TECHNISCHE UNIVERSITÄT CHEMNITZ

Faculty of Mathematics
Applied Functional Analysis
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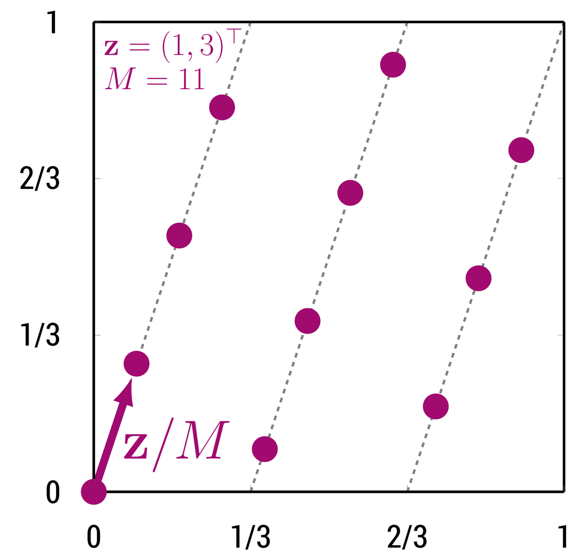
Sparse high-dimensional FFT based on rank-1 lattice sampling

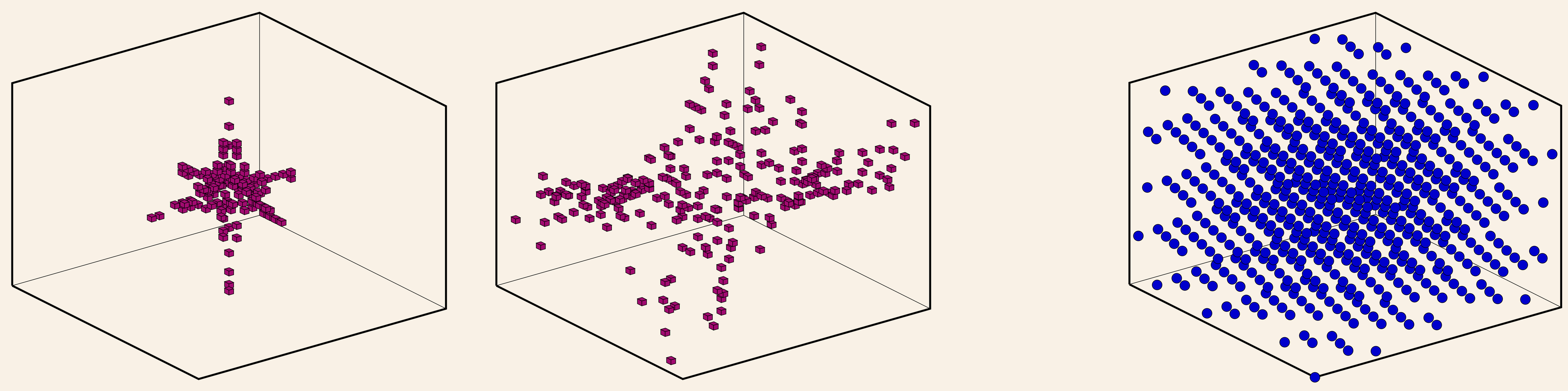


Task

We want to reconstruct a **high-dimensional** (e.g. $d = 10$) periodic signal using a trigonometric polynomial $p_I\colon \mathbb{T}^d \simeq [0,1)^d \to \mathbb{C}$,

$$p_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \hat{p}_{\boldsymbol{k}} \in \mathbb{C},$$

from samples, where $I \subset \mathbb{Z}^d$ is a suitable and generally **unstructured** frequency index set.



frequency index set I sampling nodes

Known frequency index set I

Rank-1 lattice as sampling nodes

Use nodes $\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1}$ of **rank-1 lattice**

$$\Lambda(\boldsymbol{z}, M) := \{\boldsymbol{x}_j : j = 0, \ldots, M-1\}$$

as sampling nodes, where $\boldsymbol{z} \in \mathbb{Z}^d$ and $M \in \mathbb{N}$.

Theorem: reconstructing rank-1 lattice

Let $I \subset \mathbb{Z}^d$, $|I| < \infty$, be arbitrary. Then there exists a rank-1 lattice $\Lambda(\boldsymbol{z}, M)$ with size $M \in \mathbb{N}$,

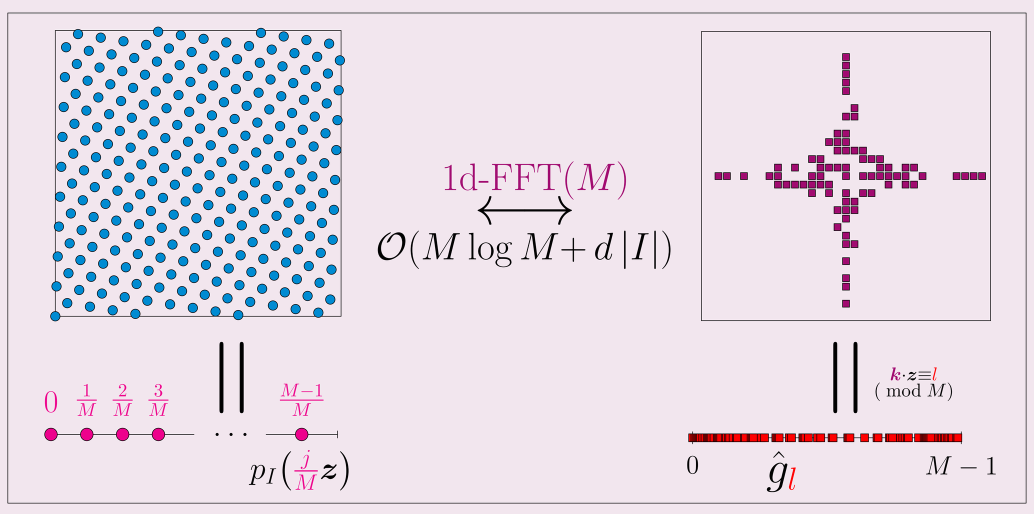
$$|I| \le M \le \max\left\{\tfrac{2}{3}|I|^2, \max\{3\|\boldsymbol{k}\|_\infty : \boldsymbol{k} \in I\}\right\},$$

which allows the reconstruction of $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, from samples $p_I(\boldsymbol{x}_j)$, $j = 0, \ldots, M-1$.
The generating vector $\boldsymbol{z}$ can be found using a simple component-by-component construction.

Computation

Compute $\hat{p}_{\boldsymbol{k}}, \boldsymbol{k} \in I$, by a single **1d-FFT** of length M:

- $(\hat{g}_\ell)_{\ell=0}^{M-1} := \mathrm{FFT}\left(p_I(\boldsymbol{x}_j)_{j=0}^{M-1}\right)$
- $\hat{p}_{\boldsymbol{k}} = \hat{g}_{\boldsymbol{k}\cdot\boldsymbol{z} \bmod M}$, $\boldsymbol{k} \in I$



Approximation results

- for functions from generalized Sobolev spaces of isotropic and dominating mixed smoothness

See also the references in

L. Kämmerer.
High dimensional fast Fourier transform based on rank-1 lattice sampling.
Dissertation. Universitätsverlag Chemnitz, 2014.

L. Kämmerer, D. Potts, and T. Volkmer.
Approximation of multivariate periodic functions by trigonometric polynomials based on sampling along rank-1 lattice with generating vector of Korobov form.
J. Complexity 31, 424 – 456, 2014.

G. Byrenheid, L. Kämmerer, T. Ullrich, and T. Volkmer.
Non-optimality of rank-1 lattice sampling in spaces of hybrid mixed smoothness.
arXiv:1510.08336, Preprint, 2015.

Contact

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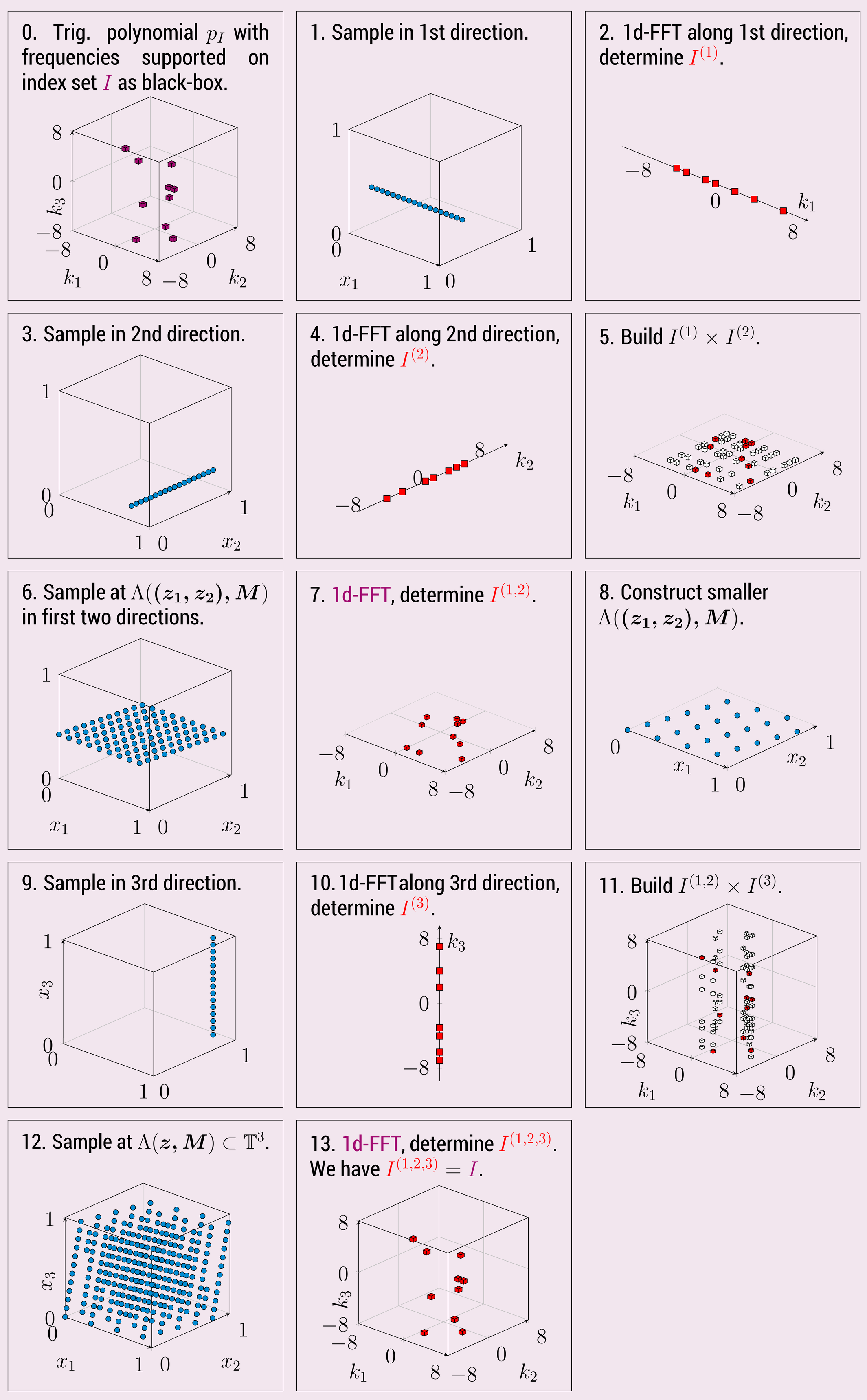
Unknown frequency index set I

Dimension-incremental method

Adaptively construct the index set of frequencies belonging to the largest (or non-zero) Fourier coefficients in a **dimension incremental** way.
Compute **projected Fourier coefficients** from samples along **rank-1 lattices** and then determine **frequency locations**.

- MATLAB implementation available
- samples: $\mathcal{O}(d\,|I|^2 N)$,
  arithm. operations: $\mathcal{O}(d\,|I|^3 + d\,|I|^2 N \log(|I|\,N))$
  (if $I \subset ([-N, N]^d \cap \mathbb{Z}^d)$ and $|I| \gtrsim \sqrt{N}$)

Example: Reconstruction of 3-dim. trigonometric polynomial



See also the references in

D. Potts and T. Volkmer.
Sparse high-dimensional FFT based on rank-1 lattice sampling.
Appl. Comput. Harm. Anal., accepted, 2015.

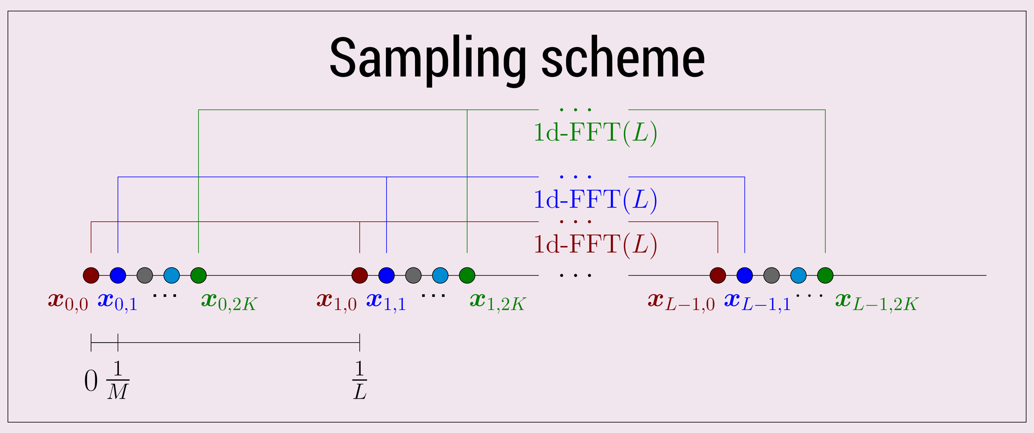
Unknown I with **less samples**

Sparse FFT via MUSIC / ESPRIT

Use shifted sampling:

Choose $K, L \in \mathbb{N}$ such that $M > (2K+1)L$. Then,

$$h_L[\ell, m] := p_I(\boldsymbol{x}_{\ell,m}), \quad \boldsymbol{x}_{\ell,m} := \left(\tfrac{\ell}{L} + \tfrac{m}{M}\right)\boldsymbol{z},$$
$$\ell = 0, \ldots, L-1, \quad m = 0, \ldots, 2K.$$

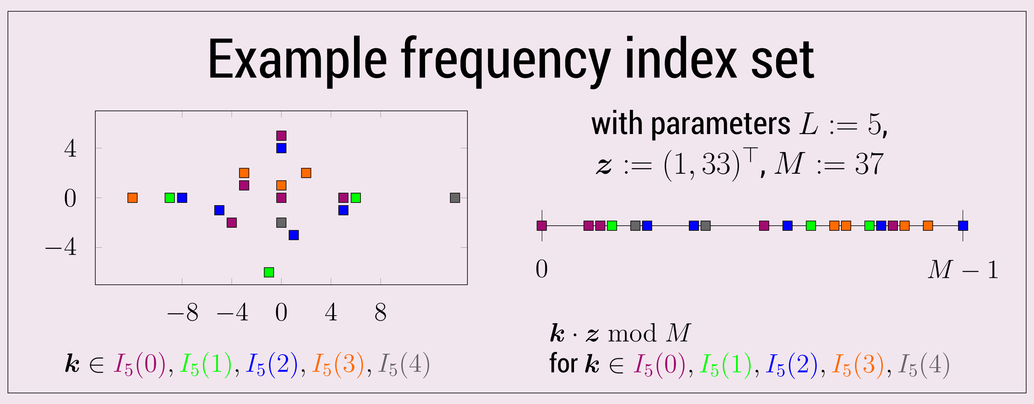


For $m = 0, \ldots, 2K$, compute 1d-FFT of length L

$$\hat{h}_L[n, m] := \sum_{\ell=0}^{L-1} h_L[\ell, m]\, \mathrm{e}^{-2\pi\mathrm{i}\ell n/L}$$
$$= \sum_{\ell=0}^{L-1} \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}(\ell/L + m/M)\,\boldsymbol{k}\cdot\boldsymbol{z}}\, \mathrm{e}^{-2\pi\mathrm{i}\ell n/L}$$
$$= L \sum_{\boldsymbol{k}\in I_L(n)} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}m\,\boldsymbol{k}\cdot\boldsymbol{z}/M},$$

$$I_L(n) := \{\boldsymbol{k} \in I : \boldsymbol{k}\cdot\boldsymbol{z} \equiv n \pmod{L}\},$$
$$n = 0, \ldots, L-1.$$

For each $n = 0, \ldots, L-1$, we interpret $\hat{h}_L[n, m]$ as sampling set of a 1-dim. trig. polynomial with frequencies $\boldsymbol{k}\cdot\boldsymbol{z} \bmod M$, $\boldsymbol{k} \in I_L(n)$, at nodes $\frac{m}{M}$ and apply Prony's method (MUSIC / ESPRIT) to determine $|I_L(n)|$, $\boldsymbol{k} \in I_L(n)$, $\hat{p}_{\boldsymbol{k}}$.



If not all frequencies are detected (e.g. $|I_L(n)| \ge K$ for some n), repeat on $p_I - \sum_{\boldsymbol{k}\in I_{\text{detected}}} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\circ}$.

- samples: $(2K+1)\, L = \mathcal{O}(|I|)$,
  arithm. operations: $\mathcal{O}(KL^2 + K^3L) = \mathcal{O}(|I|^{5/3})$
  (one iteration, $K = \mathcal{O}(|I|^{1/3})$, $L = \mathcal{O}(|I|^{2/3})$)

See also the references in

D. Potts, M. Tasche, and T. Volkmer.
Efficient spectral estimation by MUSIC and ESPRIT with application to sparse FFT.
Technische Universität Chemnitz, Preprint 2015-14, 2015.