

Approximation of high-dimensional multivariate functions by trigonometric polynomials based on rank-1 lattice sampling

Toni Volkmer

Department of Mathematics
Technische Universität Chemnitz

<http://www.tu-chemnitz.de/~tovo>



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joint work with L. Kämmerer and D. Potts

Introduction

Multivariate trigonometric polynomials

Fast evaluation at rank-1 lattices

Fast, exact and stable reconstruction

Approximate reconstruction of functions $f \in \mathcal{A}^\omega(\mathbb{T}^d)$

by sampling at rank-1 lattice nodes

by sampling at perturbed rank-1 lattice nodes

Summary

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where the Fourier coefficients of f are given by

$$\hat{f}_{\mathbf{k}} = \int_{\mathbb{T}^d} f(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} d\mathbf{x}, \quad \mathbf{k} \in \mathbb{Z}^d$$

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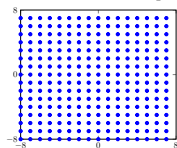
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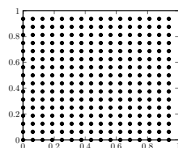
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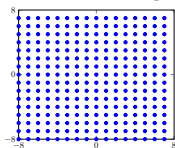
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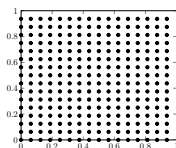
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$$\mathcal{O}(N^d \log N)$$



- problem: $|I| = (2N)^d \Rightarrow$ **curse of dimensionality**

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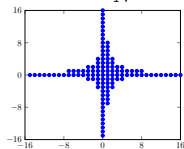
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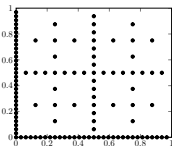
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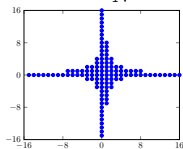
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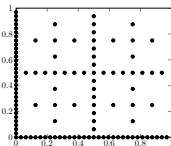
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- problem: **stability** (Kämmerer, Kunis 2011)

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- subspace of Wiener Algebra, $\omega: \mathbb{Z}^d \rightarrow [1, \infty]$,

$$\mathcal{A}^\omega(\mathbb{T}^d) := \left\{ f \in \mathcal{C}(\mathbb{T}^d) : \sum_{\mathbf{k} \in \mathbb{Z}^d} \omega(\mathbf{k}) |\hat{f}_{\mathbf{k}}| < \infty \right\}$$

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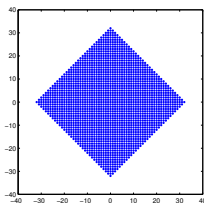
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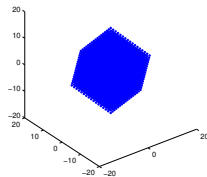


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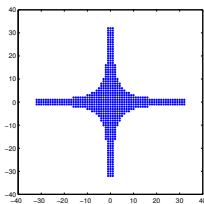


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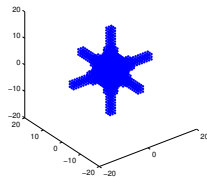


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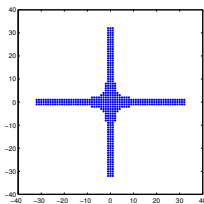


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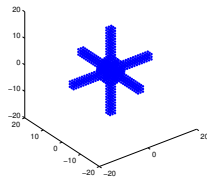


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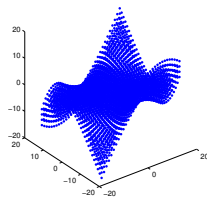


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arbitrary
index set

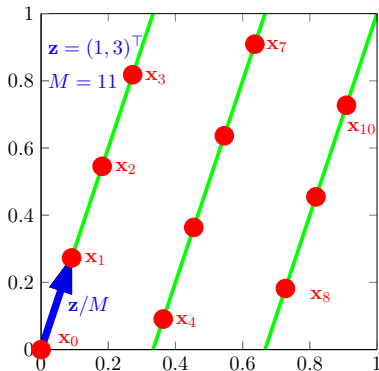
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- large dimension d

Trig. polynomials - Fast evaluation at rank-1 lattices

- rank-1 lattice: $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$

$$\mathbf{x}_j = \frac{j}{M} \mathbf{z} \bmod \mathbf{1}; j = 0, \dots, M-1$$



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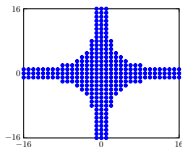
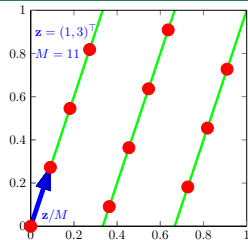
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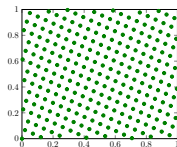
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- reformulation

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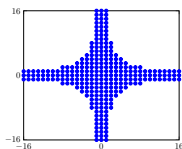
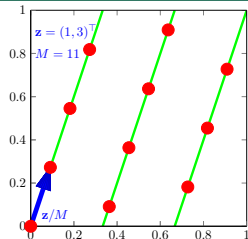
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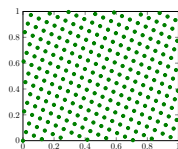
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1-dim
→
FFT

$$\mathcal{O}(M \log M + d|\mathbb{I}_N^d|)$$



$$(p(\mathbf{x}_j))_{j=0}^{M-1}$$

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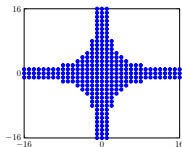
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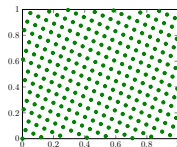
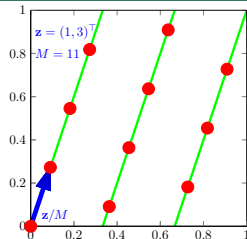
$$p(\mathbf{x}_j) = \sum_{\mathbf{k} \in I_N^d} \hat{p}_{\mathbf{k}} e^{2\pi i \frac{j \mathbf{k} \cdot \mathbf{z}}{M}} = \sum_{l=0}^{M-1} \underbrace{\left(\sum_{\substack{\mathbf{k} \in I_N^d \\ \mathbf{k} \cdot \mathbf{z} \equiv l \pmod{M}}} \hat{p}_{\mathbf{k}} \right)}_{\hat{g}_l} e^{2\pi i \frac{j \mathbf{k} \cdot \mathbf{z}}{M}} = \sum_{l=0}^{M-1} \hat{g}_l e^{2\pi i \frac{j l}{M}}$$



$$(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in I_N^d}$$

?

fast, stable, unique
reconstruction



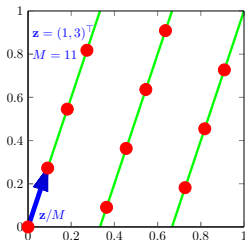
$$(p(\mathbf{x}_j))_{j=0}^{M-1}$$

Trig. polynomials - Fast reconstruction

- rank-1 lattice: $\mathbf{z} \in \mathbb{N}^d$, $M \in \mathbb{N}$

$$\mathbf{x}_j = \frac{j}{M} \mathbf{z} \bmod \mathbf{1}; \quad j = 0, \dots, M - 1$$

- reconstruction of the Fourier coefficients of $p(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{I}_N^d} \hat{p}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}$ by applying a lattice rule



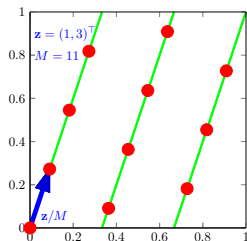
Trig. polynomials - Fast reconstruction

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$$\hat{p}_{\mathbf{k}} = \int_{\mathbb{T}^d} p(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} d\mathbf{x} \approx Q(p(\cdot)) e^{-2\pi i \mathbf{k} \cdot (\cdot)} = \underbrace{\frac{1}{M} \sum_{j=0}^{M-1} p(\mathbf{x}_j) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}_j}}_{\hat{\hat{p}}_{\mathbf{k}}}$$

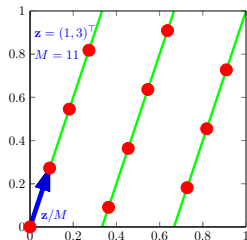


Trig. polynomials - Fast reconstruction

- rank-1 lattice: $\mathbf{z} \in \mathbb{N}^d$, $M \in \mathbb{N}$

$$\mathbf{x}_j = \frac{j}{M} \mathbf{z} \bmod \mathbf{1}; \quad j = 0, \dots, M-1$$

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$$\hat{p}_{\mathbf{k}} = \int_{\mathbb{T}^d} p(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} d\mathbf{x} \approx Q(p(\cdot) e^{-2\pi i \mathbf{k} \cdot (\cdot)}) = \underbrace{\frac{1}{M} \sum_{j=0}^{M-1} p(\mathbf{x}_j) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}_j}}_{\hat{\hat{p}}_{\mathbf{k}}}$$

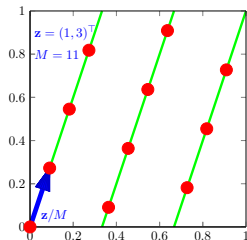
$$\Rightarrow \hat{\hat{p}}_{\mathbf{k}} = \hat{p}_{\mathbf{k}} \Leftrightarrow \mathbf{k}_1 \cdot \mathbf{z} \not\equiv \mathbf{k}_2 \cdot \mathbf{z} \pmod{M} \text{ for all } \mathbf{k}_1 \neq \mathbf{k}_2 \in \mathbb{I}_N^d$$

Trig. polynomials - Fast reconstruction

- rank-1 lattice: $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$

$$\mathbf{x}_j = \frac{j}{M} \mathbf{z} \bmod \mathbf{1}; j = 0, \dots, M-1$$

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$$\Rightarrow \hat{\hat{p}}_{\mathbf{k}} = \hat{p}_{\mathbf{k}} \Leftrightarrow \mathbf{k}_1 \cdot \mathbf{z} \not\equiv \mathbf{k}_2 \cdot \mathbf{z} \pmod{M} \text{ for all } \mathbf{k}_1 \neq \mathbf{k}_2 \in \mathbb{I}_N^d$$

$$\Rightarrow \text{reconstructing rank-1 lattice } \Lambda(\mathbf{z}, M, \mathbb{I}_N^d) \text{ for } \mathbb{I}_N^d, M \leq |\mathbb{I}_N^d|^2$$

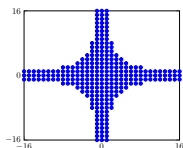
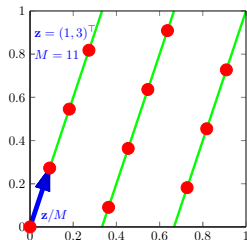
(Kammerer 2012)

Trig. polynomials - Fast reconstruction

- rank-1 lattice: $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$

$$\mathbf{x}_j = \frac{j}{M} \mathbf{z} \bmod \mathbf{1}; j = 0, \dots, M-1$$

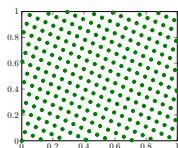
- reconstruction of the Fourier coefficients of $p(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{I}_N^d} \hat{p}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}$ by applying a lattice rule



$$(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in \mathbb{I}_N^d}$$

1-dim
←
iFFT

$$\mathcal{O}(|\mathbb{I}_N^d|^2 \log |\mathbb{I}_N^d| + d|\mathbb{I}_N^d|)$$



$$(p(\mathbf{x}_j))_{j=0}^{M-1}$$

⇒ reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathbb{I}_N^d)$ for $\mathbb{I}_N^d, M \leq |\mathbb{I}_N^d|^2$

(Kämmerer 2012)

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - rank-1 lattice nodes

- reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathbf{I}_N^d)$:
 $\mathbf{x}_j = \frac{j}{M} \mathbf{z} \bmod \mathbf{1}; j = 0, \dots, M - 1; \quad M \leq |\mathbf{I}_N^d|^2$
- approximate reconstruction of the Fourier coefficients of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ applying a lattice rule

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - rank-1 lattice nodes

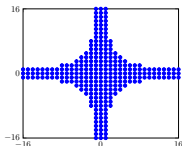
- reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathbf{I}_N^d)$:
 $\mathbf{x}_j = \frac{j}{M}\mathbf{z} \bmod \mathbf{1}; j = 0, \dots, M-1; M \leq |\mathbf{I}_N^d|^2$
- approximate reconstruction of the Fourier coefficients of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ applying a lattice rule

$$\hat{f}_{\mathbf{k}} = \int_{\mathbb{T}^d} f(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} d\mathbf{x} \approx Q(f(\cdot) e^{-2\pi i \mathbf{k} \cdot (\cdot)}) = \underbrace{\frac{1}{M} \sum_{j=0}^{M-1} f(\mathbf{x}_j) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}_j}}_{\hat{\hat{f}}_{\mathbf{k}}}$$

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - rank-1 lattice nodes

- reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathbf{I}_N^d)$:
 $\mathbf{x}_j = \frac{j}{M} \mathbf{z} \bmod \mathbf{1}$; $j = 0, \dots, M-1$; $M \leq |\mathbf{I}_N^d|^2$
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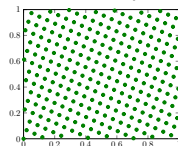
$$\hat{f}_{\mathbf{k}} = \int_{\mathbb{T}^d} f(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} d\mathbf{x} \approx Q(f(\cdot) e^{-2\pi i \mathbf{k} \cdot (\cdot)}) = \underbrace{\frac{1}{M} \sum_{j=0}^{M-1} f(\mathbf{x}_j) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}_j}}_{\hat{\hat{f}}_{\mathbf{k}}}$$



$$(\hat{\hat{f}}_{\mathbf{k}})_{\mathbf{k} \in \mathbf{I}_N^d}$$

1-dim
←
iFFT

$$\mathcal{O}(M \log M + d|\mathbf{I}_N^d|)$$

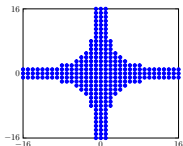


$$(f(\mathbf{x}_j))_{j=0}^{M-1}$$

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - rank-1 lattice nodes

- reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathbf{I}_N^d)$:
 $\mathbf{x}_j = \frac{j}{M} \mathbf{z} \bmod \mathbf{1}$; $j = 0, \dots, M-1$; $M \leq |\mathbf{I}_N^d|^2$
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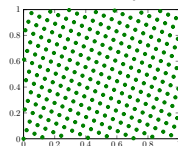
$$\hat{f}_{\mathbf{k}} = \int_{\mathbb{T}^d} f(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} d\mathbf{x} \approx Q(f(\cdot) e^{-2\pi i \mathbf{k} \cdot (\cdot)}) = \underbrace{\frac{1}{M} \sum_{j=0}^{M-1} f(\mathbf{x}_j) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}_j}}_{\hat{f}_{\mathbf{k}}}$$



$$(\hat{f}_{\mathbf{k}})_{\mathbf{k} \in \mathbf{I}_N^d}$$

1-dim
 \longleftarrow
 iFFT

$$\mathcal{O}(M \log M + d|\mathbf{I}_N^d|)$$



$$(f(\mathbf{x}_j))_{j=0}^{M-1}$$

- approximation $\tilde{S}_{\mathbf{I}_N^d} f$ of f by $\tilde{S}_{\mathbf{I}_N^d} f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbf{I}_N^d} \hat{f}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}$

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - rank-1 lattice nodes

Example: ℓ_1 -ball

weights: $\omega(\mathbf{k}) := \omega^{\alpha,0}(\mathbf{k}) := \max(1, \|\mathbf{k}\|_1)^\alpha$

function space:

$$\mathcal{A}^\omega(\mathbb{T}^d) = \mathcal{A}^{\alpha,0}(\mathbb{T}^d) := \left\{ f : \|f|_{\mathcal{A}^{\alpha,0}(\mathbb{T}^d)}\| := \sum_{\mathbf{k} \in \mathbb{Z}^d} \omega^{\alpha,0}(\mathbf{k}) |\hat{f}_{\mathbf{k}}| < \infty \right\}$$

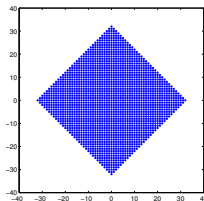
index set: $|\mathbb{I}_N^d| \in \mathcal{O}(N^d)$

$$\mathbb{I}_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d : \max(1, \|\mathbf{k}\|_1) \leq N \right\}$$

arithmetic complexity: $\mathcal{O}(N^d \log N)$

error estimate:

$$\|f - \tilde{S}_{\mathbb{I}_N^d} f|_{L_\infty(\mathbb{T}^d)}\| \leq 2N^{-\alpha} \|f|_{\mathcal{A}^{\alpha,0}(\mathbb{T}^d)}\|$$



Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - rank-1 lattice nodes

Example: hyperbolic cross

weights: $\omega(\mathbf{k}) := \omega^{0,\beta}(\mathbf{k}) := \prod_{s=1}^d \max(1, |k_s|)^\beta$

function space:

$$\mathcal{A}^\omega(\mathbb{T}^d) = \mathcal{A}^{0,\beta}(\mathbb{T}^d) := \left\{ f : \|f\|_{\mathcal{A}^{0,\beta}(\mathbb{T}^d)} := \sum_{\mathbf{k} \in \mathbb{Z}^d} \omega^{0,\beta}(\mathbf{k}) |\hat{f}_{\mathbf{k}}| < \infty \right\}$$

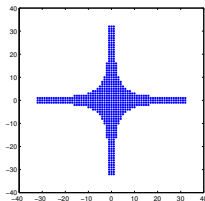
index set: $|I_N^d| \in \mathcal{O}(N \log^{d-1} N)$

$$I_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d : \prod_{s=1}^d \max(1, |k_s|) \leq N \right\}$$

arithmetic complexity: $\mathcal{O}(N^2 \log^{d-1} N)$

error estimate:

$$\|f - \tilde{S}_{I_N^d} f\|_{L_\infty(\mathbb{T}^d)} \leq 2N^{-\beta} \|f\|_{\mathcal{A}^{0,\beta}(\mathbb{T}^d)}$$



Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - rank-1 lattice nodes

Example: energy-norm based hyperbolic cross

weights: $\omega^{\alpha,\beta}(\mathbf{k}) := \max(1, \|\mathbf{k}\|_1)^\alpha \prod_{s=1}^d \max(1, |k_s|)^\beta$

function space:

$$\mathcal{A}^\omega(\mathbb{T}^d) = \mathcal{A}^{\alpha,\beta}(\mathbb{T}^d) := \left\{ f : \|f\|_{\mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)} := \sum_{\mathbf{k} \in \mathbb{Z}^d} \omega^{\alpha,\beta}(\mathbf{k}) |\hat{f}_{\mathbf{k}}| < \infty \right\}$$

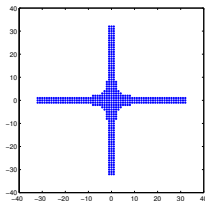
index set: for $0 < -\alpha < \beta$, $|\mathbb{I}_N^d| \in \mathcal{O}(N)$

$$\mathbb{I}_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d : \max(1, \|\mathbf{k}\|_1)^{\frac{\alpha}{\alpha+\beta}} \prod_{s=1}^d \max(1, |k_s|)^{\frac{\beta}{\alpha+\beta}} \leq N \right\}$$

arithmetic complexity: $\mathcal{O}(N^2 \log N)$

error estimate:

$$\|f - \tilde{S}_{\mathbb{I}_N^d} f\|_{L_\infty(\mathbb{T}^d)} \leq 2N^{-\alpha-\beta} \|f\|_{\mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)}$$



Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - rank-1 lattice nodes

Example: energy-norm based hyperbolic cross

weights: $\omega^{\alpha,\beta}(\mathbf{k}) := \max(1, \|\mathbf{k}\|_1)^\alpha \prod_{s=1}^d \max(1, |k_s|)^\beta$

Hilbert space:

$$\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d) := \left\{ f : \|f\|_{\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)} := \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d} \omega^{\alpha,\beta}(\mathbf{k})^2 |\hat{f}_{\mathbf{k}}|^2} < \infty \right\}$$

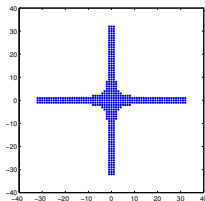
index set: for $0 < r - \alpha < \beta - t$, $|\mathbb{I}_N^d| \in \mathcal{O}(N)$

$$\mathbb{I}_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d : \max(1, \|\mathbf{k}\|_1)^{\frac{\alpha-r}{\alpha-r+\beta-t}} \prod_{s=1}^d \max(1, |k_s|)^{\frac{\beta-t}{\alpha-r+\beta-t}} \leq N \right\}$$

arithmetic complexity: $\mathcal{O}(N^2 \log N)$

error estimate: $\beta > t \geq 0$, $\lambda > 1/2$

$$\|f - \tilde{S}_{\mathbb{I}_N^d} f\|_{\mathcal{H}^{r,t}(\mathbb{T}^d)} \leq c N^{r-\alpha+t-\beta} \|f\|_{\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)}$$

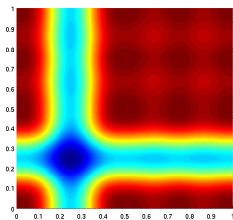


Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - Numerical test

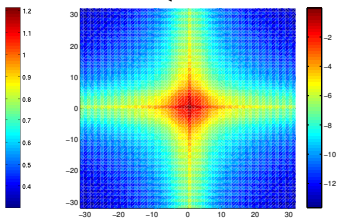
- function $f \in \mathcal{H}^{0, \frac{7}{2} - \epsilon}(\mathbb{T}^d)$, $\epsilon > 0$,

$$f(\mathbf{x}) := \prod_{s=1}^d \left(4 + \operatorname{sgn} \left(x_s - \frac{1}{2} \right) \sin(2\pi x_s)^3 + \operatorname{sgn} \left(x_s - \frac{1}{2} \right) \sin(2\pi x_s)^4 \right),$$

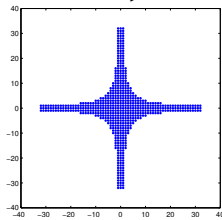
- hyperbolic cross index set $I_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d : \prod_{s=1}^d \max(1, |k_s|) \leq N \right\}$



$f((x_1, x_2)^\top)$



$\log_{10} \left| \hat{f}_{(k_1, k_2)^\top} \right|$



I_N^d

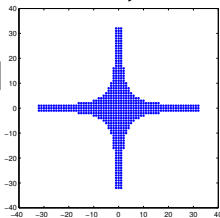
Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - Numerical test

- function $f \in \mathcal{H}^{0, \frac{7}{2} - \epsilon}(\mathbb{T}^d)$, $\epsilon > 0$,

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- hyperbolic cross index set $I_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d : \prod_{s=1}^d \max(1, |k_s|) \leq N \right\}$
- error estimate: $\tilde{\epsilon} > 0$, $\lambda > 1/2$

$$\|f - \tilde{S}_{I_N^d} f\|_{L^2(\mathbb{T}^d)} \leq c N^{-3+\tilde{\epsilon}} \|f\|_{\mathcal{H}^{0, 3-\tilde{\epsilon}+\lambda}(\mathbb{T}^d)}$$



Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - Numerical test

- function $f \in \mathcal{H}^{0, \frac{7}{2} - \epsilon}(\mathbb{T}^d)$, $\epsilon > 0$,

$$f(\mathbf{x}) := \prod_{s=1}^d \left(4 + \operatorname{sgn} \left(x_s - \frac{1}{2} \right) \sin(2\pi x_s)^3 + \operatorname{sgn} \left(x_s - \frac{1}{2} \right) \sin(2\pi x_s)^4 \right),$$

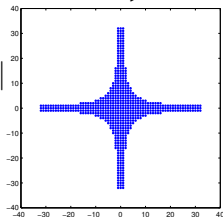
- hyperbolic cross index set $I_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d : \prod_{s=1}^d \max(1, |k_s|) \leq N \right\}$
- error estimate: $\tilde{\epsilon} > 0$, $\lambda > 1/2$

$$\|f - \tilde{S}_{I_N^d} f\|_{L^2(\mathbb{T}^d)} \leq c N^{-3+\tilde{\epsilon}} \|f\|_{\mathcal{H}^{0, 3-\tilde{\epsilon}+\lambda}(\mathbb{T}^d)}$$

- we compute the relative $L^2(\mathbb{T}^d) = \mathcal{H}^{0,0}(\mathbb{T}^d)$

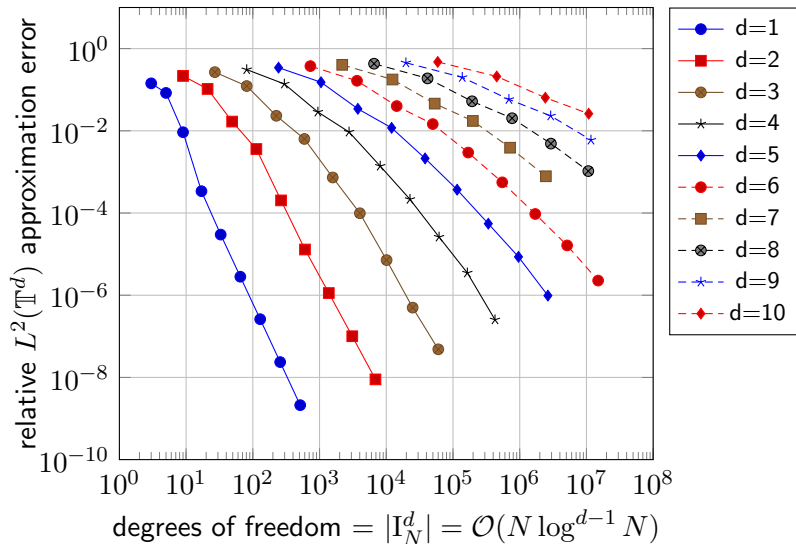
- i.e., $\|f - \tilde{S}_{I_N^d} f\|_{L^2(\mathbb{T}^d)} / \|f\|_{L^2(\mathbb{T}^d)}$
- corresponds to the above error estimate with $r = t = 0$ up to a “constant” since

$$\frac{\|f - \tilde{S}_{I_N^d} f\|_{\mathcal{H}^{0,0,\gamma}(\mathbb{T}^d)}}{\|f\|_{\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)}} = \underbrace{\frac{\|f\|_{L^2(\mathbb{T}^d)}}{\|f\|_{\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)}}}_{\leq 1} \frac{\|f - \tilde{S}_{I_N^d} f\|_{L^2(\mathbb{T}^d)}}{\|f\|_{L^2(\mathbb{T}^d)}}$$



Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - Numerical test

sampling at rank-1 lattice nodes

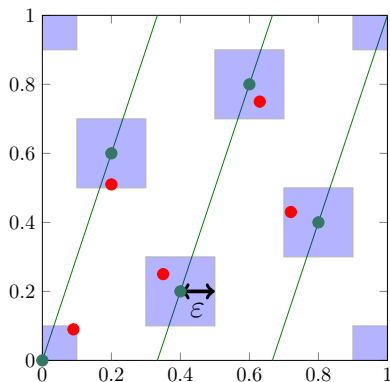


Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - Numerical test

d	N	$ I_N^d $	M	$\frac{M}{ I_N^d }$	rel. $L^2(\mathbb{T}^d)$ error
6	64	1 709 857	31 829 977	18.6	9.5e-05
6	128	5 137 789	192 757 285	37.5	1.6e-05
6	256	14 977 209	1 400 567 254	93.5	2.3e-06
8	8	768 609	6 027 975	7.8	2.0e-02
8	16	2 935 521	49 768 670	17.0	4.9e-03
8	32	10 665 297	359 896 131	33.7	1.0e-03
10	2	452 709	4 315 343	9.5	2.1e-01
10	4	2 421 009	30 780 958	12.7	6.4e-02
10	8	10 819 089	194 144 634	17.9	2.6e-02

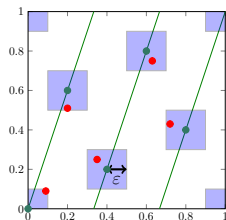
Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - perturbed rank-1 lat.

- reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathbb{I}_N^d)$:
 $\mathbf{x}_j = \frac{j}{M}\mathbf{z} \bmod \mathbf{1}$; $j = 0, \dots, M - 1$
- perturbed rank-1 lattice nodes \mathbf{y}_j :
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Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - perturbed rank-1 lat.

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$$(\hat{f}_{\mathbf{k}})_{\mathbf{k} \in \mathbb{I}_N^d} := \arg \min_{\hat{\mathbf{g}} \in \mathbb{C}^{|\mathbb{I}_N^d|}} \|\mathbf{A} \hat{\mathbf{g}} - (f(\mathbf{y}_j))_{j=0}^{M-1}\|_2, \quad \mathbf{A} := (e^{2\pi i \mathbf{k} \cdot \mathbf{y}_j})_{j=0}^{M-1}; \mathbf{k} \in \mathbb{I}_N^d$$

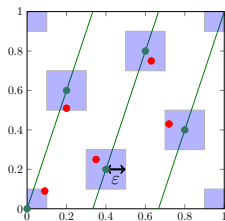
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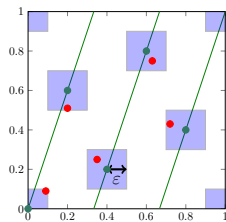
- stability? $\varepsilon < \frac{\ln 2}{2\pi d N} \implies \kappa(\mathbf{A}) \leq \frac{e^{2\pi d N \varepsilon}}{2 - e^{2\pi d N \varepsilon}}$

(Kämmerer, Potts, V. 2013)



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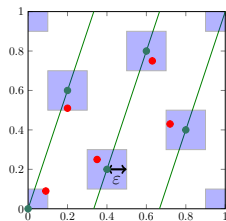


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- compute $(\hat{f}_{\mathbf{k}})_{\mathbf{k} \in \mathbb{I}_N^d}$ with CG-type method in K iterations using Taylor expansion of degree $m-1$ to approximate \mathbf{A} , arithmetic complexity $\mathcal{O}(K m^d |\mathbb{I}_N^d|^2 \log |\mathbb{I}_N^d|)$

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - perturbed rank-1 lat.

Example: energy-norm based hyperbolic cross

$$\text{weights: } \omega^{\alpha,\beta}(\mathbf{k}) := \max(1, \|\mathbf{k}\|_1)^\alpha \prod_{s=1}^d \max(1, |k_s|)^\beta$$

Hilbert space:

$$\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d) := \left\{ f : \|f\|_{\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)} := \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d} \omega^{\alpha,\beta}(\mathbf{k})^2 |\hat{f}_{\mathbf{k}}|^2} < \infty \right\}$$

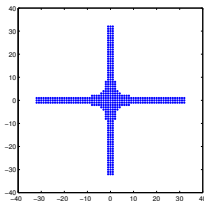
index set: for $0 < r - \alpha < \beta - t$, $|\mathbb{I}_N^d| \in \mathcal{O}(N)$

$$\mathbb{I}_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d : \max(1, \|\mathbf{k}\|_1)^{\frac{\alpha-r}{\alpha-r+\beta-t}} \prod_{s=1}^d \max(1, |k_s|)^{\frac{\beta-t}{\alpha-r+\beta-t}} \leq N \right\}$$

complexity: $\mathcal{O}(K m^d |\mathbb{I}_N^d|^2 \log |\mathbb{I}_N^d|) = \mathcal{O}(K m^d N^2 \log N)$

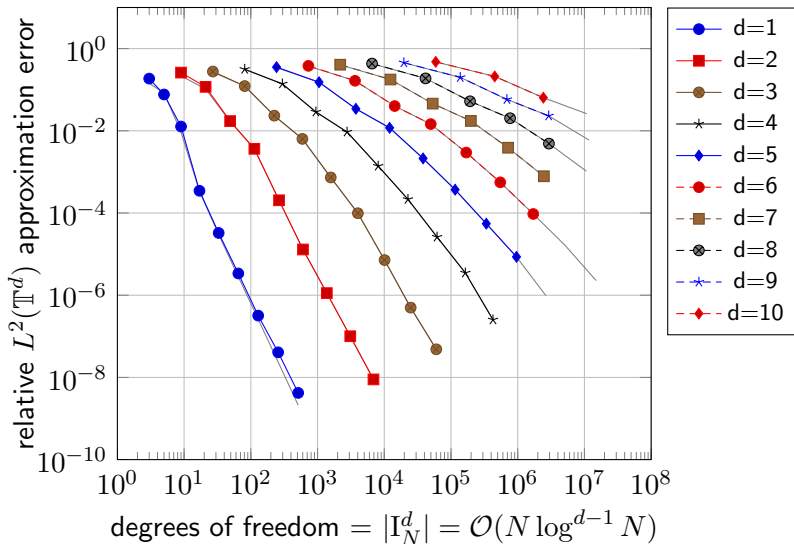
error estimate: $\beta > t \geq 0$, $\lambda > 1/2$, $\varepsilon < \frac{\ln 2}{2\pi d N}$, $0 < \alpha + \beta \leq m$

$$\|f - \tilde{S}_{\mathbb{I}_N^d} f\|_{\mathcal{H}^{r,t}(\mathbb{T}^d)} \leq \tilde{c} N^{r-\alpha+t-\beta} \|f\|_{\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)}$$



Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - Numerical test

perturbed rank-1 lattice nodes, $\|\mathbf{y}_j - \mathbf{x}_j\|_\infty < \frac{\ln 2}{2\pi d N}$, $m = 3$



Summary

- **fast** and **perfectly stable** approximate reconstruction of (multivariate) functions $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ by sampling along **rank-1 lattice** nodes using a single one-dimensional iFFT arithmetic complexity $\mathcal{O}(|\mathbb{I}_N^d|^2 \log |\mathbb{I}_N^d|)$

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Further theoretical results and numerical examples in



Kämmerer, L., Potts, D., Volkmer, T.

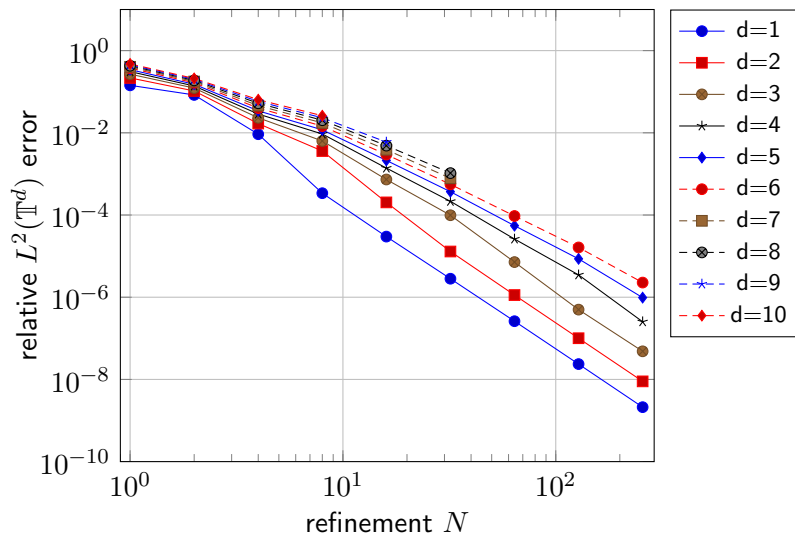
Approximation of multivariate functions by trigonometric polynomials based on rank-1 lattice sampling.

DFG-Schwerpunktprogramm 1324, Preprint 145, 2013.

(<http://www.tu-chemnitz.de/~tovo>)

Numerical results

sampling at rank-1 lattice nodes



Numerical results

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