Approximation of multivariate functions by sampling along (perturbed) rank-1 lattice nodes

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joint work with L. Kämmerer and D. Potts

Content

Introduction

Multivariate trigonometric polynomials

Fast evaluation at rank-1 lattices

Fast, exact and stable reconstruction

Approximate reconstruction of functions $f \in \mathcal{A}^\omega(\mathbb{T}^d)$

by sampling at rank-1 lattice nodes

by sampling at perturbed rank-1 lattice nodes

Summary

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- $f : \mathbb{T}^d \to \mathbb{C}$ multivariate continuous function

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$$p(\mathbf{x}) := \sum_{\mathbf{k}\in\mathrm{I}} \hat{p}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \qquad \hat{p}_{\mathbf{k}} \in \mathbb{C}$$

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- e.g., approximate f using its Fourier partial sum $S_{\mathrm{I}}f$,

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  where the Fourier coefficients of f are given by

$$\hat{f}_{\mathbf{k}} = \int_{\mathbb{T}^d} f(\mathbf{x}) \mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{x}, \qquad \mathbf{k} \in \mathbb{Z}^d$$

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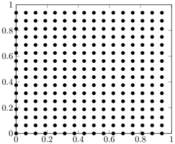
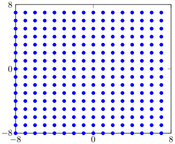
- e.g., $\mathrm{I} = \mathbb{Z}^d \cap [-N, N)^d$ full grid, $\mathbf{y}_j$ equispaced nodes



$$(f(\mathbf{y}_j))_{j=0}^{|\mathrm{I}|-1} \ \underset{\text{FFT}}{\overset{\text{d-dim}}{\longleftrightarrow}}\ (\hat{p}_{\mathbf{k}})_{\mathbf{k}\in\mathrm{I}}$$



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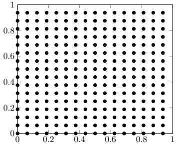

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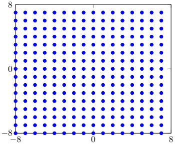
$$\mathcal{O}(N^d \log N)$$



  - problem: $|\mathrm{I}| = (2N)^d \Rightarrow$ curse of dimensionality

Introduction - Function Spaces and Index Sets I

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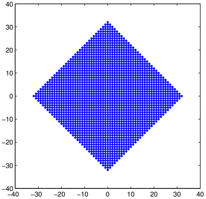

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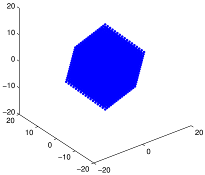

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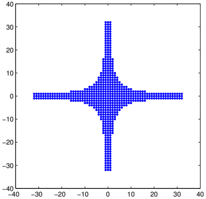

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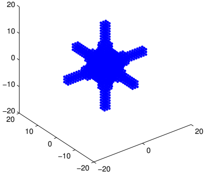

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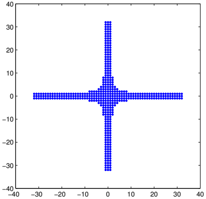

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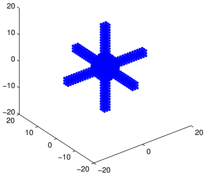

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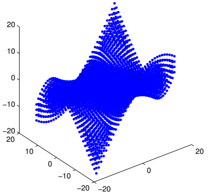

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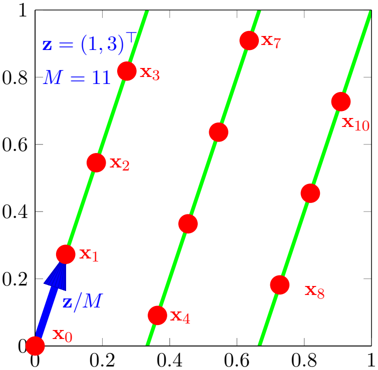
- large dimension d

Trig. polynomials - Fast evaluation at rank-1 lattices

- rank-1 lattice: $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$

$$\mathbf{x}_j = \frac{j}{M}\mathbf{z} \bmod \mathbf{1};\ j = 0, \ldots, M-1$$



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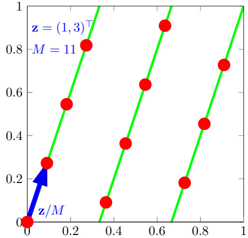

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- reformulation

$$p(\mathbf{x}_j) = \sum_{\mathbf{k} \in \mathrm{I}_N^d} \hat{p}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \frac{j \mathbf{k}\cdot\mathbf{z}}{M}} = \sum_{l=0}^{M-1} \underbrace{\left(\sum_{\substack{\mathbf{k} \in \mathrm{I}_N^d \\ \mathbf{k}\cdot\mathbf{z} \equiv l \pmod{M}}} \hat{p}_{\mathbf{k}} \right)}_{\hat{g}_l} \mathrm{e}^{2\pi \mathrm{i} \frac{j \mathbf{k}\cdot\mathbf{z}}{M}} = \sum_{l=0}^{M-1} \hat{g}_l \mathrm{e}^{2\pi \mathrm{i} \frac{jl}{M}}$$

- complexity $\mathcal{O}(M \log M + d|\mathrm{I}_N^d|)$ applying 1D FFT

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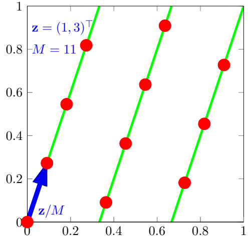

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Aim: stable and unique reconstruction of $\hat{p}_{\mathbf{k}}$

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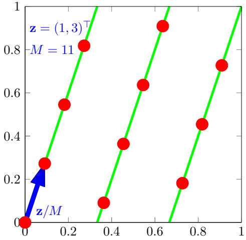
- reconstruction of the Fourier coefficients of $p(\mathbf{x}) = \sum_{\mathbf{k}\in \mathrm{I}_N^d} \hat{p}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{x}}$ by applying a lattice rule



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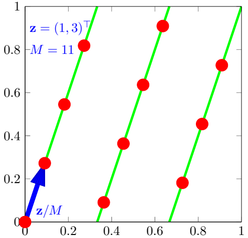

$$\hat{p}_{\mathbf{k}} = \int_{\mathbb{T}^d} p(\mathbf{x}) \mathrm{e}^{-2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}} \mathrm{d}\mathbf{x} \approx \mathrm{Q}(p(\cdot)\mathrm{e}^{-2\pi \mathrm{i} \mathbf{k} \cdot (\cdot)}) = \underbrace{\frac{1}{M} \sum_{j=0}^{M-1} p(\mathbf{x}_j) \mathrm{e}^{-2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}_j}}_{\hat{\tilde{p}}_{\mathbf{k}}}$$

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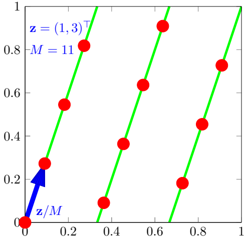

$$\hat{p}_{\mathbf{k}} = \int_{\mathbb{T}^d} p(\mathbf{x})\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\mathrm{d}\mathbf{x} \approx \mathrm{Q}(p(\cdot)\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot(\cdot)}) = \underbrace{\frac{1}{M}\sum_{j=0}^{M-1} p(\mathbf{x}_j)\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}_j}}_{\hat{\tilde{p}}_{\mathbf{k}}}$$

$\Rightarrow\; \hat{\tilde{p}}_{\mathbf{k}} = \hat{p}_{\mathbf{k}} \Leftrightarrow \mathbf{k}_1 \cdot \mathbf{z} \not\equiv \mathbf{k}_2 \cdot \mathbf{z} \pmod{M}$ for all $\mathbf{k}_1 \neq \mathbf{k}_2 \in \mathrm{I}_N^d$

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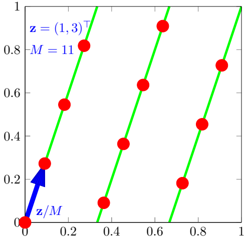
- rank-1 lattice: $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$

$$\mathbf{x}_j = \frac{j}{M}\mathbf{z} \bmod \mathbf{1};\ j = 0, \ldots, M-1$$

- reconstruction of the Fourier coefficients of $p(\mathbf{x}) = \sum_{\mathbf{k}\in I_N^d} \hat{p}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k}\cdot\mathbf{x}}$ by applying a lattice rule



$$\hat{p}_{\mathbf{k}} = \int_{\mathbb{T}^d} p(\mathbf{x}) \mathrm{e}^{-2\pi \mathrm{i} \mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{x} \approx \mathrm{Q}(p(\cdot)\mathrm{e}^{-2\pi \mathrm{i}\mathbf{k}\cdot(\cdot)}) = \underbrace{\frac{1}{M}\sum_{j=0}^{M-1} p(\mathbf{x}_j)\mathrm{e}^{-2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{x}_j}}_{\hat{\hat{p}}_{\mathbf{k}}}$$

$\Rightarrow$ $\hat{\hat{p}}_{\mathbf{k}} = \hat{p}_{\mathbf{k}} \Leftrightarrow \mathbf{k}_1 \cdot \mathbf{z} \not\equiv \mathbf{k}_2 \cdot \mathbf{z} \pmod{M}$ for all $\mathbf{k}_1 \neq \mathbf{k}_2 \in I_N^d$

$\Rightarrow$ reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, I_N^d)$ for I_N^d,
$M \leq |I_N^d|^2$ (Kämmerer 2012),
complexity $\mathcal{O}(M \log M + d|I_N^d|)$ applying 1D iFFT

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - rank-1 lattice nodes

- reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathrm{I}_N^d)$:
  $\mathbf{x}_j = \frac{j}{M}\mathbf{z} \bmod \mathbf{1};\ j = 0, \ldots, M-1$
- approximate reconstruction of the Fourier coefficients of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ applying a lattice rule

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - rank-1 lattice nodes

- reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathrm{I}_N^d)$:
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$$\hat{f}_{\mathbf{k}} = \int_{\mathbb{T}^d} f(\mathbf{x}) \mathrm{e}^{-2\pi \mathrm{i} \mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{x} \approx \mathrm{Q}(f(\cdot)\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot(\cdot)}) = \underbrace{\frac{1}{M}\sum_{j=0}^{M-1} f(\mathbf{x}_j)\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}_j}}_{\hat{\hat{f}}_{\mathbf{k}}}$$

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - rank-1 lattice nodes

- reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathrm{I}_N^d)$:
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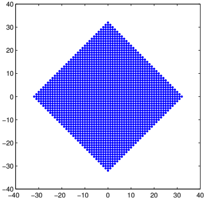
- approximation $\tilde{S}_{\mathrm{I}_N^d} f$ of f by

$$\tilde{S}_{\mathrm{I}_N^d} f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathrm{I}_N^d} \hat{\tilde{f}}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

Reconstruction of $f \in \mathcal{A}^{\omega}(\mathbb{T}^d)$ - rank-1 lattice nodes



Example: ℓ_1-ball
weights: $\omega(\mathbf{k}) := \omega^{\alpha,0}(\mathbf{k}) := \max(1, \|\mathbf{k}\|_1)^{\alpha}$

Sobolev space:

$$\mathcal{A}^{\omega}(\mathbb{T}^d) = \mathcal{A}^{\alpha,0}(\mathbb{T}^d) := \left\{ f \colon \|f|\mathcal{A}^{\alpha,0}(\mathbb{T}^d)\| := \sum_{\mathbf{k}\in\mathbb{Z}^d} \omega^{\alpha,0}(\mathbf{k})|\hat{f}_{\mathbf{k}}| < \infty \right\}$$

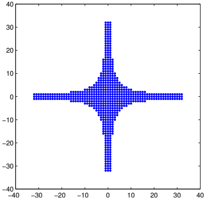
index set: $|\mathrm{I}_N^d| \in \mathcal{O}\left(N^d\right)$

$$\mathrm{I}_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d \colon \max(1, \|\mathbf{k}\|_1) \le N \right\}$$

arithmetic complexity: $\mathcal{O}\left(N^d \log N\right)$
error estimate:

$$\|f - \tilde{S}_{\mathrm{I}_N^d} f|L_{\infty}(\mathbb{T}^d)\| \le 2N^{-\alpha}\|f|\mathcal{A}^{\alpha,0}(\mathbb{T}^d)\|$$

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - rank-1 lattice nodes



Example: hyperbolic cross

weights: $\omega(\mathbf{k}) := \omega^{0,\beta}(\mathbf{k}) := \prod_{s=1}^{d} \max(1, |k_s|)^\beta$

Sobolev space:

$$\mathcal{A}^\omega(\mathbb{T}^d) = \mathcal{A}^{0,\beta}(\mathbb{T}^d) := \left\{ f \colon \|f|\mathcal{A}^{0,\beta}(\mathbb{T}^d)\| := \sum_{\mathbf{k}\in\mathbb{Z}^d} \omega^{0,\beta}(\mathbf{k})|\hat{f}_{\mathbf{k}}| < \infty \right\}$$

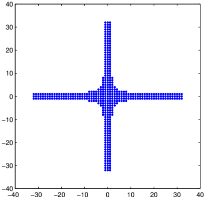
index set: $|\mathrm{I}_N^d| \in \mathcal{O}\left(N \log^{d-1} N\right)$

$$\mathrm{I}_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d \colon \prod_{s=1}^{d} \max(1, |k_s|) \leq N \right\}$$

arithmetic complexity: $\mathcal{O}\left(N^2 \log^{d-1} N\right)$
error estimate:

$$\|f - \tilde{S}_{\mathrm{I}_N^d} f | L_\infty(\mathbb{T}^d)\| \leq 2N^{-\beta} \|f|\mathcal{A}^{0,\beta}(\mathbb{T}^d)\|$$

Reconstruction of $f \in \mathcal{A}^{\omega}(\mathbb{T}^d)$ - rank-1 lattice nodes



Example: energy norm based hyperbolic cross

weights: $\omega^{\alpha,\beta}(\mathbf{k}) := \max(1, \|\mathbf{k}\|_1)^{\alpha} \prod_{s=1}^{d} \max(1, |k_s|)^{\beta}$

Sobolev space:

$$\mathcal{A}^{\omega}(\mathbb{T}^d) = \mathcal{A}^{\alpha,\beta}(\mathbb{T}^d) := \left\{ f \colon \|f|\mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)\| := \sum_{\mathbf{k}\in\mathbb{Z}^d} \omega^{\alpha,\beta}(\mathbf{k})|\hat{f}_{\mathbf{k}}| < \infty \right\}$$

index set: for $0 < -\alpha < \beta$, $|\mathrm{I}_N^d| \in \mathcal{O}\left(N\right)$

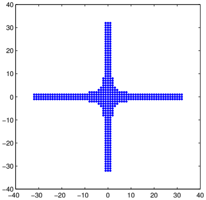
$$\mathrm{I}_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d \colon \max(1, \|\mathbf{k}\|_1)^{\frac{\alpha}{\alpha+\beta}} \prod_{s=1}^{d} \max(1, |k_s|)^{\frac{\beta}{\alpha+\beta}} \leq N \right\}$$

arithmetic complexity: $\mathcal{O}\left(N^2 \log N\right)$

error estimate:

$$\|f - \tilde{S}_{\mathrm{I}_N^d} f | L_{\infty}(\mathbb{T}^d)\| \leq 2N^{-\alpha-\beta} \|f|\mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)\|$$

Reconstruction of $f \in \mathcal{A}^{\omega}(\mathbb{T}^d)$ - rank-1 lattice nodes



Example: energy norm based hyperbolic cross

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Hilbert space:

$$\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d) := \left\{ f \colon \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\| := \sqrt{\sum_{\mathbf{k}\in\mathbb{Z}^d} \omega^{\alpha,\beta}(\mathbf{k})^2 |\hat{f}_{\mathbf{k}}|^2} < \infty \right\}$$

index set: for $0 < r - \alpha < \beta - t$, $|\mathrm{I}_N^d| \in \mathcal{O}\left(N\right)$

$$\mathrm{I}_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d \colon \max(1, \|\mathbf{k}\|_1)^{\frac{\alpha-r}{\alpha-r+\beta-t}} \prod_{s=1}^{d} \max(1, |k_s|)^{\frac{\beta-t}{\alpha-r+\beta-t}} \leq N \right\}$$

arithmetic complexity: $\mathcal{O}\left(N^2 \log N\right)$

error estimate: $\beta > t \geq 0$, $\lambda > 1/2$

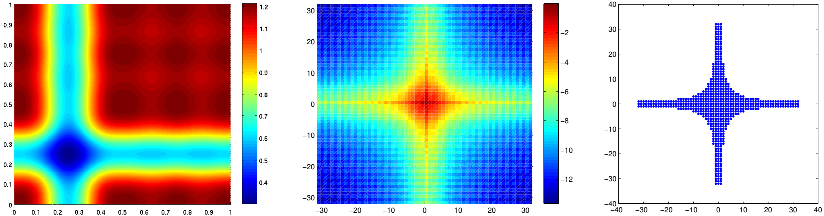
$$\|f - \tilde{S}_{\mathrm{I}_N^d} f|\mathcal{H}^{r,t}(\mathbb{T}^d)\| \leq c\, N^{r-\alpha+t-\beta} \|f|\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)\|$$

Reconstruction of $f \in \mathcal{A}^{\omega}(\mathbb{T}^d)$ - Numerical test

- function $f \in \mathcal{H}^{0,\frac{7}{2}-\epsilon}(\mathbb{T}^d)$, $\epsilon > 0$,

$$f(\mathbf{x}) := \prod_{s=1}^{d} \left(4 + \operatorname{sgn}\left(x_s - \frac{1}{2}\right) \sin\left(2\pi x_s\right)^3 + \operatorname{sgn}\left(x_s - \frac{1}{2}\right) \sin\left(2\pi x_s\right)^4\right),$$

- hyperbolic cross index set $I_N^d := \left\{\mathbf{k} \in \mathbb{Z}^d \colon \prod_{s=1}^{d} \max(1, |k_s|) \leq N\right\}$



$f\left((x_1, x_2)^\top\right)$ $\qquad$ $\log_{10}\left|\hat{f}_{(k_1,k_2)^\top}\right|$

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - Numerical test

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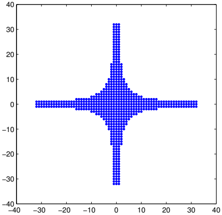
- hyperbolic cross index set $\mathrm{I}_N^d := \left\{\mathbf{k} \in \mathbb{Z}^d \colon \prod_{s=1}^{d} \max(1, |k_s|) \le N\right\}$
- error estimate: $\tilde{\epsilon} > 0$, $\lambda > 1/2$

$$\|f - \tilde{S}_{\mathrm{I}_N^d} f | L^2(\mathbb{T}^d)\| \le c\, N^{-3+\tilde{\epsilon}} \|f | \mathcal{H}^{0,3-\tilde{\epsilon}+\lambda}(\mathbb{T}^d)\|$$



Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - Numerical test

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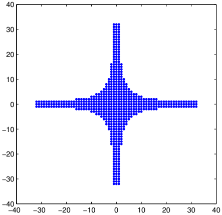
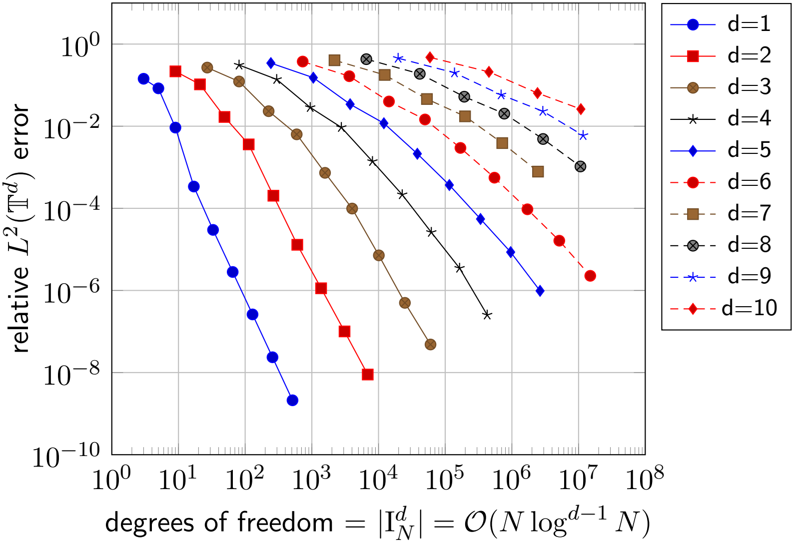


- we compute the relative $L^2(\mathbb{T}^d) = \mathcal{H}^{0,0}(\mathbb{T}^d)$
  - i.e., $\|f - \tilde{S}_{\mathrm{I}_N^d} f|L^2(\mathbb{T}^d)\| / \|f|L^2(\mathbb{T}^d)\|$
  - corresponds to the above error estimate with $r = t = 0$ up to a "constant" since

$$\frac{\|f - \tilde{S}_{\mathrm{I}_N^d} f|\mathcal{H}^{0,0,\gamma}(\mathbb{T}^d)\|}{\|f|\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)\|} = \underbrace{\frac{\|f|L^2(\mathbb{T}^d)\|}{\|f|\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)\|}}_{\le 1} \; \frac{\|f - \tilde{S}_{\mathrm{I}_N^d} f|L^2(\mathbb{T}^d)\|}{\|f|L^2(\mathbb{T}^d)\|}$$

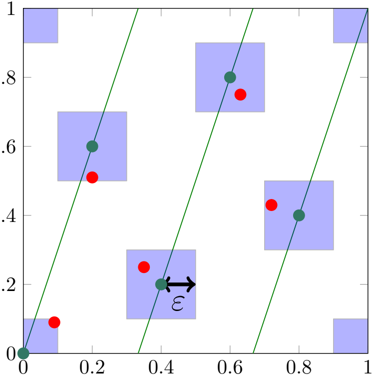
Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - Numerical test

sampling at rank-1 lattice nodes



Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - perturbed rank-1 lat.

- reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathrm{I}_N^d)$:
  $\mathbf{x}_j = \frac{j}{M}\mathbf{z} \bmod \mathbf{1}$; $j = 0, \ldots, M-1$
- perturbed rank-1 lattice nodes $\mathbf{y}_j$:
  $\|\mathbf{y}_j - \mathbf{x}_j\|_\infty \leq \varepsilon$, perturbation $\varepsilon \geq 0$



Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - perturbed rank-1 lat.

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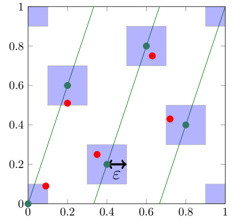
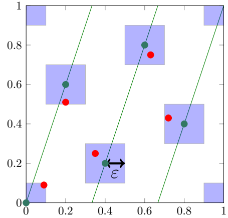


$$\left(\hat{\tilde{f}}_{\mathbf{k}}\right)_{\mathbf{k}\in\mathrm{I}_N^d} := \underset{\hat{\mathbf{g}}\in\mathbb{C}^{|\mathrm{I}_N^d|}}{\arg\min} \left\|\mathbf{A}\,\hat{\mathbf{g}} - (f(\mathbf{y}_j))_{j=0}^{M-1}\right\|_2, \ \mathbf{A} := \left(\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{y}_j}\right)_{j=0;\ \mathbf{k}\in\mathrm{I}_N^d}^{M-1}$$

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - perturbed rank-1 lat.



- reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathrm{I}_N^d)$:
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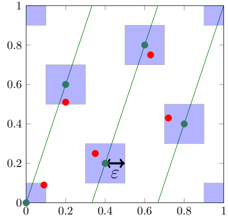
- stability? $\varepsilon < \frac{\ln 2}{2\pi d N} \implies \kappa(\mathbf{A}) \leq \frac{\mathrm{e}^{2\pi d N \varepsilon}}{2 - \mathrm{e}^{2\pi d N \varepsilon}}$ (Kämmerer,Potts,V.)

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - perturbed rank-1 lat.



- reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathrm{I}_N^d)$:
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- perturbed rank-1 lattice nodes $\mathbf{y}_j$:
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$$\big(\hat{\tilde{f}}_{\mathbf{k}}\big)_{\mathbf{k}\in\mathrm{I}_N^d} := \operatorname*{arg\,min}_{\hat{\mathbf{g}}\in\mathbb{C}^{|\mathrm{I}_N^d|}} \|\mathbf{A}\,\hat{\mathbf{g}} - (f(\mathbf{y}_j))_{j=0}^{M-1}\|_2,\ \ \mathbf{A} := \left(\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{y}_j}\right)_{j=0;\ \mathbf{k}\in\mathrm{I}_N^d}^{M-1}$$

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Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - perturbed rank-1 lat.

- reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathrm{I}_N^d)$:
  $\mathbf{x}_j = \frac{j}{M}\mathbf{z} \bmod \mathbf{1};\ j = 0, \ldots, M-1$
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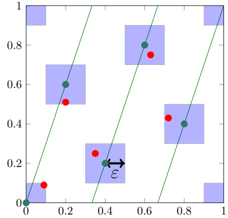

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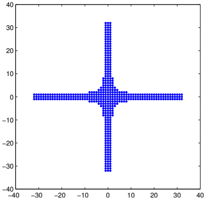
- approximation $\tilde{S}_{\mathrm{I}_N^d} f$ of f by $\tilde{S}_{\mathrm{I}_N^d} f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathrm{I}_N^d} \hat{\tilde{f}}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$
- compute $\left(\hat{\tilde{f}}_{\mathbf{k}}\right)_{\mathbf{k}\in\mathrm{I}_N^d}$ with CG-type method in K iterations using Taylor expansion of degree $m-1$ to approximate $\mathbf{A}$, arithmetic complexity $\mathcal{O}\left(K\ m^d\ |\mathrm{I}_N^d|^2 \log |\mathrm{I}_N^d|\right)$

Reconstruction of $f \in \mathcal{A}^{\omega}(\mathbb{T}^d)$ - perturbed rank-1 lat.



Example: energy norm based hyperbolic cross

weights: $\omega^{\alpha,\beta}(\mathbf{k}) := \max(1, \|\mathbf{k}\|_1)^{\alpha} \prod_{s=1}^{d} \max(1, |k_s|)^{\beta}$

Hilbert space:

$$\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d) := \left\{ f \colon \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\| := \sqrt{\sum_{\mathbf{k}\in\mathbb{Z}^d} \omega^{\alpha,\beta}(\mathbf{k})^2 |\hat{f}_{\mathbf{k}}|^2} < \infty \right\}$$

index set: for $0 < r - \alpha < \beta - t$, $|\mathrm{I}_N^d| \in \mathcal{O}\left(N\right)$

$$\mathrm{I}_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d \colon \max(1, \|\mathbf{k}\|_1)^{\frac{\alpha - r}{\alpha - r + \beta - t}} \prod_{s=1}^{d} \max(1, |k_s|)^{\frac{\beta - t}{\alpha - r + \beta - t}} \leq N \right\}$$

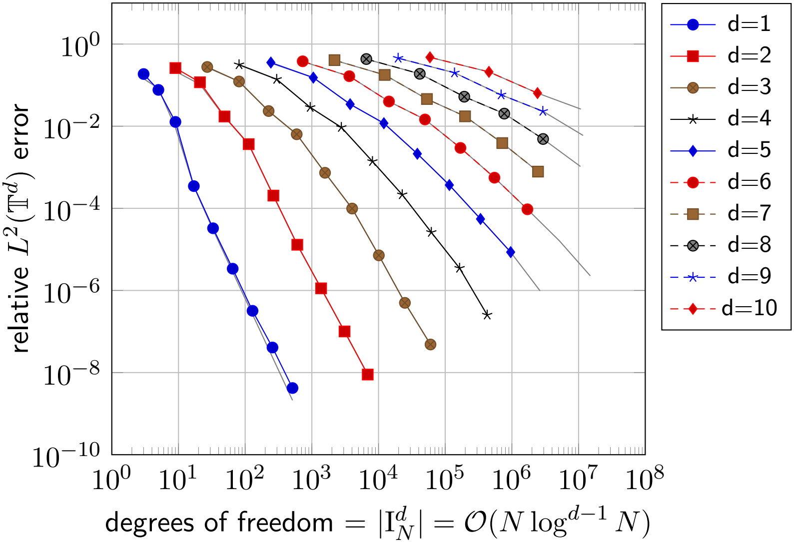
complexity: $\mathcal{O}\left(K\, m^d\, |\mathrm{I}_N^d|^2 \log |\mathrm{I}_N^d|\right) = \mathcal{O}\left(K\, m^d\, N^2 \log N\right)$

error estimate: $\beta > t \geq 0$, $\lambda > 1/2$, $\varepsilon < \frac{\ln 2}{2\pi d N}$, $0 < \alpha + \beta \leq m$

$$\|f - \tilde{S}_{\mathrm{I}_N^d} f | \mathcal{H}^{r,t}(\mathbb{T}^d)\| \leq \tilde{c}\, N^{r - \alpha + t - \beta} \|f|\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)\|$$

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - Numerical test

perturbed rank-1 lattice nodes, $\|\mathbf{y}_j - \mathbf{x}_j\|_\infty < \frac{\ln 2}{2\pi d N}$, $m = 3$



Summary

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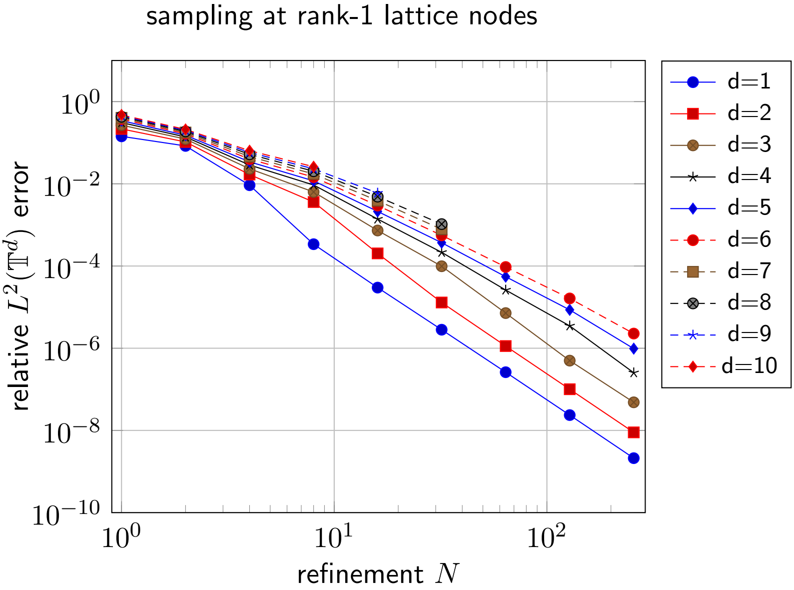
Further theoretical results and numerical examples in

Kämmerer, L., Potts, D., Volkmer, T.
Approximation of multivariate functions by trigonometric polynomials based on rank-1 lattice sampling.
DFG-Schwerpunktprogramm 1324, Preprint 145, 2013.
(http://www.tu-chemnitz.de/∼tovo)

Numerical results



Numerical results

perturbed rank-1 lattice nodes, $\|\mathbf{y}_j - \mathbf{x}_j\|_\infty < \frac{\ln 2}{2\pi d N}$, $m = 3$