Taylor and rank-1 lattice based nonequispaced fast Fourier transform

Toni Volkmer

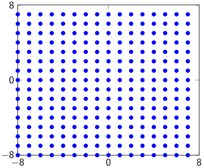
Chemnitz University of Technology
supported by DFG-SPP 1324

Road map

Introduction

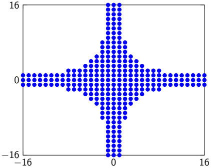
- $p\colon \mathbb{T}^d := [0,1)^d \to \mathbb{C}$ trigonometric polynomial,

$$p(\mathbf{x}) := \sum_{\mathbf{k}\in \mathrm{I}} \hat{p}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k}\mathbf{x}}, \quad \mathrm{I} \subset \mathbb{Z}^d,\ |\mathrm{I}| < \infty,\ \hat{p}_{\mathbf{k}} \in \mathbb{C}$$



I: full grid

$d = 2$



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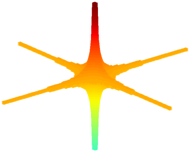

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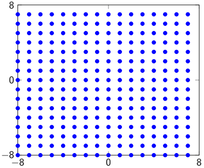

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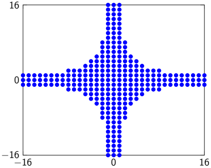

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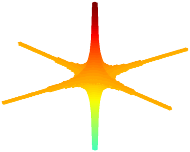

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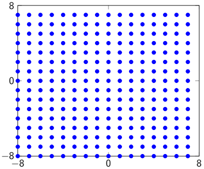

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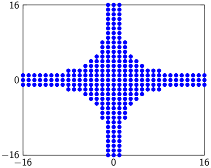

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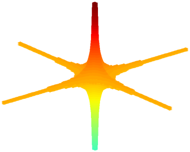

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- fast evaluation of trigonometric polynomial p at nodes $\mathbf{y}_\ell$, $\ell = 0, \ldots, L-1$?

- fast and stable reconstruction of Fourier coefficients $(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in\mathrm{I}}$ from L sampling values $(p(\mathbf{y}_\ell))_{\ell=0}^{L-1}$, $L \geq |\mathrm{I}|$?

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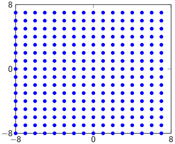
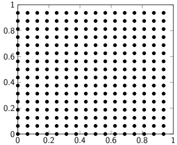
- $\mathrm{I} = \mathbb{Z}^d \cap [-N, N)^d$ full grid, $\mathbf{y}_\ell$ equispaced, $L = |\mathrm{I}|$



$(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in\mathrm{I}}$



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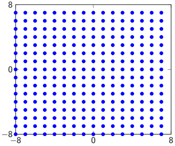

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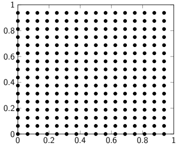
$\overset{\text{d-dim}}{\underset{\text{FFT}}{\longleftrightarrow}}$



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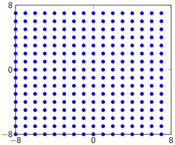

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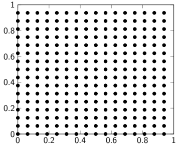

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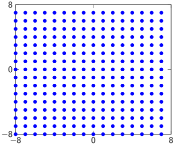

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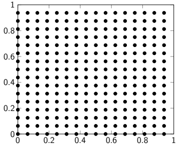

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$\Longrightarrow$ "thinner" frequency index sets I

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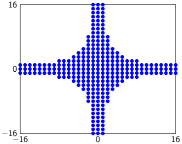

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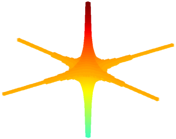

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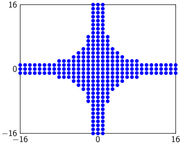

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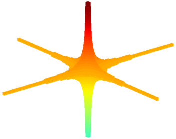

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  e.g. tensor product of smooth 1-periodic one-dimensional functions

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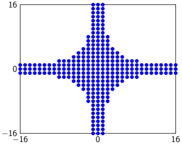

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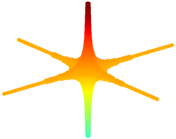

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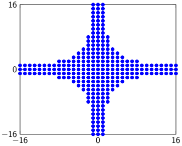
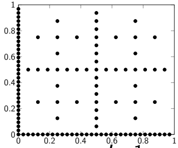


$(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in \mathrm{I}}$



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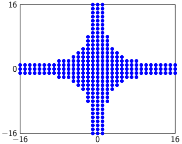

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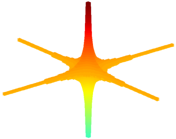

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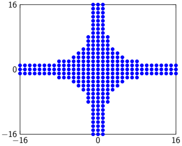

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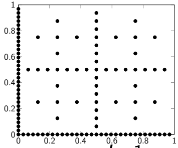

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  - Hyperbolic Cross Fast Fourier Transform
    (Baszenski, Delvos 1989; Hallatschek 1992; Gradinaru 2007; Griebel, Hamaekers 2013)

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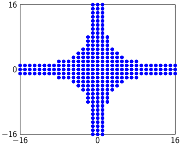

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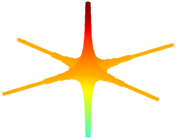

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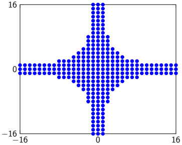

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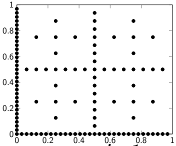

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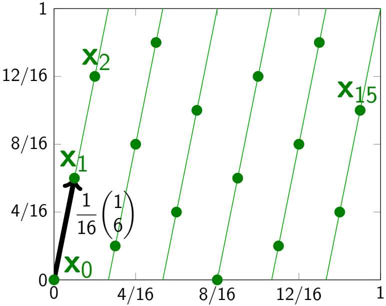
  - Hyperbolic Cross Fast Fourier Transform
    (Baszenski, Delvos 1989; Hallatschek 1992; Gradinaru 2007; Griebel, Hamaekers 2013)
  - problem: stability (Kämmerer, Kunis 2011)

Rank-1 lattices - General

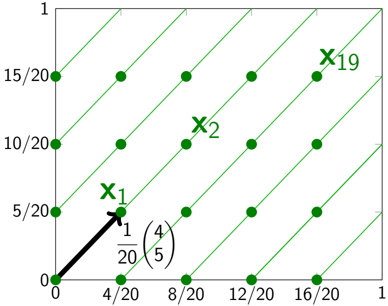
Definition (Rank-1 lattice; e.g. Sloan, Joe 1994)

Let $\mathbf{z} \in \mathbb{Z}^d$ and $M \in \mathbb{N}$ be given. We define the rank-1 lattice

$$\Lambda(\mathbf{z}, M) := \left\{ \mathbf{x}_j := \left(\frac{j}{M}\,\mathbf{z} \right) \bmod \mathbf{1} : \; j = 0, \ldots, M-1 \right\} \subset \mathbb{T}^d.$$



$\mathbf{z} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$, $M = 16$

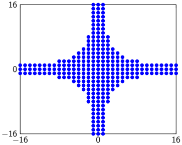


$\mathbf{z} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, $M = 20$

Rank-1 lattices - Lattice based FFT

- evaluation of trigonometric polynomial $p(\mathbf{x}) = \sum_{\mathbf{k}\in\mathrm{I}} \hat{p}_{\mathbf{k}}\, \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{x}}$ at rank-1 lattice nodes $\mathbf{x}_j = (j\mathbf{z}/M) \bmod 1,\ j = 0, \ldots, M-1,$

$$p(\mathbf{x}_j) = \sum_{\mathbf{k}\in\mathrm{I}} \hat{p}_{\mathbf{k}}\, \mathrm{e}^{2\pi\mathrm{i}j\,\mathbf{k}\mathbf{z}/M}$$



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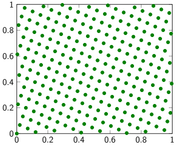

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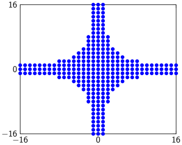

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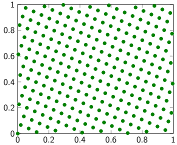

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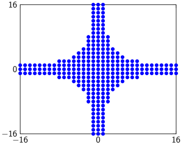
$$p(\mathbf{x}_j) = \sum_{\mathbf{k}\in\mathrm{I}} \hat{p}_{\mathbf{k}}\, \mathrm{e}^{2\pi\mathrm{i}j\,\mathbf{kz}/M} = \sum_{r=0}^{M-1} \left(\sum_{\substack{\mathbf{k}\in\mathrm{I} \\ \mathbf{kz}\equiv r \pmod M}} \hat{p}_{\mathbf{k}} \right) \mathrm{e}^{2\pi\mathrm{i}j\,r/M}$$



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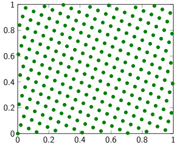

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Rank-1 lattices - Lattice based FFT

- evaluation of trigonometric polynomial $p(\mathbf{x}) = \sum_{\mathbf{k}\in \mathrm{I}} \hat{p}_{\mathbf{k}}\, \mathrm{e}^{2\pi \mathrm{i} \mathbf{k}\mathbf{x}}$ at rank-1 lattice nodes $\mathbf{x}_j = (j\mathbf{z}/M) \bmod 1,\ j = 0, \ldots, M-1,$

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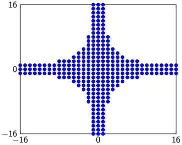
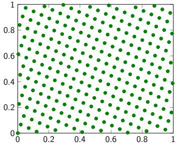
by 1-dim FFT(M) (e.g. Li, Hickernell 2003; Kämmerer 2012)



$(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in \mathrm{I}}$

$\xrightarrow[\text{FFT}]{\text{1-dim}}$

$\mathcal{O}(\ M \log M \ + d|\mathrm{I}|)$

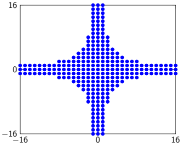


$(p(\mathbf{x}_j))_{j=0}^{M-1}$

Rank-1 lattices - Reconstruction

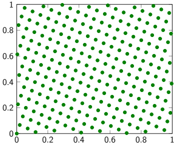
- reconstruction of Fourier coefficients $(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in\mathrm{I}}$ from sampling at rank-1 lattice nodes $\mathbf{x}_j = (j\mathbf{z}/M) \bmod 1$, $j = 0, \ldots, M-1$,

$$p(\mathbf{x}_j) = \sum_{\mathbf{k}\in\mathrm{I}} \hat{p}_{\mathbf{k}}\, \mathrm{e}^{2\pi\mathrm{i}j\,\mathbf{k}\mathbf{z}/M} = \sum_{r=0}^{M-1} \left(\sum_{\substack{\mathbf{k}\in\mathrm{I} \\ \mathbf{k}\mathbf{z} \equiv r \pmod{M}}} \hat{p}_{\mathbf{k}} \right) \mathrm{e}^{2\pi\mathrm{i}j\,r/M}$$



$(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in\mathrm{I}}$

$\overset{?}{\longleftarrow}$



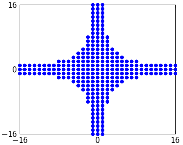
$(p(\mathbf{x}_j))_{j=0}^{M-1}$

Rank-1 lattices - Reconstruction

- reconstruction of Fourier coefficients $(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in\mathrm{I}}$ from sampling at rank-1 lattice nodes $\mathbf{x}_j = (j\mathbf{z}/M) \bmod 1$, $j = 0, \dots, M-1$,

$$p(\mathbf{x}_j) = \sum_{\mathbf{k}\in\mathrm{I}} \hat{p}_{\mathbf{k}}\, \mathrm{e}^{2\pi\mathrm{i} j\, \mathbf{k}\mathbf{z}/M} = \sum_{r=0}^{M-1} \left(\sum_{\substack{\mathbf{k}\in\mathrm{I} \\ \mathbf{k}\mathbf{z} \equiv r \pmod M}} \hat{p}_{\mathbf{k}} \right) \mathrm{e}^{2\pi\mathrm{i} j\, r/M}$$

- perfectly stable, $\kappa(\mathbf{A}) = 1$, $\mathbf{A} := (\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{x}_j})_{j=0;\,\mathbf{k}\in\mathrm{I}}^{M-1}$, if inner sum contains at most one summand



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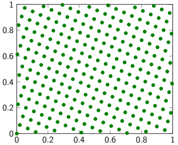

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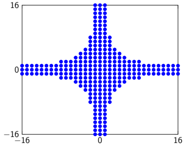
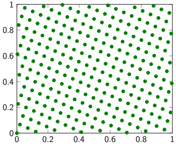
  $\Rightarrow$ reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathrm{I})$ for index set I
  - exists and can be constructed with CBC algorithm (Kämmerer 2012)
  - $\mathrm{I} = H_N^d$: exists with $M \leq C\, N^2 \log^{d-2} N$



$(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in\mathrm{I}}$

$\xleftarrow[\text{FFT}]{\text{1-dim}}$

$\mathcal{O}(N^2 \log^{d-1} N)$



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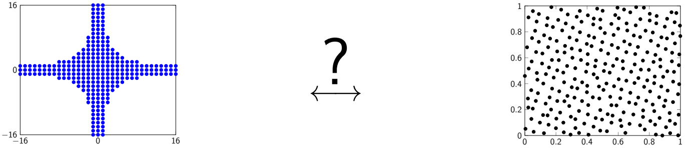


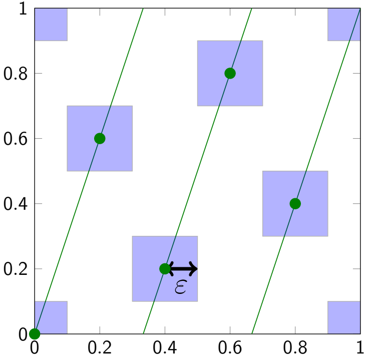
$(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in\mathrm{I}}$ $\qquad$ $(p(\mathbf{y}_j))_{j=0}^{M-1}$

Perturbed rank-1 lattices

- rank-1 lattice $\Lambda(\mathbf{z}, M)$
- set of admissible evaluation nodes

$$\mathcal{Y}_\varepsilon := \{\mathbf{y} \in \mathbb{T}^d : \exists \mathbf{x}_{j'} \in \Lambda(\mathbf{z}, M) : \min_{\mathbf{k}\in\mathbb{Z}^d} \|\mathbf{y} - \mathbf{x}_{j'} + \mathbf{k}\|_\infty \leq \varepsilon\},$$

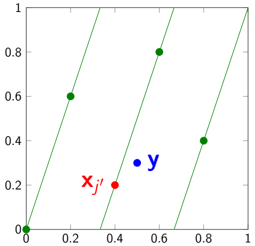
for a pertubation parameter $\varepsilon \geq 0$



$$\mathbf{z} = \begin{pmatrix}1\\3\end{pmatrix},\ M = 5,\ \varepsilon = 0.1$$

Taylor and rank-1 lattice based NFFT - Method

- frequency index set I, rank-1 lattice $\Lambda(\mathbf{z}, M)$, set $\mathcal{Y}_\varepsilon$
- approximate $p(\mathbf{y}) = \sum_{\mathbf{k}\in\mathrm{I}} \hat{p}_{\mathbf{k}} \, \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{y}}$ by Taylor expansion $s_m(\mathbf{y})$, $\mathbf{y} \in \mathcal{Y}_\varepsilon$, at closest rank-1 lattice node $\mathbf{x}_{j'}$



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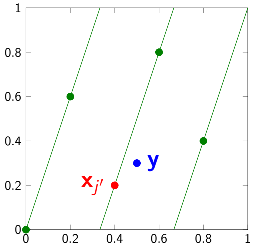
$$s_m(\mathbf{y}) \quad = \quad p(\mathbf{x}_{j'}) + \sum_{0<|\boldsymbol{\nu}|<m} \frac{(\mathbf{y}-\mathbf{x}_{j'})^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \; (D^{\boldsymbol{\nu}} p)(\mathbf{x}_{j'})$$



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$$
\begin{aligned}
s_m(\mathbf{y}) &= p(\mathbf{x}_{j'}) + \sum_{0<|\boldsymbol{\nu}|<m} \frac{(\mathbf{y}-\mathbf{x}_{j'})^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \, (D^{\boldsymbol{\nu}} p)(\mathbf{x}_{j'}) \\
&= \sum_{0\le|\boldsymbol{\nu}|<m} \frac{(\mathbf{y}-\mathbf{x}_{j'})^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \sum_{\mathbf{k}\in\mathrm{I}} (2\pi\mathrm{i}\mathbf{k})^{\boldsymbol{\nu}}\, \hat{p}_{\mathbf{k}}\, \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{x}_{j'}},
\end{aligned}
$$

Taylor and rank-1 lattice based NFFT - Method

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\end{aligned}
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for fixed $\boldsymbol{\nu} \in \mathbb{N}_0^d$, compute $(D^{\boldsymbol{\nu}} p)(\mathbf{x}_j)$ for all $\mathbf{x}_j$, $j = 0, \ldots, M-1$,

with 1-dim FFT(M) in $\mathcal{O}(M \log M + d|\mathrm{I}|)$

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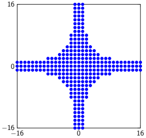
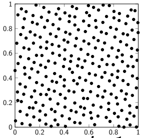


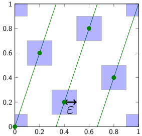
$(\hat{p}_\mathbf{k})_{\mathbf{k}\in\mathrm{I}}$

Taylor and rank-1 lattice $\longrightarrow$ based NFFT

$\mathcal{O}\left(m^d (L + M \log M + d|\mathrm{I}|)\right)$



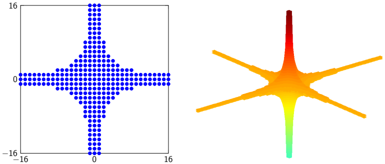
$(p(\mathbf{y}_\ell))_{\ell=0}^{L-1}$



$\mathbf{y}_\ell \in \mathcal{Y}_\varepsilon$

Theorem (V. 2013)

Let H_N^d,
be given,



H_N^d

Taylor and rank-1 lattice based NFFT - Error estimates

Theorem (V. 2013)

Let H_N^d, $p(\mathbf{x}) := \sum_{\mathbf{k}\in H_N^d} \hat{p}_{\mathbf{k}}\, \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{x}}$,
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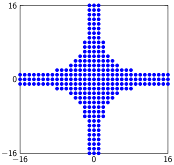

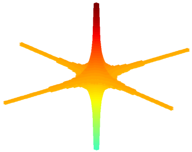

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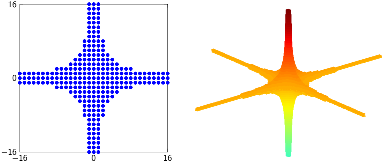
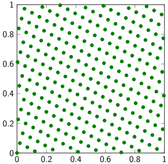


H_N^d



$\Lambda(\mathbf{z}, M)$

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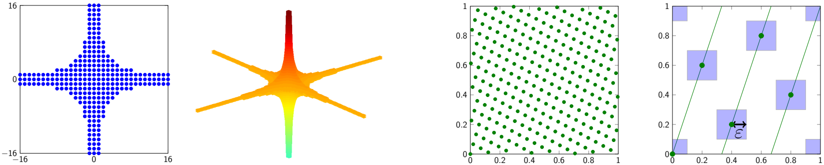

H_N^d $\Lambda(\mathbf{z}, M)$ $\mathcal{Y}_\varepsilon$

Taylor and rank-1 lattice based NFFT - Error estimates

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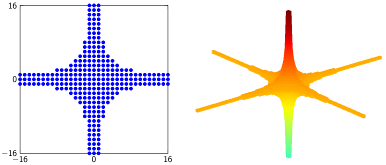
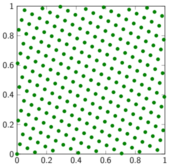
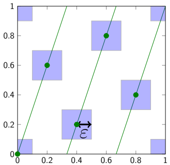


H_N^d



$\Lambda(\mathbf{z}, M)$



$\mathcal{Y}_\varepsilon$

Taylor and rank-1 lattice based NFFT - Error estimates

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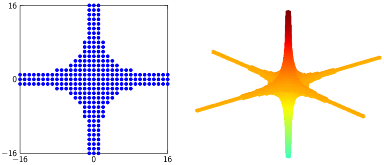
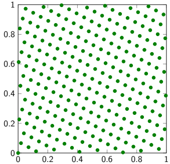
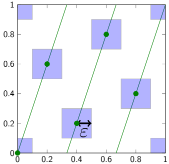


H_N^d



$\Lambda(\mathbf{z}, M)$



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- error estimates can be generalized to other finite frequency index sets $\mathrm{I} \subset \mathbb{Z}^d$, e.g. ℓ_1 balls, energy-based hyperbolic crosses, ...

Taylor and rank-1 lattice based NFFT - Error estimates

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- error estimates can be generalized to other finite frequency index sets $\mathrm{I} \subset \mathbb{Z}^d$, e.g. ℓ_1 balls, energy-based hyperbolic crosses, ...
- numerical results:
  V.: Taylor and rank-1 lattice based nonequispaced fast Fourier transform.

Approx. reconstruction - perturbed rank-1 lattice nodes

- H_N^d, $p(\mathbf{x}) := \sum_{\mathbf{k}\in H_N^d} \hat{p}_{\mathbf{k}}\, \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{x}}$, $\Lambda(\mathbf{z}, M, H_N^d)$

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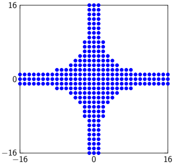
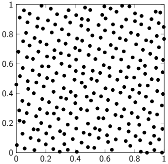
- $\left(p(\mathbf{y}_j)\right)_{j=0}^{M-1} = \mathbf{A}\left(\hat{p}_{\mathbf{k}}\right)_{\mathbf{k}\in H_N^d}$, $\quad \mathbf{A} := \left(\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{y}_j}\right)_{j=0;\ \mathbf{k}\in H_N^d}^{M-1}$

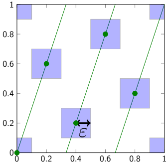


$\left(\hat{p}_{\mathbf{k}}\right)_{\mathbf{k}\in H_N^d}$

$\longrightarrow$



$\left(p(\mathbf{y}_j)\right)_{j=0}^{M-1}$



Approx. reconstruction - perturbed rank-1 lattice nodes

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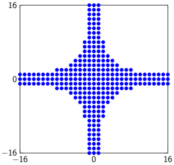
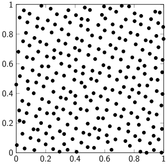
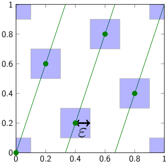


$\left(\hat{p}_{\mathbf{k}}\right)_{\mathbf{k}\in H_N^d}$

$\longleftarrow$

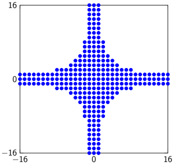


$\left(p(\mathbf{y}_j)\right)_{j=0}^{M-1}$



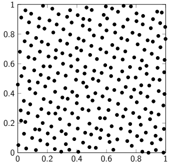
- approximately reconstruct Fourier coefficients $\hat{p}_{\mathbf{k}}$ from M sampling values $p(\mathbf{y}_j)$ at perturbed rank-1 lattice nodes $\mathbf{y}_j$

- H_N^d, $p(\mathbf{x}) := \sum_{\mathbf{k}\in H_N^d} \hat{p}_{\mathbf{k}}\, \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{x}}$, $\Lambda(\mathbf{z}, M, H_N^d)$
- $\left(p(\mathbf{y}_j)\right)_{j=0}^{M-1} = \mathbf{A}\left(\hat{p}_{\mathbf{k}}\right)_{\mathbf{k}\in H_N^d}$, $\quad \mathbf{A} := \left(\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{y}_j}\right)_{j=0;\,\mathbf{k}\in H_N^d}^{M-1}$

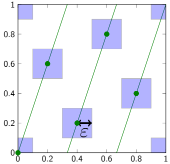


$\left(\hat{p}_{\mathbf{k}}\right)_{\mathbf{k}\in H_N^d}$

$\longleftarrow$



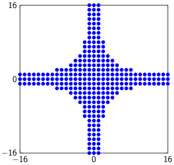
$\left(p(\mathbf{y}_j)\right)_{j=0}^{M-1}$



- approximately reconstruct Fourier coefficients $\hat{p}_{\mathbf{k}}$ from M sampling values $p(\mathbf{y}_j)$ at perturbed rank-1 lattice nodes $\mathbf{y}_j$
- $\mathbf{A}^*\mathbf{A}\left(\hat{p}_{\mathbf{k}}\right)_{\mathbf{k}\in H_N^d} = \mathbf{A}^*\left(p(\mathbf{y}_j)\right)_{j=0}^{M-1}$

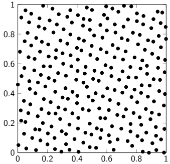
Approx. reconstruction - perturbed rank-1 lattice nodes

- H_N^d, $p(\mathbf{x}) := \sum_{\mathbf{k}\in H_N^d} \hat{p}_{\mathbf{k}}\, \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{x}}$, $\Lambda(\mathbf{z}, M, H_N^d)$
- $\left(p(\mathbf{y}_j)\right)_{j=0}^{M-1} = \mathbf{A}\left(\hat{p}_{\mathbf{k}}\right)_{\mathbf{k}\in H_N^d}$, $\quad \mathbf{A} := \left(\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{y}_j}\right)_{j=0;\ \mathbf{k}\in H_N^d}^{M-1}$

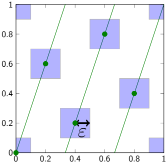


$\left(\hat{p}_{\mathbf{k}}\right)_{\mathbf{k}\in H_N^d}$

$\longleftarrow$



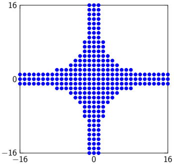
$\left(p(\mathbf{y}_j)\right)_{j=0}^{M-1}$



- approximately reconstruct Fourier coefficients $\hat{p}_{\mathbf{k}}$ from M sampling values $p(\mathbf{y}_j)$ at perturbed rank-1 lattice nodes $\mathbf{y}_j$
- $\mathbf{A}^*\mathbf{A}(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in H_N^d} = \mathbf{A}^*\left(p(\mathbf{y}_j)\right)_{j=0}^{M-1}$
- stability?

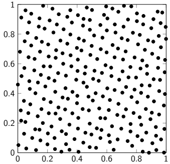
Approx. reconstruction - perturbed rank-1 lattice nodes

- H_N^d, $p(\mathbf{x}) := \sum_{\mathbf{k}\in H_N^d} \hat{p}_{\mathbf{k}}\, \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{x}}$, $\Lambda(\mathbf{z}, M, H_N^d)$
- $\left(p(\mathbf{y}_j)\right)_{j=0}^{M-1} = \mathbf{A}\left(\hat{p}_{\mathbf{k}}\right)_{\mathbf{k}\in H_N^d}$, $\quad \mathbf{A} := \left(\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{y}_j}\right)_{j=0;\ \mathbf{k}\in H_N^d}^{M-1}$

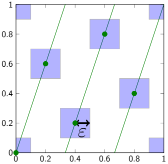


$(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in H_N^d}$

$\longleftarrow$



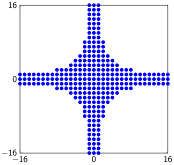
$\left(p(\mathbf{y}_j)\right)_{j=0}^{M-1}$



- approximately reconstruct Fourier coefficients $\hat{p}_{\mathbf{k}}$ from M sampling values $p(\mathbf{y}_j)$ at perturbed rank-1 lattice nodes $\mathbf{y}_j$
- $\mathbf{A}^*\mathbf{A}(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in H_N^d} = \mathbf{A}^*\left(p(\mathbf{y}_j)\right)_{j=0}^{M-1}$
- stability? $\varepsilon < \frac{\ln 2}{2\pi d N}$, $\kappa(\mathbf{A}) \leq \frac{\mathrm{e}^{2\pi d N\varepsilon}}{2-\mathrm{e}^{2\pi d N\varepsilon}} < \frac{2}{2-\mathrm{e}^{2\pi d N\varepsilon}}$ (Kämmerer,Potts,V.)

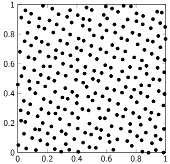
Approx. reconstruction - perturbed rank-1 lattice nodes

- H_N^d, $p(\mathbf{x}) := \sum_{\mathbf{k}\in H_N^d} \hat{p}_{\mathbf{k}}\, \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{x}}$, $\Lambda(\mathbf{z}, M, H_N^d)$
- $\left(p(\mathbf{y}_j)\right)_{j=0}^{M-1} = \mathbf{A}\left(\hat{p}_{\mathbf{k}}\right)_{\mathbf{k}\in H_N^d}$, $\quad \mathbf{A} := \left(\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{y}_j}\right)_{j=0;\ \mathbf{k}\in H_N^d}^{M-1}$

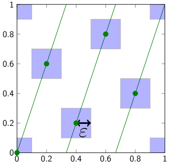


$\left(\hat{p}_{\mathbf{k}}\right)_{\mathbf{k}\in H_N^d}$

Taylor and rank-1 lattice $\longleftarrow$ based NFFT + CGNR/LSQR



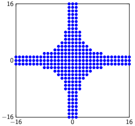
$\left(p(\mathbf{y}_j)\right)_{j=0}^{M-1}$



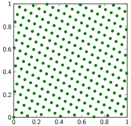
- approximately reconstruct Fourier coefficients $\hat{p}_{\mathbf{k}}$ from M sampling values $p(\mathbf{y}_j)$ at perturbed rank-1 lattice nodes $\mathbf{y}_j$
- $\mathbf{A}^*\mathbf{A}(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in H_N^d} = \mathbf{A}^*\left(p(\mathbf{y}_j)\right)_{j=0}^{M-1}$
- stability? $\varepsilon < \frac{\ln 2}{2\pi d N}$, $\kappa(\mathbf{A}) \le \frac{\mathrm{e}^{2\pi d N\varepsilon}}{2-\mathrm{e}^{2\pi d N\varepsilon}} < \frac{2}{2-\mathrm{e}^{2\pi d N\varepsilon}}$ (Kämmerer,Potts,V.)
- compute $(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in H_N^d}$ with CGNR or LSQR method in K iterations arithmetic complexity $\mathcal{O}\left(K\, m^d (N^2 \log^{d-1} N)\right)$

Summary

- fast, exact evaluation



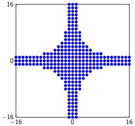
$(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in\mathrm{I}}$ $\xrightarrow[\text{FFT}]{\text{1-dim}}$



$(p(\mathbf{x}_j))_{j=0}^{M-1}$ (e.g. Li, Hickernell 2003; Kämmerer 2012)

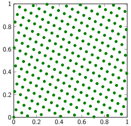
Summary

- fast, exact and stable reconstruction



$\left(\hat{p}_{\mathbf{k}}\right)_{\mathbf{k}\in\mathrm{I}}$

$\xleftarrow[\text{FFT}]{\text{1-dim}}$

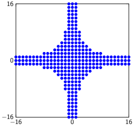


$\left(p(\mathbf{x}_j)\right)_{j=0}^{M-1}$

(Kämmerer 2012)

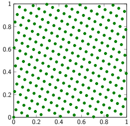
Summary

- fast, exact and stable reconstruction



$(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in\mathrm{I}}$

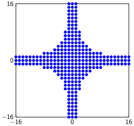
1-dim $\longleftarrow$ FFT



$(p(\mathbf{x}_j))_{j=0}^{M-1}$

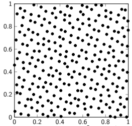
(Kämmerer 2012)

- fast algorithm for approximate evaluation + error estimates

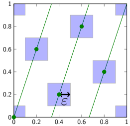


$(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in H_N}$

Taylor and rank-1 lattice $\longrightarrow$ based NFFT



$(p(\mathbf{y}_\ell))_{\ell=0}^{L-1}$

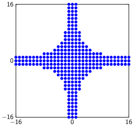


$\mathcal{Y}_\varepsilon$

(V. 2013)

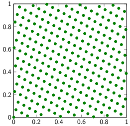
Summary

- fast, exact and stable reconstruction



$(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in\mathrm{I}}$

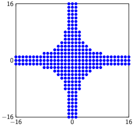
1-dim $\longleftarrow$ FFT



$(p(\mathbf{x}_j))_{j=0}^{M-1}$

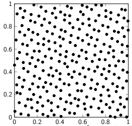
(Kämmerer 2012)

- fast algorithm for approximate evaluation + error estimates

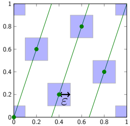


$(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in H_N}$

Taylor and rank-1 lattice $\longrightarrow$ based NFFT



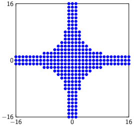
$(p(\mathbf{y}_\ell))_{\ell=0}^{L-1}$



$\mathcal{Y}_\varepsilon$

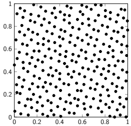
(V. 2013)

- fast and stable approximate reconstruction

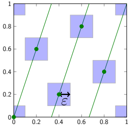


$(\hat{p}_{\mathbf{k}})_{\mathbf{k}\in H_N}$

Taylor and rank-1 lattice $\longleftarrow$ based NFFT + CGNR/LSQR



$(p(\mathbf{y}_j))_{j=0}^{M-1}$



(Kämmerer, Potts, V. 2013)