.de/~tovo>)