Approximation of multivariate periodic functions by trigonometric polynomials based on sampling along rank-1 lattice with generating vector of Korobov form

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In this paper, we present error estimates for the approximation of multivariate periodic functions in periodic Hilbert spaces of isotropic and dominating mixed smoothness by trigonometric polynomials. The approximation is based on sampling of the multivariate functions on rank-1 lattices. We use reconstructing rank-1 lattices with generating vectors of Korobov form for the sampling and generalize the technique from [25], in order to show that the aliasing error of that approximation is of the same order as the error of the approximation using the partial sum of the Fourier series. The main advantage of our method is that the computation of the Fourier coefficients of such a trigonometric polynomial, which we use as approximant, is based mainly on a one-dimensional fast Fourier transform, cf. [16, 13]. This means that the arithmetic complexity of the computation depends only on the cardinality of the support of the trigonometric polynomial in the frequency domain. Numerical results are presented up to dimension $d = 10$.

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1 Introduction

We approximate functions $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ from the Hilbert space

$$\mathcal{H}^\omega(\mathbb{T}^d) := \left\{ f \in L^1(\mathbb{T}^d) \colon \|f|\mathcal{H}^\omega(\mathbb{T}^d)\| := \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \omega(\boldsymbol{k})^2 |\hat{f}_{\boldsymbol{k}}|^2} < \infty \right\},$$

where $\omega\colon \mathbb{Z}^d \to (c,\infty]$, $c > 0$, is a weight function, by trigonometric polynomials p with frequencies supported on an index set $I \subset \mathbb{Z}^d$ of finite cardinality, $p(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$. Thereby, we are especially interested in the higher-dimensional cases, i.e., $d \geq 4$. As usual, we denote the Fourier coefficients of the function f by

$$\hat{f}_{\boldsymbol{k}} := \int_{\mathbb{T}^d} f(\boldsymbol{x}) \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}} \mathrm{d}\boldsymbol{x}, \quad \boldsymbol{k} \in \mathbb{Z}^d.$$

We remark that for the special choice $\omega \equiv 1$, we have $\mathcal{H}^\omega(\mathbb{T}^d) = L^2(\mathbb{T}^d)$. One theoretical possibility to obtain such a trigonometric polynomial p is to formally approximate the function f by the Fourier partial sum

$$S_I f := \sum_{\boldsymbol{k}\in I} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\circ},$$

where $I \subset \mathbb{Z}^d$ is a frequency index set of finite cardinality. Since $S_I f$ is the truncated Fourier series of the function f, this approximation causes a truncation error $\|f - S_I f\|$, where $\|\cdot\|$ is an arbitrarily chosen norm. For a function $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ we choose a frequency index set $I = I_N := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \omega(\boldsymbol{k})^{1/\nu} \leq N\}$ of refinement $N \in \mathbb{R}$, $N \geq 1$, $\nu > 0$, and obtain

$$\|f - S_{I_N} f|L^2(\mathbb{T}^d)\| \leq N^{-\nu} \|f|\mathcal{H}^\omega(\mathbb{T}^d)\|,$$

see Lemma 3.3. We stress the fact that $S_{I_N} f$ is the best approximation of the function f with respect to the $L^2(\mathbb{T}^d)$ norm in the space $\Pi_{I_N} := \operatorname{span}\{\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\circ} \colon \boldsymbol{k} \in I_N\}$ of trigonometric polynomials with frequencies supported on the index set I_N and that the operator $S_{I_N} \colon L^1(\mathbb{T}^d) \to \Pi_{I_N}$ only depends on the frequency index set I_N. A similar estimate for the special case of product weights can be found in [18].

Since, in general, we do not know the Fourier coefficients $\hat{f}_{\boldsymbol{k}}$, we are going to approximate the function f from samples using the approximated Fourier partial sum

$$\tilde{S}_{I_N} f := \sum_{\boldsymbol{k}\in I_N} \tilde{\hat{f}}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\circ}.$$

We compute the approximated Fourier coefficients $\tilde{\hat{f}}_{\boldsymbol{k}} \in \mathbb{C}$, $\boldsymbol{k} \in I_N$, of the function f using sampling values. Therefore, we assume the function f to be continuous. We sample f along a rank-1 lattice and we compute the approximated Fourier coefficients $\tilde{\hat{f}}_{\boldsymbol{k}}$ by the rank-1 lattice rule

$$\tilde{\hat{f}}_{\boldsymbol{k}} := \frac{1}{M} \sum_{j=0}^{M-1} f(\boldsymbol{x}_j)\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j} \quad \text{for } \boldsymbol{k} \in I_N, \tag{1.1}$$

where the sampling nodes $\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1}$ are the nodes of a so-called reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N)$ with generating vector $\boldsymbol{z} \in \mathbb{Z}^d$ and rank-1 lattice size $M \in \mathbb{N}$ for the frequency index set I_N, see Section 2.2 for the definition. Lattice rules have extensively been investigated for the integration of functions of many variables for a long time, cf. e.g., [23, 4, 5] and the extensive reference list therein. Especially, rank-1 lattice rules have also been studied for the approximation of multivariate functions of suitable smoothness, cf. [25, 19, 18, 20]. Furthermore, there exist already comprehensive tractability results for numerical integration and approximation using rank-1 lattices, see [21, 18].

Since we consider the partial sum $\tilde{S}_{I_N} f$ of the approximated Fourier coefficients $\tilde{\hat{f}}_{\boldsymbol{k}}$ instead of the Fourier partial sum $S_{I_N} f$ of Fourier coefficients $\hat{f}_{\boldsymbol{k}}$, we obtain an additional error. As in [16], we estimate the approximation error $\|f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\|$ using the triangle inequality $\|f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\| \leq \|f - S_{I_N} f | L^2(\mathbb{T}^d)\| + \|S_{I_N} f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\|$, where $\|f - S_{I_N} f | L^2(\mathbb{T}^d)\|$ is called the truncation error and $\|S_{I_N} f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\|$ is called the aliasing error.

In this paper, we consider frequency index sets I_N of special structure and show that there exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N)$ of reasonable size M, see Section 2.2 for the definition, such that the order of the aliasing error $\|S_{I_N} f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\|$ is bounded by the order of the truncation error $\|f - S_{I_N} f | L^2(\mathbb{T}^d)\|$. To this end, we use the highly structured rank-1 lattice rules with generating vector of Korobov form. This allows us to generalize the ideas of V. N. Temlyakov, see [25], in order to estimate the aliasing error. We consider, similar to [7] and as in [16], continuous functions f from the Hilbert space

$$\mathcal{H}^{\omega}(\mathbb{T}^d) = \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d) := \left\{ f \in L^1(\mathbb{T}^d) \colon \|f | \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\| := \sqrt{\sum\nolimits_{\boldsymbol{k} \in \mathbb{Z}^d} \omega^{\alpha,\beta}(\boldsymbol{k})^2 |\hat{f}_{\boldsymbol{k}}|^2} < \infty \right\},$$

where the parameter $\beta \in \mathbb{R}$, $\beta \geq 0$, characterizes the dominating mixed smoothness, the parameter $\alpha \in \mathbb{R}$, $\alpha > -\beta$, characterizes the isotropic smoothness, and the weights $\omega(\boldsymbol{k}) = \omega^{\alpha,\beta}(\boldsymbol{k})$ are given by

$$\omega^{\alpha,\beta}(\boldsymbol{k}) := \max(1, \|\boldsymbol{k}\|_1)^{\alpha} \prod_{s=1}^{d} \max(1, |k_s|)^{\beta}, \ \boldsymbol{k} := (k_1, \ldots, k_d)^{\top}.$$

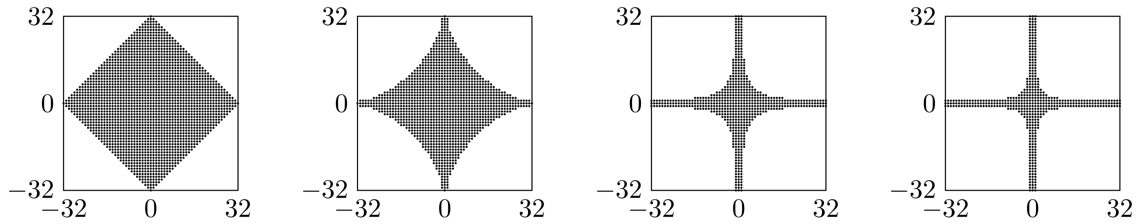
We remark that one can use various equivalent weights $\omega(\boldsymbol{k})$ which have different approximation properties for large dimensions d, cf. [17]. Furthermore, we define the corresponding frequency index sets $I_N = I_N^{d,T}$, $N \in \mathbb{R}$, $N \geq 1$, $T \in \mathbb{R}$, $-\infty < T < 1$, by

$$I_N^{d,T} := \left\{ \boldsymbol{k} \in \mathbb{Z}^d \colon \omega^{-\frac{T}{1-T}, \frac{1}{1-T}}(\boldsymbol{k}) = \max(1, \|\boldsymbol{k}\|_1)^{-\frac{T}{1-T}} \prod_{s=1}^{d} \max(1, |k_s|)^{\frac{1}{1-T}} \leq N \right\}.$$

In the cases $0 < T < 1$, the frequency index sets $I_N^{d,T}$ are called energy-norm based hyperbolic crosses, see [2, 3], and in the case $T = 0$ symmetric hyperbolic crosses. As a natural extension for $T = -\infty$, we define the frequency index set $I_N^{d,-\infty}$ as the d-dimensional ℓ_1-ball of size N,

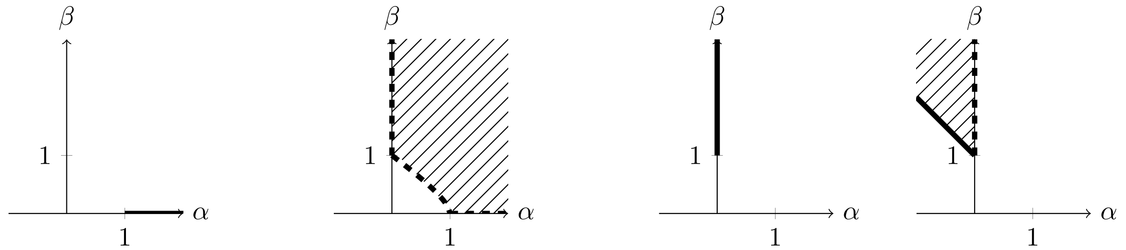
$$I_N^{d,-\infty} := \left\{ \boldsymbol{k} \in \mathbb{Z}^d \colon \max(1, \|\boldsymbol{k}\|_1) \leq N \right\}.$$

Figure 1.1 illustrates the frequency index sets $I_N^{d,T}$ in the two-dimensional case for different choices $-\infty \leq T < 1$ of the parameter T. The cardinalities of the frequency index sets $I_N^{d,T}$



l_1-ball $I_{32}^{2,-\infty}$ index set $I_{32}^{2,-5}$ symmetric hyperbolic cross $I_{32}^{2,0}$ energy-norm based hyperbolic cross $I_{32}^{2,1/2}$

Figure 1.1: Two-dimensional frequency index sets $I_{32}^{2,T}$ for $T \in \{-\infty, -5, 0, \frac{1}{2}\}$.



$\alpha > \frac{d}{2}$ and $\beta = 0$ $(T = -\infty)$

$\beta > 0$ and $\alpha > \max\left(0, (\frac{1}{4} - \beta + \frac{1}{4}\sqrt{8\beta+1})d\right)$ $(-\infty < T < 0)$

$\alpha = 0$ and $\beta > 1$ $(T = 0)$

$\alpha < 0$ and $\beta > 1 - \alpha$ $(0 < T < 1)$

Figure 1.2: Visualization of the admissible values of α and β in the case $d = 2$, such that (1.3) and (1.4) are valid. We set the corresponding values $T := -\alpha/\beta$.

are given in Lemma 4.1, which reads for fixed $d \in \mathbb{N}$ and $T := -\alpha/\beta$ as follows

$$\left| I_N^{d,-\alpha/\beta} \right| = \begin{cases} \Theta(N^d) & \text{for } \alpha > 0 \text{ and } \beta = 0 \quad (\Longleftrightarrow T = -\infty), \\ \Theta(N^{d\frac{\beta+\alpha}{d\beta+\alpha}}) & \text{for } \alpha > 0 \text{ and } \beta > 0 \quad (\Longleftrightarrow -\infty < T < 0), \\ \Theta(N \log^{d-1} N) & \text{for } \alpha = 0 \text{ and } \beta > 0 \quad (\Longleftrightarrow T = 0), \\ \Theta(N) & \text{for } \alpha < 0 \text{ and } \beta > -\alpha \quad (\Longleftrightarrow 0 < T < 1). \end{cases} \tag{1.2}$$

In this setting, we obtain that the $L^2(\mathbb{T}^d)$ truncation error is bounded by

$$\|f - S_{I_N^{d,-\alpha/\beta}} f | L^2(\mathbb{T}^d)\| \le N^{-(\alpha+\beta)} \, \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|,$$

see Lemma 4.4. The main result of this paper is, that for fixed dimension $d \in \mathbb{N}$, $d \ge 2$ there exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N^{d,-\alpha/\beta})$ with generating vector $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top \in \mathbb{Z}^d$ of Korobov form and size

$$M = \begin{cases} \mathcal{O}(N^d) & \text{for } \alpha > \frac{d}{2} \text{ and } \beta = 0, \\ \mathcal{O}(N^{d\frac{(2d\beta+\alpha)(\beta+\alpha)}{(d\beta+\alpha)^2}}) & \text{for } \beta > 0 \text{ and } \alpha > \max\left(0, (\frac{1}{4} - \beta + \frac{1}{4}\sqrt{8\beta+1})d\right), \\ \mathcal{O}(N^2 \log^{d-1} N) & \text{for } \alpha = 0 \text{ and } \beta > 1, \\ \mathcal{O}(N^2) & \text{for } \alpha < 0 \text{ and } \beta > 1 - \alpha, \end{cases} \tag{1.3}$$

such that the aliasing error is bounded by

$$\|S_{I_N^{d,-\alpha/\beta}} f - \tilde{S}_{I_N^{d,-\alpha/\beta}} f | L^2(\mathbb{T}^d)\| \leq C(d,\alpha,\beta)\; N^{-(\alpha+\beta)}\; \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|, \tag{1.4}$$

where $C(d,\alpha,\beta) > 0$ is a constant which only depends on d, α, β. In the cases where $\alpha \geq 0$, we obtain estimate (1.4) from Theorem 4.7 and the lower bound for the size M in (1.3) due to (4.5) and (4.1). For $\alpha < 0$, we infer estimate (1.4) from Theorem 4.10 and the lower bound for the size M in (1.3) due to (4.6) and (4.1). Figure 1.2 visualizes the different cases for the admissible values of the isotropic smoothness α and the dominating mixed smoothness β in (1.3) and (1.4) in the two-dimensional case and gives the corresponding values of the parameter T. In Figure 1.3, the admissible values of α and β are shown for the cases $d = 2, 6, 10$. Comparing the number M of sampling nodes $\boldsymbol{x}_j$ in (1.3) and the number

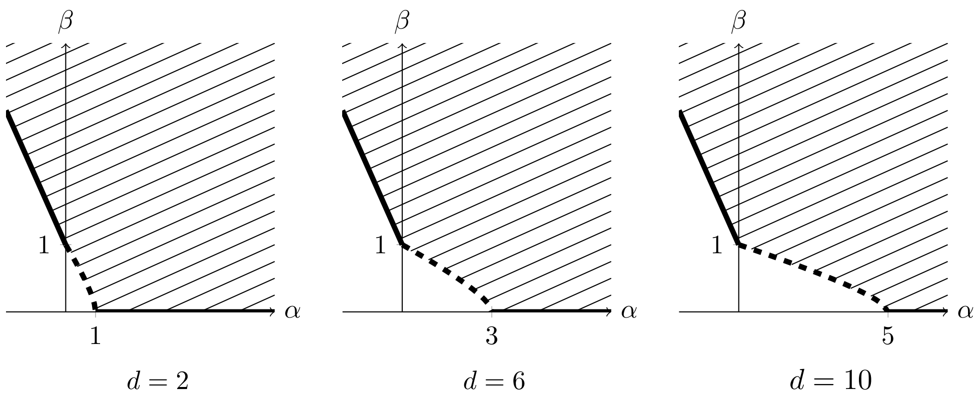


Figure 1.3: Visualization of the admissible values of α and β in the cases $d = 2, 6, 10$, such that (1.3) and (1.4) are valid.

$|I_N^{d,-\alpha/\beta}|$ of frequency indices in (1.2), our results yield in general an oversampling, i.e.,

$$\frac{M}{\left|I_N^{d,-\alpha/\beta}\right|} = \begin{cases} \mathcal{O}(1) & \text{for } \alpha > \frac{d}{2} \text{ and } \beta = 0, \\ \mathcal{O}\left(N^{\frac{d^2\beta(\beta+\alpha)}{(d\beta+\alpha)^2}}\right) & \text{for } \beta > 0 \text{ and } \alpha > \max\left(0, (\frac{1}{4} - \beta + \frac{1}{4}\sqrt{8\beta+1})d\right), \\ \mathcal{O}(N) & \text{for } \alpha = 0 \text{ and } \beta > 1, \\ \mathcal{O}(N) & \text{for } \alpha < 0 \text{ and } \beta > 1 - \alpha, \end{cases}$$

for fixed d, α, and β. In the case $\alpha > \frac{d}{2}$ and $\beta = 0$, where the frequency index sets $I_N^{d,-\infty}$ are l_1-balls, the asymptotic order of M and $|I_N^{d,-\infty}|$ in N is obviously identical. Considering the case $\alpha < 0$ and $\beta > 1 - \alpha$, where the frequency index sets $I_N^{d,-\alpha/\beta}$ are energy-norm based hyperbolic crosses, we obtain a gap between M and $|I_N^{d,-\alpha/\beta}|$ in the asymptotic order in N. However, this gap is necessary in order to obtain an orthogonal Fourier transform as given by (1.1), cf. [12, Lemma 2.1]. Note that in the case $\alpha = 0$, the oversampling factors $M/|I_N^{d,0}|$, i.e., ratios of the rank-1 lattice sizes M and the cardinalities of the symmetric hyperbolic cross index sets $I_N^{d,0}$ are still moderate for reasonable problem sizes compared to the asymptotic statement $\mathcal{O}(N)$ in (1.3) and (1.2), see Table 5.1.

Let us mention that sampling on (generalized) sparse grids, see [26, 1, 30, 10, 27, 2, 24, 3, 6, 22, 11, 7], is another intensively studied approach used to approximate functions of the classes $\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$. One advantage of the spare grids method is that only $|I_N^{d,-\alpha/\beta}|$ many samples are required. Furthermore, for $\alpha = 0$, there exists a fast algorithm for computing the approximation of the Fourier partial sum $S_{I_N^{d,0}}$ of a function f in $\mathcal{O}(N \log^d N)$ arithmetic operations. However, the computation may be numerically unstable in this setting, cf. [14]. Known upper bounds for the approximation errors are discussed in Section 4.3. We stress again, that the outstanding property of the sampling method (1.1) discussed in this paper is that the computation of the approximated Fourier coefficients $\tilde{\hat{f}}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I_N^{d,0}$, is perfectly stable and takes $\mathcal{O}(N^2 \log^d N)$ arithmetic operations, since it is mainly based on a one-dimensional fast Fourier transform (FFT), cf. [19] and [15].

The paper is organized as follows: We discuss the exact reconstruction of trigonometric polynomials from samples along a rank-1 lattice in Section 2 and prove the existence of a special rank-1 lattice with certain properties. Based on these special properties, we show general estimates for the aliasing error for general frequency index sets I_N in Section 3. Then, in Section 4, we consider the approximation error $\|f - \tilde{S}_{I_N^{d,-\alpha/\beta}} f | L^2(\mathbb{T}^d)\|$. Therefore, we present the estimates for the truncation error in Section 4.1. In Section 4.2, we prove the results (1.3) and (1.4). We compare these results with previously known ones in Section 4.3. Finally, we present numerical tests in Section 5 in order to illustrate the theoretical results and we give some concluding remarks in Section 6.

Notation. As usual, $\mathbb{Z}$ denotes the integers, $\mathbb{N}$ the natural numbers, $\mathbb{R}$ the real numbers, $\mathbb{C}$ the complex numbers and i the imaginary unit. We denote the torus by $\mathbb{T} \simeq [0,1)$, where opposite sides are identified with each other, and we use the letter $d \in \mathbb{N}$ for the dimension. Typically, the letter I denotes a subset of $\mathbb{Z}^d$ of finite cardinality and we use I as a frequency index set. We use the notation I_N, $N \in \mathbb{R}$, $N \geq 1$, to express that we have defined a sequence of frequency index sets depending on a refinement parameter N and we often have the inclusions $I_{N'} \subset I_{N''}$ for $N' \leq N''$. Furthermore, the vector $\boldsymbol{x} := (x_1, \ldots, x_d)^\top$ is usually taken from the d-dimensional torus $\mathbb{T}^d$, the vectors $\boldsymbol{z}$, $\boldsymbol{k}$ and $\boldsymbol{h}$ are taken from $\mathbb{Z}^d$. For a vector $\boldsymbol{a} \in \mathbb{R}^d$, we define the p-norm of $\boldsymbol{a}$ by $\|\boldsymbol{a}\|_p := (\sum_{t=1}^d |a_t|^p)^{1/p}$ for $1 \leq p < \infty$ and $\|\boldsymbol{a}\|_\infty := \max_{t=1}^d |a_t|$. By the expression $\boldsymbol{k}\boldsymbol{z}$ for two arbitrary d-dimensional vectors $\boldsymbol{k} := (k_1, \ldots, k_d)^\top$ and $\boldsymbol{z} := (z_1, \ldots, z_d)^\top$, we mean the scalar product $\boldsymbol{k}\boldsymbol{z} := \sum_{t=1}^d k_t z_t$. The space of all (complex-valued) functions on the d-dimensional torus $\mathbb{T}^d$ for which the p-th power of the absolute value is Lebesgue integrable is denoted by $L^p(\mathbb{T}^d)$, $1 \leq p < \infty$, and the norm $\|f|L^p(\mathbb{T}^d)\|$ of a function $f \in L^p(\mathbb{T}^d)$ is defined by $\|f|L^p(\mathbb{T}^d)\| := \left(\int_{\mathbb{T}^d} |f(\boldsymbol{x})|^p \mathrm{d}\boldsymbol{x}\right)^{1/p}$.

2 Approximation based on rank-1 lattice sampling

2.1 Reconstruction of trigonometric polynomials from samples

As already discussed in Section 1, we approximate a function $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ using a trigonometric polynomial p. Here, we use the following approach from [13]. For a given frequency index set $I \subset \mathbb{Z}^d$ of finite cardinality, we exactly reconstruct the Fourier coefficients $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, of an arbitrarily chosen trigonometric polynomial $p(\boldsymbol{x}) := \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$ with frequencies supported on I from sampling values $p(\boldsymbol{x}_j)$. As sampling nodes $\boldsymbol{x}_j$, $j = 0, \ldots, M-1$, we use the nodes of a rank-1 lattice $\Lambda(\boldsymbol{z}, M) := \{\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1} \colon j = 0, \ldots, M-1\}$ with generating vector $\boldsymbol{z} \in \mathbb{Z}^d$ of size $M \in \mathbb{N}$. Formally, the Fourier coefficients $\hat{p}_{\boldsymbol{k}}$ are given by

$$\hat{p}_{\boldsymbol{k}} := \int_{\mathbb{T}^d} p(\boldsymbol{x})\, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}} \mathrm{d}\boldsymbol{x}$$

and we approximate this integral by the (rank-1) lattice rule

$$\frac{1}{M}\sum_{j=0}^{M-1} p(\boldsymbol{x}_j)\mathrm{e}^{-2\pi \mathrm{i}\boldsymbol{k}\boldsymbol{x}_j} = \frac{1}{M}\sum_{j=0}^{M-1} p\left(\frac{j}{M}\boldsymbol{z}\right) \mathrm{e}^{-2\pi \mathrm{i} j \boldsymbol{k}\boldsymbol{z}/M} =: \tilde{\hat{p}}_{\boldsymbol{k}}.$$

Now, we ask for the exactness of this cubature formula, i.e., when is $\hat{p}_{\boldsymbol{k}} = \tilde{\hat{p}}_{\boldsymbol{k}}$ for all $\boldsymbol{k} \in I$. Since we have

$$\tilde{\hat{p}}_{\boldsymbol{k}} = \frac{1}{M}\sum_{j=0}^{M-1}\sum_{\boldsymbol{k}'\in I} \hat{p}_{\boldsymbol{k}'}\, \mathrm{e}^{2\pi \mathrm{i} j \boldsymbol{k}'\boldsymbol{z}/M} \mathrm{e}^{-2\pi \mathrm{i} j \boldsymbol{k}\boldsymbol{z}/M} = \sum_{\boldsymbol{k}'\in I} \hat{p}_{\boldsymbol{k}'} \frac{1}{M}\sum_{j=0}^{M-1} \mathrm{e}^{2\pi \mathrm{i} j(\boldsymbol{k}'-\boldsymbol{k})\boldsymbol{z}/M},$$

we need the condition

$$\frac{1}{M}\sum_{j=0}^{M-1} \mathrm{e}^{2\pi \mathrm{i} j(\boldsymbol{k}'-\boldsymbol{k})\boldsymbol{z}/M} = \begin{cases} 1 & \text{for } \boldsymbol{k} = \boldsymbol{k}' \\ 0 & \text{for } \boldsymbol{k} \neq \boldsymbol{k}', \boldsymbol{k}, \boldsymbol{k}' \in I, \end{cases} \tag{2.1}$$

to be fulfilled. This is the case if and only if

$$(\boldsymbol{k}' - \boldsymbol{k})\boldsymbol{z} \not\equiv 0 \pmod{M}\ \forall \boldsymbol{k}, \boldsymbol{k}' \in I, \boldsymbol{k} \neq \boldsymbol{k}', \tag{2.2}$$

$$\iff \boldsymbol{k}\boldsymbol{z} \not\equiv \boldsymbol{k}'\boldsymbol{z} \pmod{M}\ \forall \boldsymbol{k}, \boldsymbol{k}' \in I, \boldsymbol{k} \neq \boldsymbol{k}', \tag{2.3}$$

see [13, Section 2]. Introducing the difference set $\mathcal{D}(I)$ for the index set I, $\mathcal{D}(I) := \{\boldsymbol{k} - \boldsymbol{k}' \colon \boldsymbol{k}, \boldsymbol{k}' \in I\}$, we can rewrite the above conditions to

$$\boldsymbol{m}\boldsymbol{z} \not\equiv 0 \pmod{M}\ \forall \boldsymbol{m} \in \mathcal{D}(I) \setminus \{\boldsymbol{0}\}. \tag{2.4}$$

2.2 Reconstructing rank-1 lattices

A rank-1 lattice $\Lambda(\boldsymbol{z}, M)$ which fulfills one of the (equivalent) *reconstruction properties* (2.1),(2.2),(2.3),(2.4) for a given frequency index set I will be called *reconstructing rank-1 lattice*

$$\Lambda(\boldsymbol{z}, M, I) := \left\{\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1} \colon j = 0, \ldots, M-1, \text{ and condition (2.2) is fulfilled}\right\}.$$

Under mild assumptions, e.g., $I \subset \mathbb{Z}^d \cap (-M/2, M/2)^d$, there always exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I)$ of size $\frac{|\mathcal{D}(I)|}{2} \leq M \leq |\mathcal{D}(I)|$ due to [13, Corollary 1] and Bertrand's postulate.

Now, we use reconstructing rank-1 lattices $\Lambda(\boldsymbol{z}, M, I)$ as sampling scheme for approximating functions from a Hilbert space $\mathcal{H}^\omega(\mathbb{T}^d)$ by trigonometric polynomials with frequencies supported on the index set I. In detail, one samples such a function f at all nodes of a reconstructing rank-1 lattice and then applies a normal equation in order to compute the approximated Fourier coefficients $\tilde{\hat{f}}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, of the approximating trigonometric polynomial $\tilde{S}_I f := \sum_{\boldsymbol{k}\in I} \tilde{\hat{f}}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi \mathrm{i}\boldsymbol{k}\circ}$. We remark that we can compute the approximated Fourier coefficients

$\tilde{\hat{f}}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, from (1.1) in $\mathcal{O}(M \log M + d|I|)$ arithmetic operations using a single one-dimensional fast Fourier transform of length M and by computing the scalar products $\boldsymbol{kz}$ for $\boldsymbol{k} \in I$, cf. [19], [15] and [16, Algorithm 2]. For a given frequency index set $I \subset \mathbb{Z}^d \cap (-|I|, |I|)^d$, a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I)$ can be constructed using a component-by-component approach, see [13], and the arithmetic complexity is $\mathcal{O}(d|I|M) \lesssim \mathcal{O}(d|I|^3)$.

Another approach is to use generating vectors $\boldsymbol{z}$ of Korobov form, $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top \in \mathbb{Z}^d$. For a given frequency index set I, an essential task is to find a suitable rank-1 lattice size M, such that there exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I)$ with generating vector $\boldsymbol{z}$ of Korobov form. The search for such a generating vector $\boldsymbol{z}$ takes $\mathcal{O}(d|I|M)$ arithmetic operations if the rank-1 lattice size M is given. Our theoretical considerations in Lemma 2.1 yield a possible choice for M if we set $\mathcal{I}_N := \mathcal{D}(I_N)$.

2.3 Known results from [25]

V.N. Temlyakov investigated the approximation of functions of dominating mixed smoothness by trigonometric polynomials with frequencies supported on hyperbolic cross index sets using function samples along rank-1 lattices of Korobov form, cf. [25]. The considered function spaces and hyperbolic cross index sets in [25] are equivalent to $\mathcal{H}^{0,\beta}(\mathbb{T}^d)$ and $I_N^{d,0}$ in this paper, respectively. Especially, for dominating mixed smoothness parameters $\beta > 1$, there exists a reconstructing rank-1 lattice of Korobov form with size $M = \mathcal{O}(N^2 \log^{d-1} N)$, such that the error estimate

$$\|f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\| \leq C N^{-\beta} \, \|f | \mathcal{H}^{0,\beta}(\mathbb{T}^d)\|$$

is valid for all functions $f \in \mathcal{H}^{0,\beta}(\mathbb{T}^d)$, where the constant $C \geq 1$ does not depend on the refinement N. Essential ingredients for this result are [25, Lemma 1 and Theorem 2]. Using the ideas in the proofs of [25], we develop wide generalizations of [25, Lemma 1 and Theorem 2] in Lemma 2.1 and Theorem 3.4 to (almost) arbitrary frequency index sets. In Section 4, we apply these general statements in order to extend the above estimate for the approximation error to the much more general error estimate

$$\|f - \tilde{S}_{I_N^{d,-\alpha/\beta}} f | L^2(\mathbb{T}^d)\| \leq (1 + C(d, \alpha, \beta)) \, N^{-(\alpha+\beta)} \, \|f | \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|$$

for all functions $f \in \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$, where the smoothness parameters α,β and the rank-1 lattice size M satisfy (1.3).

2.4 Existence of a special reconstructing rank-1 lattice

In this section, we prove a generalization of [25, Lemma 1] and [28, Lemma 4.1]. Conceptually, we consider a frequency index set $I \subset \mathbb{Z}^d$ of finite cardinality which is used as the support in frequency domain for the approximation of a function $f \in C(\mathbb{T}^d)$. We approximate f by a trigonometric polynomial $p \in \Pi_I$ based on sampling values $f(\boldsymbol{x}_j)$. For the theoretical considerations we define the difference set of I by $\mathcal{D}(I) := \{\boldsymbol{k} - \boldsymbol{k}' \colon \boldsymbol{k}, \boldsymbol{k}' \in I\}$ and use a suitable superset $\mathcal{I} \supset \mathcal{D}(I)$ of finite cardinality.

Typically, the error of the approximation p of the function f mainly depends on the frequency index set I. In general, increasing the frequency index set results in decreasing the approximation error. Therefore, one usually introduces a nested sequence of frequency index sets $I_N \subset \mathbb{Z}^d$, $N \in \mathbb{R}$, i.e., $I_{N'} \subset I_{N''}$ for $N' \le N''$. Correspondingly, we use a sequence of suitable supersets $\mathcal{I}_N \supset \mathcal{D}(I_N)$ of the difference sets of I_N, $N \in \mathbb{R}$.

As sampling nodes $\boldsymbol{x}_j$, we use the nodes $\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1}$, $j = 0, \ldots, M-1$, of reconstructing rank-1 lattices $\Lambda(\boldsymbol{z}, M, I_N)$ with generating vector $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top \in \mathbb{Z}^d$ of Korobov form, i.e., the condition $\boldsymbol{m}\boldsymbol{z} \not\equiv 0 \pmod{M}$ has to be fulfilled for all $\boldsymbol{m} \in \mathcal{D}(I_N) \setminus \{\boldsymbol{0}\}$ or one of the other equivalent conditions (2.1),(2.2),(2.3).

Lemma 2.1. *Let a sequence of frequency index sets* $\mathcal{I}_N \subset \mathbb{Z}^d$, $d \in \mathbb{N}$, *of finite cardinality* $|\mathcal{I}_N|$ *be given, which may depend on the refinement* $N \in \mathbb{R}$, $N \ge 1$. *For fixed refinement* $N \in \mathbb{R}$, $N \ge 1$, *and arbitrarily chosen parameter* $\kappa \in \mathbb{R}$, $\kappa > 0$, *let* $M \in \mathbb{N}$ *be a prime such that*

$$M > \frac{d\,|\mathcal{I}_N|}{1 - 2^{-\kappa}} + 1 \tag{2.5}$$

and

$$\mathcal{I}_N \cap M\mathbb{Z}^d = \{\boldsymbol{0}\}. \tag{2.6}$$

For an arbitrarily chosen monotonic increasing function $\varphi : \mathbb{N} \cup \{0\} \to [1, \infty)$ *with* $\varphi(0) = 1$, *we define the shells* $F_l(N) := \mathcal{I}_{N \cdot \varphi(l)} \setminus \mathcal{I}_{N \cdot \varphi(l-1)}$, $N \in \mathbb{R}$, $N \ge 1$, $l \in \mathbb{N}$, *and for each* $a \in \{1, \ldots, M-1\}$ *the sets*

$$M_a^l := \{\boldsymbol{m} \in F_l(N)\colon m_1 + m_2\, a + \ldots + m_d\, a^{d-1} \equiv 0 \pmod{M} \text{ and } \boldsymbol{m} \ne M\boldsymbol{m}'\ \forall \boldsymbol{m}' \in \mathbb{Z}^d\}.$$

Then, there exists a number $a \in \{1, \ldots, M-1\}$, *such that*

$$m_1 + m_2\, a + \ldots + m_d\, a^{d-1} \not\equiv 0 \pmod{M} \text{ for all } \boldsymbol{m} \in \mathcal{I}_N \setminus \{\boldsymbol{0}\} \tag{2.7}$$

and

$$|M_a^l| \le A_l^N := |F_l(N)| d 2^{(l+1)\kappa} (2^\kappa - 1)^{-1} (M-1)^{-1}, \quad l \in \mathbb{N}. \tag{2.8}$$

Proof. This proof is a generalization of the proofs of [25, Lemma 1] and [28, Lemma 4.1]. We remark that $F_l(N) = \emptyset$ may occur for some or all $l \in \mathbb{N}$ and then also $M_a^l = \emptyset$ follows. The idea is to prove that the number of integers $a \in \{1, \ldots, M-1\}$ for which the statements (2.7) and (2.8) of the lemma are not valid is less than $M-1$ and consequently, at least one $a \in \{1, \ldots, M-1\}$ fulfills the statement. We consider the congruence

$$m_1 + m_2\, a + \ldots + m_d\, a^{d-1} \equiv 0 \pmod{M}. \tag{2.9}$$

For a fixed frequency $\boldsymbol{m} \in \mathbb{Z}^d$, we denote the set of natural numbers $a \in \{1, \ldots, M-1\}$ which are solutions of congruence (2.9) by $A_M(\boldsymbol{m})$, i.e.,

$$A_M(\boldsymbol{m}) := \{a \in \{1, \ldots, M-1\}\colon m_1 + m_2\, a + \ldots + m_d\, a^{d-1} \equiv 0 \pmod{M}\}.$$

Let a frequency $\boldsymbol{m} \in \mathcal{I}_N \setminus \{\boldsymbol{0}\}$ be given. Due to condition (2.6), at least one component fulfills $m_{s'} \not\equiv 0 \pmod{M}$ and we can apply Lagrange's Theorem from number theory. This yields that the congruence (2.9) has at most $d-1$ roots modulo M. Therefore, we have

$$|A_M(\boldsymbol{m})| \le d - 1 < d \tag{2.10}$$

for all $\boldsymbol{m} \in \mathcal{I}_N \setminus \{\boldsymbol{0}\}$. Next, we estimate the number of integers $a \in \{1, \ldots, M-1\}$ for which the relation (2.7) is not valid for at least one $\boldsymbol{m} \in \mathcal{I}_N \setminus \{\boldsymbol{0}\}$. Therefore, we denote by G_0 the set of numbers $a \in \{1, \ldots, M-1\}$ which are solutions of congruence (2.9) for at least one frequency $\boldsymbol{m} \in \mathcal{I}_N \setminus \{\boldsymbol{0}\}$,

$$G_0 = \bigcup_{\boldsymbol{m} \in \mathcal{I}_N \setminus \{\boldsymbol{0}\}} A_M(\boldsymbol{m}).$$

Since $|A_M(\boldsymbol{m})| < d$ by (2.10) and due to (2.5), we obtain

$$|G_0| \leq \sum_{\boldsymbol{m} \in \mathcal{I}_N \setminus \{\boldsymbol{0}\}} |A_M(\boldsymbol{m})| < d\,|\mathcal{I}_N| < (M-1)(1-2^{-\kappa}). \tag{2.11}$$

This means, for any $a \in \{1, \ldots, M-1\} \setminus G_0$, the relations (2.7) are valid and $|\{1, \ldots, M-1\} \setminus G_0| > M-1-(M-1)(1-2^{-\kappa}) = (M-1)2^{-\kappa} > 0$.
Next, we consider the inequalities (2.8). For each $l \in \mathbb{N}$, we estimate the number of integers $a \in \{1, \ldots, M-1\}$ for which $|M_a^l| > A_l^N$, i.e., for which the inequalities (2.8) are not fulfilled. Therefore, we define the sets $G_l := \{a \in \{1, \ldots, M-1\} \colon |M_a^l| > A_l^N\}$, $l \in \mathbb{N}$. If $F_l(N) = \emptyset$, then obviously $|G_l| = 0$. Otherwise for $F_l(N) \neq \emptyset$, we have

$$\sum_{a \in G_l} |M_a^l| > \sum_{a \in G_l} A_l^N = |G_l|\, A_l^N. \tag{2.12}$$

We note that estimate (2.10) is also true for all $\boldsymbol{m} \in M_a^l$ due to Lagrange's Theorem from number theory, i.e., there exist at most $d-1$ many numbers $a \in \{1, \ldots, M-1\}$ satisfying (2.9) for fixed $\boldsymbol{m} \in M_a^l$. Consequently, for fixed $\boldsymbol{m} \in M_a^l$, there exist at most $d-1$ sets M_a^l which contain $\boldsymbol{m}$. Thus, each $\boldsymbol{m} \in F_l(N)$ can belong to at most $d-1$ different sets M_a^l and therefore

$$\sum_{a \in G_l} |M_a^l| \leq (d-1)\,|F_l(N)| < d\,|F_l(N)|. \tag{2.13}$$

Comparing (2.12) and (2.13), we obtain $|G_l|\, A_l^N < d\,|F_l(N)|$ and by inserting the definition of A_l^N from (2.8), we infer

$$|G_l| < d\,|F_l(N)|/A_l^N = 2^{-(l+1)\kappa}(2^\kappa - 1)(M-1) = 2^{-l\kappa}(M-1)(1-2^{-\kappa}), \quad l \in \mathbb{N}, \tag{2.14}$$

if $F_l(N) \neq \emptyset$. Altogether, relation (2.11) as well as relation (2.14) if $F_l(N) \neq \emptyset$ and $|G_l| = 0$ if $F_l(N) = \emptyset$ yield

$$\begin{aligned}\sum_{l=0}^{\infty} |G_l| < \sum_{l=0}^{\infty} 2^{-l\kappa}(M-1)(1-2^{-\kappa}) &= (M-1)(1-2^{-\kappa}) \sum_{l=0}^{\infty} (2^{-\kappa})^l \\ &= (M-1)(1-2^{-\kappa}) \frac{1}{1-2^{-\kappa}} = M-1.\end{aligned}$$

This means that the number of integers $a \in \{1, \ldots, M-1\}$ for which the statement of the lemma is not valid is less than $M-1$. Since the cardinality $|\{1, \ldots, M-1\}| = M-1$, there exists at least one $a \in \{1, \ldots, M-1\}$ for which relations (2.7) and (2.8) are valid. ■

3 Aliasing error for rank-1 lattice sampling and general frequency index sets

Based on Lemma 2.1, we prove general statements for the aliasing error for arbitrary frequency index sets $I \subset \mathbb{Z}^d$ of finite cardinality. We are going to use the results of this section extensively in Section 4. The following lemma was proven in [25], see [25, Property 2°].

Lemma 3.1. *Let the dimension $d \in \mathbb{N}$, $d \geq 2$, a frequency index set $I \subset \mathbb{Z}^d$ of finite cardinality and a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I)$ with the nodes $\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1}$, $j = 0, \ldots, M-1$, be given. We denote the Dirichlet kernel with frequencies supported on the index set I by $D_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$. For an arbitrary vector $\boldsymbol{b} := (b_0, \ldots, b_{M-1})^\top \in \mathbb{C}^M$, we have*

$$\left\| \frac{1}{M} \sum_{j=0}^{M-1} b_j D_I(\circ - \boldsymbol{x}_j) \middle| L_2(T^d) \right\| \leq \left(\frac{1}{M} \sum_{j=0}^{M-1} |b_j|^2 \right)^{1/2} = \|\boldsymbol{b}/\sqrt{M}\|_2. \tag{3.1}$$

Additionally, for an arbitrary trigonometric polynomial $p\colon \mathbb{T}^d \to \mathbb{C}$ with frequencies supported on the index set I, $p(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$, $\hat{p}_{\boldsymbol{k}} \in \mathbb{C}$, we have $\tilde{S}_I p = p$.

Proof. Due to $\int_{\mathbb{T}^d} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}} \mathrm{d}\boldsymbol{x} = \begin{cases} 1 & \text{for } \boldsymbol{k} = \boldsymbol{0}, \\ 0 & \text{for } \boldsymbol{k} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}, \end{cases}$ we obtain

$$\begin{aligned} \left\| \frac{1}{M} \sum_{j=0}^{M-1} b_j D_I(\circ - \boldsymbol{x}_j) \middle| L_2(\mathbb{T}^d) \right\|^2 &= \frac{1}{M^2} \int_{\mathbb{T}^d} \left| \sum_{j=0}^{M-1} b_j D_I(\boldsymbol{x} - \boldsymbol{x}_j) \right|^2 \mathrm{d}\boldsymbol{x} \\ = \frac{1}{M^2} \sum_{0 \leq j, j' \leq M-1} b_j D_I(\boldsymbol{x}_{j'} - \boldsymbol{x}_j) \overline{b_{j'}}. \end{aligned}$$

We can rewrite this as a quadratic form of the vector $\boldsymbol{b}/\sqrt{M}$ and the matrix $\boldsymbol{\mathcal{D}} = \left(\mathcal{D}_{j',j}\right)_{j',j=0}^{M-1}$ with elements $\mathcal{D}_{j',j} := \frac{1}{M} D_I(\boldsymbol{x}_{j'} - \boldsymbol{x}_j)$, $\left\| \frac{1}{M} \sum_{j=0}^{M-1} b_j D_I(\circ - \boldsymbol{x}_j) \middle| L_2(T^d) \right\|^2 = \left(\frac{\boldsymbol{b}}{\sqrt{M}}\right)^{\mathrm{H}} \boldsymbol{\mathcal{D}} \left(\frac{\boldsymbol{b}}{\sqrt{M}}\right)$.
Next, we consider the matrix $\boldsymbol{\mathcal{D}}^2 = \boldsymbol{\mathcal{D}} \cdot \boldsymbol{\mathcal{D}} := \left((\mathcal{D}^2)_{j',j}\right)_{j',j=0}^{M-1}$ with the elements $(\mathcal{D}^2)_{j',j}$. We obtain by using the reconstructing property (2.1) of the reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I)$ that $(\mathcal{D}^2)_{j',j} = \frac{1}{M^2} \sum_{\rho=0}^{M-1} D_I(\boldsymbol{x}_{j'} - \boldsymbol{x}_\rho) D_I(\boldsymbol{x}_\rho - \boldsymbol{x}_j) \overset{(2.1)}{=} \frac{1}{M} \sum_{\boldsymbol{k}\in I} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}(\boldsymbol{x}_{j'} - \boldsymbol{x}_j)} = \mathcal{D}_{j',j}$, i.e., $\boldsymbol{\mathcal{D}}^2 = \boldsymbol{\mathcal{D}}$. Furthermore, we have $\boldsymbol{\mathcal{D}}^{\mathrm{H}} = \boldsymbol{\mathcal{D}}$, where $\boldsymbol{\mathcal{D}}^{\mathrm{H}}$ is the adjoint of the matrix $\boldsymbol{\mathcal{D}}$. Therefore, $\boldsymbol{\mathcal{D}} = \boldsymbol{\mathcal{D}}^2 = \boldsymbol{\mathcal{D}}^{\mathrm{H}}\boldsymbol{\mathcal{D}}$ follows. Consequently, we infer

$$\left\| \frac{1}{M} \sum_{j=0}^{M-1} b_j D_I(\circ - \boldsymbol{x}_j) \middle| L_2(T^d) \right\|^2 \leq \|\boldsymbol{\mathcal{D}}\|_2^2 \, \left\| \left(\boldsymbol{b}/\sqrt{M}\right) \right\|_2^2 = \sigma_{\max}(\boldsymbol{\mathcal{D}})^2 \, \left\| \boldsymbol{b}/\sqrt{M} \right\|_2^2,$$

where $\sigma_{\max}(\boldsymbol{\mathcal{D}})$ denotes the largest singular value of the matrix $\boldsymbol{\mathcal{D}}$. Last, we show $\sigma_{\max}(\boldsymbol{\mathcal{D}}) \leq 1$. Let $\boldsymbol{\mathcal{D}} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{H}}$ be a singular value decomposition of the matrix $\boldsymbol{\mathcal{D}}$, where $\boldsymbol{U}$, $\boldsymbol{V}$ are unitary matrices and $\boldsymbol{\Sigma} = \mathrm{diag}\left((\sigma_1, \ldots, \sigma_M)\right)$ is a diagonal matrix of the singular values $\sigma_j \geq 0$, $j = 1, \ldots, M$, of the matrix $\boldsymbol{\mathcal{D}}$. Then, we infer from $\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{H}} = \boldsymbol{\mathcal{D}} = \boldsymbol{\mathcal{D}}^2 = \boldsymbol{\mathcal{D}}^{\mathrm{H}}\boldsymbol{\mathcal{D}} = \boldsymbol{U}\boldsymbol{\Sigma}^2\boldsymbol{V}^{\mathrm{H}}$ that $\sigma_j^2 = \sigma_j$, $j = 1, \ldots, M$. Therefore, each singular value $\sigma_j \in \{0, 1\}$ and we obtain $\sigma_{\max}(\boldsymbol{\mathcal{D}}) \leq 1$.
In order to show the statement $\tilde{S}_I p = p$ for an arbitrary trigonometric polynomial p with frequencies supported on the index set I, $p(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$, we only need the reconstruction property (2.1) to be fulfilled. Since we required that the sampling nodes $\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1}$,

$j=0,\dots,M-1$, are from a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z},M,I)$ for the frequency index set I, this reconstruction property is valid by definition. Consequently, we infer

$$\begin{aligned}\tilde{S}_I\, p(\boldsymbol{x}) &= \sum_{\boldsymbol{k}\in I}\frac{1}{M}\sum_{j=0}^{M-1} p\left(\boldsymbol{x}_j\right)\,\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}\\ &= \sum_{\boldsymbol{k}\in I}\left(\sum_{\boldsymbol{k}'\in I}\hat{p}_{\boldsymbol{k}'}\,\frac{1}{M}\sum_{j=0}^{M-1}\mathrm{e}^{2\pi\mathrm{i}j(\boldsymbol{k}'-\boldsymbol{k})\boldsymbol{z}/M}\right)\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}} \overset{(2.1)}{=} \sum_{\boldsymbol{k}\in I}\hat{p}_{\boldsymbol{k}}\,\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}} = p(\boldsymbol{x})\end{aligned}$$

for all $\boldsymbol{x}\in\mathbb{T}^d$. ∎

Lemma 3.2. *Let the dimension $d\in\mathbb{N}$, $d\geq 2$, a function $f\in C(\mathbb{T}^d)$ with absolutely convergent Fourier series, a frequency index set $I\subset\mathbb{Z}^d$ of finite cardinality and a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z},M,I)$ with the nodes $\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1}$, $j=0,\dots,M-1$, be given. Additionally, we define shells $U_l\subset\mathbb{Z}^d$, $l\in\mathbb{N}\cup\{0\}$, with the properties $U_{l'}\cap U_{l''}=\emptyset$ for $l'\neq l''$ and $\mathrm{supp}\hat{f}\setminus I\subset\bigcup_{l=0}^{\infty}U_l$, where $\mathrm{supp}\hat{f} := \{\boldsymbol{k}\in\mathbb{Z}^d\colon \hat{f}_{\boldsymbol{k}}\neq 0\}$. Then, we have*

$$\|\tilde{S}_I\,(f-S_If)\,|L^2(\mathbb{T}^d)\| \leq \sum_{l=0}^{\infty}\sigma_l, \qquad \sigma_l := \left(\frac{1}{M}\sum_{j=0}^{M-1}|S_{U_l}f(\boldsymbol{x}_j)|^2\right)^{1/2}.$$

Proof. By definition, we have

$$\begin{aligned}\tilde{S}_I\,(f-S_If) &= \sum_{\boldsymbol{h}\in I}\frac{1}{M}\sum_{j=0}^{M-1}(f-S_If)\;(\boldsymbol{x}_j)\,\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{h}\boldsymbol{x}_j}\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{h}\circ}\\ &= \frac{1}{M}\sum_{j=0}^{M-1}S_{\mathrm{supp}\hat{f}\setminus I}\,f\;(\boldsymbol{x}_j)\;D_I(\circ-\boldsymbol{x}_j).\end{aligned}$$

Due to $S_{\mathrm{supp}\hat{f}\setminus I}\,f = \sum_{\boldsymbol{k}\in\mathrm{supp}\hat{f}\setminus I}\hat{f}_{\boldsymbol{k}}\,\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\circ} = \sum_{l=0}^{\infty}\sum_{\boldsymbol{k}\in U_l}\hat{f}_{\boldsymbol{k}}\,\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\circ}$, we obtain

$$\begin{aligned}\tilde{S}_I\,(f-S_If) &= \sum_{l=0}^{\infty}\frac{1}{M}\sum_{j=0}^{M-1}\sum_{\boldsymbol{k}\in U_l}\hat{f}_{\boldsymbol{k}}\,\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}\;D_I(\circ-\boldsymbol{x}_j)\\ &= \sum_{l=0}^{\infty}\frac{1}{M}\sum_{j=0}^{M-1}S_{U_l}f(\boldsymbol{x}_j)\;D_I(\circ-\boldsymbol{x}_j).\end{aligned}$$

We apply the Minkowski inequality and Lemma 3.1 with $b_j := S_{U_l}f\;(\boldsymbol{x}_j)$. This yields

$$\begin{aligned}\|\tilde{S}_I\,(f-S_If)\,|L^2(\mathbb{T}^d)\| &= \left\|\sum_{l=0}^{\infty}\frac{1}{M}\sum_{j=0}^{M-1}S_{U_l}f\;(\boldsymbol{x}_j)\;D_I(\circ-\boldsymbol{x}_j)\middle|L^2(\mathbb{T}^d)\right\|\\ &\leq \sum_{l=0}^{\infty}\left\|\frac{1}{M}\sum_{j=0}^{M-1}S_{U_l}f\;(\boldsymbol{x}_j)\;D_I(\circ-\boldsymbol{x}_j)\middle|L^2(\mathbb{T}^d)\right\|\\ &\overset{(3.1)}{\leq} \sum_{l=0}^{\infty}\left(\frac{1}{M}\sum_{j=0}^{M-1}|S_{U_l}f\;(\boldsymbol{x}_j)|^2\right)^{1/2} = \sum_{l=0}^{\infty}\sigma_l\end{aligned}$$

and the assertion follows. ■

Analogously to [18, 16], we estimate the truncation error $f - S_{I_N} f$ in the L^2 norm in

Lemma 3.3. *Let the dimension $d \in \mathbb{N}$, a weight function $\omega : \mathbb{Z}^d \to (0, \infty]$, a smoothness parameter $\nu > 0$, the sequence of frequency index sets $I_N := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \omega(\boldsymbol{k})^{1/\nu} \leq N\}$ of refinement $N \in \mathbb{R}$, $N \geq 1$, and a function $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ be given. Then, the truncation error is bounded by*

$$\|f - S_{I_N} f | L^2(\mathbb{T}^d)\|^2 \leq N^{-\nu} \, \|f | \mathcal{H}^\omega(\mathbb{T}^d)\|.$$

Proof. We have

$$\mathbb{Z}^d \setminus I_N = \{\boldsymbol{k} \in \mathbb{Z}^d \colon \omega(\boldsymbol{k})^{1/\nu} > N\} = \{\boldsymbol{k} \in \mathbb{Z}^d \colon \frac{1}{\omega(\boldsymbol{k})^{1/\nu}} < \frac{1}{N}\} = \{\boldsymbol{k} \in \mathbb{Z}^d \colon \frac{1}{\omega(\boldsymbol{k})^2} < N^{-2\nu}\}$$

and this yields the assertion since

$$\begin{aligned}
\|f - S_{I_N} f | L^2(\mathbb{T}^d)\|^2 &= \sum_{\boldsymbol{k} \in \mathbb{Z}^d \setminus I_N} \frac{\omega(\boldsymbol{k})^2}{\omega(\boldsymbol{k})^2} |\hat{f}_{\boldsymbol{k}}|^2 \leq \max_{\boldsymbol{k} \in \mathbb{Z}^d \setminus I_N} \frac{1}{\omega(\boldsymbol{k})^2} \sum_{\boldsymbol{k} \in \mathbb{Z}^d \setminus I_N} \omega(\boldsymbol{k})^2 |\hat{f}_{\boldsymbol{k}}|^2 \\
&\leq N^{-2\nu} \sum_{\boldsymbol{k} \in \mathbb{Z}^d \setminus I_N} \omega(\boldsymbol{k})^2 |\hat{f}_{\boldsymbol{k}}|^2 \leq N^{-2\nu} \, \|f | \mathcal{H}^\omega(\mathbb{T}^d)\|^2.
\end{aligned}$$

■

Theorem 3.4. *Let the dimension $d \in \mathbb{N}$, $d \geq 2$, a function $f \in C(\mathbb{T}^d) \cap \mathcal{H}^\omega(\mathbb{T}^d)$ with absolutely convergent Fourier series, a smoothness parameter $\nu > 0$ and the sequence of frequency index sets $I_N := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \omega(\boldsymbol{k})^{1/\nu} \leq N\}$ with refinement $N \in \mathbb{R}$, $N \geq 1$, be given, where $\omega : \mathbb{Z}^d \to (0, \infty]$ is a weight function such that all frequency index sets I_N are of finite cardinality. Furthermore, let $\mathcal{I}_N \subset \mathbb{Z}^d$ be a nested sequence of frequency index sets with refinement $N \in \mathbb{R}$, $N \geq 1$,*

$$\mathcal{I}_{N'} \subset \mathcal{I}_{N''} \text{ for } N' \leq N'', \tag{3.2}$$

such that $|\mathcal{I}_N| < \infty$ and the inclusion $\mathcal{I}_N \supset \mathcal{D}(I_N) := \{\boldsymbol{k} - \boldsymbol{k}' \colon \boldsymbol{k}, \boldsymbol{k}' \in I_N\}$ is valid for all $N \in \mathbb{R}$, $N \geq 1$. For each fixed $N \in \mathbb{R}$, $N \geq 1$, let a parameter $\kappa > 0$ and a prime number $M \in \mathbb{N}$,

$$M > \frac{d\,|\mathcal{I}_N|}{1 - 2^{-\kappa}} + 1, \tag{3.3}$$

be given. Additionally, let the inequality

$$|\{\boldsymbol{m} \in \mathcal{I}_{N2^l} \colon \exists \boldsymbol{m}' \in \mathbb{Z}^d \text{ such that } \boldsymbol{m} = M\boldsymbol{m}'\}| \leq C \frac{|\mathcal{I}_{N2^l}|}{M} \psi(l) + 1 \quad \forall l \in \mathbb{N} \tag{3.4}$$

be valid, where $\psi : [0, \infty) \to [1, \infty)$ and $C > 0$ is a constant which does not depend on N or M. Then, there exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N)$ with generating vector $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top \in \mathbb{Z}^d$ of Korobov form such that the aliasing error is bounded by

$$\begin{aligned}
\|S_{I_N} f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\| &\leq 2^\nu \, N^{-\nu} \, \|f | \mathcal{H}^\omega(\mathbb{T}^d)\| \\
&\quad \cdot \sum_{l=0}^{\infty} \sqrt{2\,(2 + (1 - 2^{-\kappa})\, C \, \psi(l+1))} \, 2^{(l+1)(\frac{\kappa}{2} - \nu)} \sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}}.
\end{aligned}$$

Proof. This proof is a generalization of [25, Theorem 2]). Since inequality (3.3) is valid, we apply Lemma 2.1 and obtain that there exists a number $a \in \{1, \ldots, M-1\}$ which fulfills properties (2.7) and (2.8). Since property (2.7) is valid, the rank-1 lattice $\Lambda(\boldsymbol{z}, M)$ with the generating vector $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top$ and the nodes $\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1}$, $j = 0, \ldots, M-1$, is a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N)$. We use this special rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N)$ for computing the approximated Fourier coefficients $\tilde{\hat{f}}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I_N$, from the sampling values $f(\boldsymbol{x}_j)$. Since the Fourier partial sum $S_{I_N} f$ of the function f is a trigonometric polynomial with frequencies supported on the index set I_N and by applying Lemma 3.1, we obtain $\tilde{S}_{I_N}(S_{I_N} f) = S_{I_N} f$. This yields $S_{I_N} f - \tilde{S}_{I_N} f = \tilde{S}_{I_N}(f - S_{I_N} f)$. Next, we set the shells $U_l := I_{N2^{l+1}} \setminus I_{N2^l}$, $l = 0, 1, \ldots$, and consequently, the property $U_l \cap U_{l'} = \emptyset \ \forall l \neq l'$ is valid. We apply Lemma 3.2 and we obtain $\|\tilde{S}_{I_N}(f - S_{I_N} f) | L^2(\mathbb{T}^d)\| \leq \sum_{l=0}^{\infty} \sigma_l$, where

$$\sigma_l := \left(\frac{1}{M} \sum_{j=0}^{M-1} |S_{U_l} f(\boldsymbol{x}_j)|^2 \right)^{1/2}, \; l \in \mathbb{N} \cup \{0\}.$$

Next, we estimate

$$\sigma_l^2 \leq B_l \sum_{\boldsymbol{k} \in U_l} |\hat{f}_{\boldsymbol{k}}|^2,$$

with numbers $B_l \geq 0$, which have to be determined. We have

$$\sigma_l^2 = \frac{1}{M} \sum_{j=0}^{M-1} \left| \sum_{\boldsymbol{k} \in U_l} \hat{f}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}_j} \right|^2 = \frac{1}{M} \sum_{j=0}^{M-1} \sum_{\boldsymbol{k},\boldsymbol{h} \in U_l} \hat{f}_{\boldsymbol{k}} \overline{\hat{f}_{\boldsymbol{h}}} \mathrm{e}^{2\pi \mathrm{i}(\boldsymbol{k}-\boldsymbol{h})\boldsymbol{x}_j} = \sum_{\boldsymbol{k},\boldsymbol{h} \in U_l} \hat{f}_{\boldsymbol{k}} \overline{\hat{f}_{\boldsymbol{h}}} \, \Delta_M(\boldsymbol{k} - \boldsymbol{h}),$$

where

$$\Delta_M(\boldsymbol{m}) := \frac{1}{M} \sum_{j=0}^{M-1} \mathrm{e}^{2\pi \mathrm{i} j \boldsymbol{m}\boldsymbol{z}/M} = \begin{cases} 1 & \text{for } m_1 + m_2\, a + \ldots + m_d\, a^{d-1} \equiv 0 \pmod{M}, \\ 0 & \text{for } m_1 + m_2\, a + \ldots + m_d\, a^{d-1} \not\equiv 0 \pmod{M}. \end{cases}$$

For fixed frequency $\boldsymbol{k} \in U_l$, we define the set of frequencies

$$\theta_{\ell,\boldsymbol{k}} := \{\boldsymbol{h} \in U_l \colon \Delta_M(\boldsymbol{k} - \boldsymbol{h}) = 1\},$$

and by applying the Cauchy Schwarz inequality twice, we obtain

$$\begin{aligned} \sigma_l^2 = \sum_{\boldsymbol{k} \in U_l} \hat{f}_{\boldsymbol{k}} \sum_{\boldsymbol{h} \in \theta_{\ell,\boldsymbol{k}}} \overline{\hat{f}_{\boldsymbol{h}}} \quad &\leq \quad \left(\sum_{\boldsymbol{k} \in U_l} |\hat{f}_{\boldsymbol{k}}|^2 \right)^{1/2} \left(\sum_{\boldsymbol{k} \in U_l} \left| \sum_{\boldsymbol{h} \in \theta_{\ell,\boldsymbol{k}}} \overline{\hat{f}_{\boldsymbol{h}}} \right|^2 \right)^{1/2} \\ &\leq \quad \left(\sum_{\boldsymbol{k} \in U_l} |\hat{f}_{\boldsymbol{k}}|^2 \right)^{1/2} \left(\sum_{\boldsymbol{k} \in U_l} \left(\sum_{\boldsymbol{h} \in \theta_{\ell,\boldsymbol{k}}} 1 \cdot |\hat{f}_{\boldsymbol{h}}| \right)^2 \right)^{1/2} \\ &\leq \quad \left(\sum_{\boldsymbol{k} \in U_l} |\hat{f}_{\boldsymbol{k}}|^2 \right)^{1/2} \left(\sum_{\boldsymbol{k} \in U_l} |\theta_{\ell,\boldsymbol{k}}| \sum_{\boldsymbol{h} \in \theta_{\ell,\boldsymbol{k}}} |\hat{f}_{\boldsymbol{h}}|^2 \right)^{1/2}. \end{aligned}$$

We have $\boldsymbol{k}-\boldsymbol{h}\in\mathcal{D}(I_{N2^{l+1}})\subset\mathcal{I}_{N2^{l+1}}$ for $\boldsymbol{k},\boldsymbol{h}\in U_l$ and this yields

$$|\theta_{\ell,\boldsymbol{k}}|\le|\{\boldsymbol{m}\in\mathcal{I}_{N2^{l+1}}\colon m_1+m_2\,a+\ldots+m_d\,a^{d-1}\equiv 0\pmod M\}|.$$

We define the function $\varphi(l):=2^l$ for $l\in\mathbb{N}\cup\{0\}$. Due to property (2.8) in Lemma 2.1, we obtain

$$\begin{aligned}&\left|\{\boldsymbol{m}\in I^{d,0}_{N\,\varphi(l+1)}\colon m_1+m_2\,a+\ldots+m_d\,a^{d-1}\equiv 0\pmod M \text{ and } \boldsymbol{m}\neq M\boldsymbol{m}'\ \forall\boldsymbol{m}'\in\mathbb{Z}^d\}\right|\\&=\left|\bigcup_{j=1}^{l+1}F_j(N)\right|\le\sum_{j=1}^{l+1}A_j^N.\end{aligned}$$

Then, we have

$$|\theta_{\ell,\boldsymbol{k}}|\le B_l:=\sum_{j=1}^{l+1}A_j^N+C\frac{|\mathcal{I}_{N2^{l+1}}|}{M}\psi(l+1)+1 \tag{3.5}$$

and

$$\sigma_l^2\le\left(\sum_{\boldsymbol{k}\in U_l}|\hat{f}_{\boldsymbol{k}}|^2\right)^{1/2}\left(B_l\sum_{\boldsymbol{k}\in U_l}\sum_{\boldsymbol{h}\in\theta_{\ell,\boldsymbol{k}}}|\hat{f}_{\boldsymbol{h}}|^2\right)^{1/2}.$$

For an arbitrarily chosen $\boldsymbol{k}\in U_l$, let $\boldsymbol{h}\in\theta_{\ell,\boldsymbol{k}}$. This means, we have $(\boldsymbol{h}-\boldsymbol{k})\boldsymbol{z}\equiv 0\pmod M$. If $\boldsymbol{h}\in\theta_{\boldsymbol{k}'}$ for another $\boldsymbol{k}'\in U_l$, $\boldsymbol{k}'\neq\boldsymbol{k}$, then $(\boldsymbol{h}-\boldsymbol{k}')\boldsymbol{z}\equiv 0\pmod M$ is valid and $(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{z}\equiv 0\pmod M$ follows. This yields $\boldsymbol{k}'\in\theta_{\boldsymbol{k}}$. Especially, we have $\boldsymbol{k}\in\theta_{\ell,\boldsymbol{k}}$. Therefore, each frequency $\boldsymbol{h}'\in U_l$ is element of at most B_l many distinct sets $\theta_{\ell,\boldsymbol{k}}$. This means, we obtain

$$\sum_{\boldsymbol{k}\in U_l}\sum_{\boldsymbol{h}\in\theta_{\ell,\boldsymbol{k}}}|\hat{f}_{\boldsymbol{h}}|^2\le\sum_{\boldsymbol{k}\in U_l}B_l|\hat{f}_{\boldsymbol{k}}|^2$$

and

$$\begin{aligned}\sigma_l^2&\le\left(\sum_{\boldsymbol{k}\in U_l}|\hat{f}_{\boldsymbol{k}}|^2\right)^{1/2}\left(B_l^2\sum_{\boldsymbol{k}\in U_l}|\hat{f}_{\boldsymbol{k}}|^2\right)^{1/2}=B_l\sum_{\boldsymbol{k}\in U_l}|\hat{f}_{\boldsymbol{k}}|^2\le B_l\sum_{\boldsymbol{k}\in\mathbb{Z}^d\setminus I_{N2^l}}|\hat{f}_{\boldsymbol{k}}|^2\\&=B_l\,\|f-S_{I_{N2^l}}f|L^2(\mathbb{T}^d)\|^2\le B_l\,(N2^l)^{-2\nu}\,\|f|\mathcal{H}^\omega(\mathbb{T}^d)\|^2.\end{aligned}$$

Next, we estimate B_l. Using the inequality $\frac{1}{M-1}\le\frac{2}{M}$ for $M\ge 2$ as well as (3.5) and (2.8), we infer

$$\begin{aligned}B_l&=\sum_{j=1}^{l+1}|\mathcal{I}_{N\varphi(j)}\setminus\mathcal{I}_{N\varphi(j-1)}|d2^{(j+1)\kappa}(2^\kappa-1)^{-1}(M-1)^{-1}+C\frac{|\mathcal{I}_{N2^{l+1}}|}{M}\psi(l+1)+1\\&\le d\frac{2^\kappa}{2^\kappa-1}2^{(l+1)\kappa}\frac{2}{M}\sum_{j=1}^{l+1}|\mathcal{I}_{N2^j}\setminus\mathcal{I}_{N2^{j-1}}|+C\,\psi(l+1)\,|\mathcal{I}_{N2^{l+1}}|/M+1\\&\overset{(3.2)}{\le} d2^{(l+1)\kappa}\frac{|\mathcal{I}_{N2^{l+1}}|}{M}\left(\frac{2}{1-2^{-\kappa}}+C\,\psi(l+1)\right)+1\\&\overset{(3.3)}{\le} d2^{(l+1)\kappa}\frac{|\mathcal{I}_{N2^{l+1}}|}{\frac{d\,|\mathcal{I}_N|}{1-2^{-\kappa}}+1}\left(\frac{2}{1-2^{-\kappa}}+C\,\psi(l+1)\right)+1\\&\le 2^{(l+1)\kappa+1}\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}\left(2+(1-2^{-\kappa})\,C\,\psi(l+1)\right)\end{aligned}$$

and this yields

$$\begin{aligned}\sigma_l &\leq (N2^l)^{-\nu}\,\|f|\mathcal{H}^\omega(\mathbb{T}^d)\|\,\sqrt{B_l}\\ &\leq \sqrt{2\,(2+(1-2^{-\kappa})\,C\,\psi(l+1))}\,2^\nu\,N^{-\nu}\,\|f|\mathcal{H}^\omega(\mathbb{T}^d)\|\,2^{(l+1)(\frac{\kappa}{2}-\nu)}\,\sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}}.\end{aligned}$$

■

We remark that the inequality (3.4) needs to be checked for each specific sequence of index sets $\mathcal{I}_N$. The starting point is the weight function ω which produces a nested sequence of frequency index sets I_N and its difference sets $\mathcal{D}(I_N)$. Based on these difference sets, a nested sequence of index sets $\mathcal{I}_N \supset \mathcal{D}(I_N)$ should be chosen such that

- the cardinalities $|\mathcal{I}_N|$ are close to the cardinalities $|\mathcal{D}(I_N)|$,
- the upper and lower bound of the cardinalities $|\mathcal{I}_N|$ are known and are almost of the same order up to logarithmic gaps in N, as well as
- the inequality (3.4) can be (easily) shown.

In the next section, we demonstrate this strategy on functions from the Hilbert space $\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$ for the approximation by trigonometric polynomials with frequencies supported on the index sets $I_N = I_N^{d,T}$, $T := -\alpha/\beta$.

4 Approximation error for rank-1 lattice sampling and frequency index sets $I_N^{d,T}$

Next, we apply the general results from Section 2 and Section 3. Therefor, we use the index sets $I = I_N = I_N^{d,T}$. In the case $-\infty \leq T \leq 0$, we set $\mathcal{I}_N := I_L^{d,T}$, where $L := 2^{\frac{d-T}{1-T}}\,N^{1+\frac{d}{d-T}}$ and $\mathcal{I}_N \supset \mathcal{D}(I_N^{d,T})$, see Lemma 4.2. This means, we cover the difference set $\mathcal{D}(I_N^{d,T})$ with the index set $I_L^{d,T}$ of larger refinement $L = 2^{\frac{d-T}{1-T}}\,N^{1+\frac{d}{d-T}}$. In the case $0 < T < 1$, we set $\mathcal{I}_N := \mathcal{D}(I_N^{d,T})$. Before we estimate the truncation error $\|f - S_{I_N^{d,T}} f|L^2(\mathbb{T}^d)\|$ and the aliasing error $\|S_{I_N^{d,T}} f - \tilde{S}_{I_N^{d,T}} f|L^2(\mathbb{T}^d)\|$, we show preliminary lemmata for the cardinalities and embeddings of the frequency index sets $I_N^{d,T}$.

Lemma 4.1. *Let the dimension $d \in \mathbb{N}$, and a parameter T, $-\infty \leq T < 1$, be given. Then, the cardinalities of the frequency index sets $I_N^{d,T}$ are*

$$|I_N^{d,T}| = \begin{cases} \Theta(N^d) & \text{for } T = -\infty,\\ \Theta(N^{\frac{T-1}{T/d-1}}) & \text{for } -\infty < T < 0,\\ \Theta(N \log^{d-1} N) & \text{for } T = 0,\\ \Theta(N) & \text{for } 0 < T < 1, \end{cases} \tag{4.1}$$

for fixed parameters d and T, where the constants only depend on d and T.

Proof. We show the cardinalities for the different cases.

- Case $T = -\infty$. Since we have the inclusions $\{-\lfloor \frac{N}{d} \rfloor, \ldots, \lfloor \frac{N}{d} \rfloor\}^d \subset I_N^{d,-\infty} \subset \{-N, \ldots, N\}^d$, we infer $c_1(d) N^d \leq |I_N^{d,-\infty}| \leq C_1(d) N^d$, where $c_1(d) = d^{-d}$ and $C_1(d) = 3^d$.

- Case $-\infty < T < 0$. First, we consider the lower bound and for this, we show $I_{N^{(1-T)/(d-T)}}^{d,-\infty} \subset I_N^{d,T}$. For arbitrary $\boldsymbol{k} \in I_{N^{(1-T)/(d-T)}}^{d,-\infty}$, we have

$$N^{\frac{1-T}{d-T}} \geq \max(1, \|\boldsymbol{k}\|_1) = \max(1, \|\boldsymbol{k}\|_1)^{-\frac{T}{d-T}} \max(1, \|\boldsymbol{k}\|_1)^{1+\frac{T}{d-T}}.$$

Since $\max(1, \|\boldsymbol{k}\|_1)^d \geq \max(1, \|\boldsymbol{k}\|_\infty)^d \geq \prod_{s=1}^d \max(1, |k_s|)$, we infer

$$\begin{aligned} N^{\frac{1-T}{d-T}} &\geq \max(1, \|\boldsymbol{k}\|_1)^{-\frac{T}{d-T}} \prod_{s=1}^{d} \max(1, |k_s|)^{\frac{1}{d}(1+\frac{T}{d-T})} \\ &= \max(1, \|\boldsymbol{k}\|_1)^{-\frac{T}{1-T}\frac{1-T}{d-T}} \prod_{s=1}^{d} \max(1, |k_s|)^{\frac{1}{1-T}\frac{1-T}{d-T}} \end{aligned}$$

and consequently $\max(1, \|\boldsymbol{k}\|_1)^{-\frac{T}{1-T}} \prod_{s=1}^d \max(1, |k_s|)^{\frac{1}{1-T}} \leq N$. This means, we have $\boldsymbol{k} \in I_N^{d,T}$ and therefore we obtain $I_{N^{(1-T)/(d-T)}}^{d,-\infty} \subset I_N^{d,T}$. Since we have $|I_{N^{(1-T)/(d-T)}}^{d,-\infty}| \geq c_1(d) N^{\frac{1-T}{d-T} d}$, we obtain $|I_N^{d,T}| \geq |I_{N^{(1-T)/(d-T)}}^{d,-\infty}| \geq c_1(d) N^{\frac{1-T}{d-T} d} = c_1(d) N^{\frac{T-1}{T/d-1}}$.
Due to [8, Lemma 1], we obtain $|I_N^{d,T}| \leq C_2(d,T) N^{\frac{T-1}{T/d-1}}$, where $C_2(d,T) > 0$ is a constant depending only on d and T.

- Case $T = 0$. We apply the inclusions of [15, Lemma 2.1] and use the results from [10, Section 5.3]. This yields $c_3(d) N \log_2^{d-1} N \leq |I_N^{d,0}| \leq C_3(d) N \max(1, \log_2 N)^{d-1}$, where $c_3(d) = (8d-8)^{-d+1}$ and $C_3(d) = \frac{8}{3} \frac{(d+1)^{d-1}}{(d-1)!} 12^d$.

- Case $0 < T < 1$. Since the frequencies on the coordinate axis from $-\lfloor N \rfloor$ to $\lfloor N \rfloor$ are elements of $I_N^{d,T}$, we obtain $|I_N^{d,T}| \geq 2d\lfloor N \rfloor + 1 \geq 2d(N-1) + 1 \geq c_4(d) N$ for $N \geq 2$, where $c_4(d) = d$.
Due to [9, Lemma 4.2], we obtain $|I_N^{d,T}| \leq C_4(d,T) N$, where $C_4(d,T) > 0$ is a constant depending only on d and T.

These estimates yield the assertion. ■

Next, we show that we can cover the difference set $\mathcal{D}(I_N^{d,T})$ with the index set $I_L^{d,T}$ of larger refinement $L = 2^{\frac{d-T}{1-T}} N^{1+\frac{d}{d-T}}$.

Lemma 4.2. *Let the dimension $d \in \mathbb{N}$, and a parameter T, $-\infty \leq T \leq 0$, be given. We consider the difference set $\mathcal{D}(I_N^{d,T}) := \left\{\boldsymbol{k}' - \boldsymbol{k} \colon \boldsymbol{k}, \boldsymbol{k}' \in I_N^{d,T}\right\}$. Then, we have the inclusion*

$$\mathcal{D}(I_N^{d,T}) \subset I_{2^{\frac{d-T}{1-T}} N^{1+\frac{d}{d-T}}}^{d,T}. \tag{4.2}$$

Proof. For $\boldsymbol{k} \in I_N^{d,T}$, we have $\max(1,\|\boldsymbol{k}\|_1)^{-\frac{T}{1-T}} \prod_{s=1}^d \max(1,|k_s|)^{\frac{1}{1-T}} \leq N$ by definition. Consequently, for $\boldsymbol{k},\boldsymbol{k}' \in I_N^{d,T}$ and $-\infty \leq T < 0$, we infer

$$\begin{aligned}
&\max(1,\|\boldsymbol{k}-\boldsymbol{k}'\|_1) \prod_{s=1}^d \max(1,|k_s-k_s'|)^{-\frac{1}{T}} \\
\leq\ & \big(\max(1,\|\boldsymbol{k}\|_1)+\max(1,\|\boldsymbol{k}'\|_1)\big) \prod_{s=1}^d \big(\max(1,|k_s|)+\max(1,|k_s'|)\big)^{-\frac{1}{T}} \\
\leq\ & \big(\max(1,\|\boldsymbol{k}\|_1)+\max(1,\|\boldsymbol{k}'\|_1)\big)\ 2^{-\frac{d}{T}} \prod_{s=1}^d \max(1,|k_s|)^{-\frac{1}{T}} \max(1,|k_s'|)^{-\frac{1}{T}} \\
\leq\ & 2^{-\frac{d}{T}} N^{-\frac{1-T}{T}} \left(\prod_{s=1}^d \max(1,|k_s'|)^{-\frac{1}{T}} + \prod_{s=1}^d \max(1,|k_s|)^{-\frac{1}{T}} \right).
\end{aligned}$$

Next, we estimate dominating mixed smoothness by isotropic smoothness. Since we have $\prod_{s=1}^d \max(1,|k_s|) \leq \max(1,\|\boldsymbol{k}\|_\infty)^d \leq \max(1,\|\boldsymbol{k}\|_1)^d$ for $\boldsymbol{k} \in \mathbb{Z}^d$, we obtain

$$\begin{aligned}
\prod_{s=1}^d \max(1,|k_s|)^{-\frac{1}{T}} &= \prod_{s=1}^d \max(1,|k_s|)^{\frac{1}{d-T}} \prod_{s=1}^d \max(1,|k_s|)^{-\frac{1}{T}-\frac{1}{d-T}} \\
&\leq \max(1,\|\boldsymbol{k}\|_1)^{\frac{d}{d-T}} \prod_{s=1}^d \max(1,|k_s|)^{-\frac{d}{T(d-T)}} \\
&= \left(\max(1,\|\boldsymbol{k}\|_1)^{-\frac{T}{1-T}} \prod_{s=1}^d \max(1,|k_s|)^{\frac{1}{1-T}} \right)^{-\frac{1-T}{T}\frac{d}{d-T}} \\
&\leq N^{-\frac{1-T}{T}\frac{d}{d-T}}
\end{aligned}$$

and analogously $\prod_{s=1}^d \max(1,|k_s'|)^{-\frac{1}{T}} \leq N^{-\frac{1-T}{T}\frac{d}{d-T}}$. For $T=0$, we have

$$\prod_{s=1}^d \max(1,|k_s-k_s'|) \leq 2^d \prod_{s=1}^d \max(1,|k_s|) \prod_{s=1}^d \max(1,|k_s'|) \leq 2^d N^2.$$

These results yield

$$\max(1,\|\boldsymbol{k}-\boldsymbol{k}'\|_1)^{-\frac{T}{1-T}} \prod_{s=1}^d \max(1,|k_s-k_s'|)^{\frac{1}{1-T}} \leq 2^{\frac{d-T}{1-T}} N^{1+\frac{d}{d-T}} \text{ for all } \boldsymbol{k},\boldsymbol{k}' \in I_N^{d,T}$$

and inclusion (4.2) follows. ∎

4.1 Truncation error

We estimate the truncation error $\|f - S_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\|$, since this error is part of the approximation error $\|f - \tilde{S}_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\|$ and since we also need the result as a prerequisite for Theorem 3.4. First, we show $\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$ for $\beta \geq 0$ and $\alpha > -\beta$.

Lemma 4.3. *Let the parameter $\alpha, \beta \in \mathbb{R}$, $\beta \geq 0$, $\alpha > -\beta$ be given. Then, $\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$.*

Proof. In the case $\alpha \geq 0$, we obviously have $\omega^{\alpha,\beta}(\boldsymbol{k})$ for all $\boldsymbol{k} \in \mathbb{Z}^d$. In the case $\alpha < 0$, due to $\prod_{s=1}^d \max(1, |k_s|) \leq \max(1, \|\boldsymbol{k}\|_1)^d$ for $\boldsymbol{k} \in \mathbb{Z}^d$ and $\beta + \frac{\alpha}{d} > \alpha + \beta > 0$, we infer

$$\omega^{\alpha,\beta}(\boldsymbol{k}) := \max(1, \|\boldsymbol{k}\|_1)^\alpha \prod_{s=1}^d \max(1, |k_s|)^\beta \geq \prod_{s=1}^d \max(1, |k_s|)^{\beta + \frac{\alpha}{d}} \geq 1 \text{ for all } \boldsymbol{k} \in \mathbb{Z}^d.$$

Consequently, we obtain

$$\|f|L^2(\mathbb{T}^d)\| = \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}|^2} \leq \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \omega^{\alpha,\beta}(\boldsymbol{k})^2 |\hat{f}_{\boldsymbol{k}}|^2} = \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\| < \infty$$

for an arbitrarily chosen function $f \in \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$. ■

Next, we estimate the truncation error $\|f - S_{I_N^{d,T}} f|L^2(\mathbb{T}^d)\|$ as in the proof of [16, Theorem 3.4].

Lemma 4.4. *Let the dimension $d \in \mathbb{N}$, a function $f \in \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$ and the d-dimensional index set $I_N^{d,T}$ of refinement $N \in \mathbb{R}$, $N \geq 1$, be given, where $\beta \geq 0$, $\alpha > -\beta$ and $T := -\alpha/\beta$. Then, the truncation error is bounded by*

$$\|f - S_{I_N^{d,T}} f|L^2(\mathbb{T}^d)\| \leq N^{-(\alpha+\beta)} \, \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|. \tag{4.3}$$

More specifically, the operator norm of $\operatorname{Id} - S_{I_N^{d,T}}$ *is bounded by*

$$(N+1)^{-(\alpha+\beta)} \leq \| \operatorname{Id} - S_{I_N^{d,T}} | \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d) \to L^2(\mathbb{T}^d)\| \leq N^{-(\alpha+\beta)},$$

where Id *denotes the embedding operator from $\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$ into $L^2(\mathbb{T}^d)$.*

Proof. From Lemma 4.3, we obtain $\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$. Next, we apply Lemma 3.3 with $\omega(\boldsymbol{k}) := \omega^{\alpha,\beta}(\boldsymbol{k})$, $\nu := \alpha + \beta$ and $I_N := I_N^{d,T}$. Since $T := -\alpha/\beta$, the conditions $\beta \geq 0$ and $\alpha > -\beta$ ensure that $-\infty \leq T < 1$. Due to $\omega(\boldsymbol{k})^{1/\nu} = \max(1, \|\boldsymbol{k}\|_1)^{\frac{\alpha}{\alpha+\beta}} \prod_{s=1}^d \max(1, |k_s|)^{\frac{\beta}{\alpha+\beta}} = \omega^{-\frac{T}{1-T}, \frac{1}{1-T}}(\boldsymbol{k})$, we obtain $\|f - S_{I_N^{d,T}} f|L^2(\mathbb{T}^d)\| \leq N^{-\nu} \, \|f|\mathcal{H}^\omega(\mathbb{T}^d)\| = N^{-(\alpha+\beta)} \, \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|$. The error estimate in (4.3) verifies the upper bound of the operator norm of $\operatorname{Id} - S_{I_N^{d,T}}$. We show the lower bound by an explicit example. Let the frequency index $\boldsymbol{k} = (N+1, 0, \ldots, 0)^\top \in \mathbb{Z}^d \setminus I_N^{d,T}$ and the trigonometric polynomial $g(\boldsymbol{x}) = \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$ be given. We calculate

$$\|g - S_{I_N^{d,T}} g|L^2(\mathbb{T}^d)\| = \|g|L^2(\mathbb{T}^d)\| = (N+1)^{-(\alpha+\beta)} \|g|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|$$

and we conclude that the norm of $\operatorname{Id} - S_{I_N^{d,T}}$ is bounded from below by $(N+1)^{-(\alpha+\beta)}$. ■

4.2 Aliasing error

We are going to apply Theorem 3.4 for the frequency index sets $I_N = I_N^{d,T}$ in order to estimate the aliasing error $\|S_{I_N} f - \tilde{S}_{I_N} f|L^2(\mathbb{T}^d)\|$. Therefore, we show that condition (3.4) is fulfilled for the frequency index sets $I_N^{d,T}$ of refinements $N \in \mathbb{R}$, $N \geq 2$, and parameters $-\infty \leq T < 1$.

4.2.1 Cases $-\infty \leq T \leq 0$

Lemma 4.5. *Let the dimension $d \in \mathbb{N}$, $d \geq 2$, a parameter T, $-\infty \leq T \leq 0$, and $M \in \mathbb{N}$, $M \geq 2$, be given. Then, we have*

$$|\{\boldsymbol{m} \in I^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}} : \exists \boldsymbol{m}' \in \mathbb{Z}^d \text{ such that } \boldsymbol{m} = M\boldsymbol{m}'\}| \leq C_{\mathrm{A}}(d,T)\,|I^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}}|/M + 1$$

for all refinements $N \in \mathbb{R}$, $N \geq 1$, and levels $l \in \mathbb{N} \cup \{0\}$, where $C_{\mathrm{A}}(d,T) \geq 1$ is a constant which only depends on d and T.

Proof. We denote $A^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}} := \{\boldsymbol{m} \in I^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}} : \exists \boldsymbol{m}' \in \mathbb{Z}^d \text{ such that } \boldsymbol{m} = M\boldsymbol{m}'\}$ and we group the indices $\boldsymbol{m} \in A^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}}$, where all components are zero, exactly one component is non-zero, ..., $d-1$ components are non-zero, and all d components are non-zero. For $t = 0, \ldots, d$, we denote

$$A^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})},t} := \left\{\boldsymbol{m} \in A^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}} : \text{exactly } t \text{ components of } \boldsymbol{m} \text{ are non-zero}\right\}.$$

- Case $t = 0$. We have $A^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})},0} = \{\boldsymbol{0}\}$.

- Case $1 \leq t \leq d$. If exactly the components $m_{i_1}, \ldots, m_{i_t}$ of $\boldsymbol{m} \in A^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}}$ are non-zero, $i_1, \ldots, i_t \in \{1, \ldots, d\}$, $i_j \neq i_{j'}$ for $j \neq j'$, we have

$$\begin{aligned}
&\omega^{-\frac{T}{1-T},\frac{1}{1-T}}(\boldsymbol{m}) \\
= \;& \max\left(1, M(|m'_{i_1}| + \ldots + |m'_{i_t}|)\right)^{-\frac{T}{1-T}} \prod_{\tau=1}^{t} \max(1, M|m'_{i_\tau}|)^{\frac{1}{1-T}} \\
= \;& M^{-\frac{T}{1-T}} \max(1, |m'_{i_1}| + \ldots + |m'_{i_t}|)^{-\frac{T}{1-T}}\, M^{\frac{t}{1-T}} \prod_{\tau=1}^{t} \max(1, M|m'_{i_\tau}|)^{\frac{1}{1-T}} \\
= \;& M^{\frac{t-T}{1-T}} \omega^{-\frac{T}{1-T},\frac{1}{1-T}}(\boldsymbol{m}') \leq N\,2^{(l+1)(1+\frac{d}{d-T})} \iff \omega^{-\frac{T}{1-T},\frac{1}{1-T}}(\boldsymbol{m}') \leq \frac{N\,2^{(l+1)(1+\frac{d}{d-T})}}{M^{\frac{t-T}{1-T}}}.
\end{aligned}$$

  Since there are $\binom{d}{t}$ choices for the non-zero components and due to Lemma 4.1, we have

$$\left|A^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})},t}\right| \leq \binom{d}{t} \cdot \begin{cases} C_1(d)\, \left(\frac{N\,2^{(l+1)}}{M}\right)^t & \text{for } T = -\infty, \\ C_2(d,T)\, \frac{\left(N\,2^{(l+1)(1+\frac{d}{d-T})}\right)^{\frac{T-1}{T/t-1}}}{M^t} & \text{for } -\infty < T < 0, \\ C_3(d)\, \left(\frac{N\,2^{(l+1)2}}{M^t}\right) \log^{t-1}\left(\frac{N\,2^{(l+1)2}}{M^t}\right) & \text{for } T = 0, \end{cases}$$

  for fixed $d \in \mathbb{N}$.

This means

- for $T = -\infty$

$$\begin{aligned}
|A^{d,-\infty}_{N\,2^{l+1}}| \;&\leq\; 1 + \sum_{t=1}^{d} \binom{d}{t} C_1(d) \left(\frac{N\,2^{(l+1)}}{M}\right)^t \leq 1 + \frac{(N\,2^{(l+1)})^d}{M} C_1(d)\,(2^d - 1) \\
&\leq\; 1 + \frac{|I^{d,-\infty}_{N\,2^{l+1}}|}{M} \frac{C_1(d)}{c_1(d)}\,(2^d - 1)
\end{aligned}$$

  due to $|I^{d,-\infty}_{N\,2^{l+1}}| \geq c_1(d)(N\,2^{(l+1)})^d$ as stated in Lemma 4.1,

- for $-\infty < T < 0$

$$\begin{aligned}
\left|A^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}}\right| &\leq 1+\sum_{t=1}^{d}\binom{d}{t}C_2(d,T)\left(\frac{N\,2^{(l+1)(1+\frac{d}{d-T})}}{M^{\frac{t-T}{1-T}}}\right)^{\frac{T-1}{T/t-1}} \\
&= 1+\sum_{t=1}^{d}\binom{d}{t}C_2(d,T)\frac{\left(N\,2^{(l+1)(1+\frac{d}{d-T})}\right)^{\frac{t(1-T)}{t-T}}}{M^t} \\
&\leq 1+C_2(d,T)\frac{\left(N\,2^{(l+1)(1+\frac{d}{d-T})}\right)^{\frac{d(1-T)}{d-T}}}{M}\,(2^d-1) \\
&\leq 1+\frac{C_2(d,T)}{c_1(d)}\frac{\left|I^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}}\right|}{M}\,(2^d-1)
\end{aligned}$$

due to $|I^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}}| \geq c_1(d)\,\left(N\,2^{(l+1)(1+\frac{d}{d-T})}\right)^{\frac{T-1}{T/d-1}} = c_1(d)\,\left(N\,2^{(l+1)(1+\frac{d}{d-T})}\right)^{\frac{d(1-T)}{d-T}}$ as stated in Lemma 4.1,

- for $T=0$

$$\begin{aligned}
\left|A^{d,0}_{N\,2^{(l+1)2}}\right| &\leq 1+\sum_{t=1}^{d}\binom{d}{t}C_3(d)\left(\frac{N\,2^{(l+1)2}}{M^t}\right)\log^{t-1}\left(\frac{N\,2^{(l+1)2}}{M^t}\right) \\
&\leq 1+C_3(d)\frac{N\,2^{(l+1)2}}{M}\log^{d-1}\left(N\,2^{(l+1)2}\right)\,(2^d-1) \\
&\leq 1+\frac{\left|I^{d,0}_{N\,2^{(l+1)2}}\right|}{M}\frac{C_3(d)}{c_3(d)}\,(2^d-1)
\end{aligned}$$

due to $\left|I^{d,0}_{N\,2^{(l+1)2}}\right| \geq c_3(d)\,N\,2^{(l+1)2}\,\log^{d-1}\left(N\,2^{(l+1)2}\right)$ as stated in Lemma 4.1.

We set

$$C_{\mathrm{A}}(d,T) := (2^d-1)\cdot\begin{cases} C_1(d)/c_1(d) & \text{for } T=-\infty, \\ C_2(d,T)/c_1(d) & \text{for } -\infty<T<0, \\ C_3(d)/c_3(d) & \text{for } T=0, \end{cases}$$

and this yields the assertion. ■

Lemma 4.6. *Let the dimension $d\in\mathbb{N}$, $d\geq 2$ and a function $f\in\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$ be given, where $\alpha,\beta\geq 0$ and $\alpha > d(\frac{1}{2}-\beta)$. Then, the function f has an absolutely converging Fourier series,*

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d}|\hat{f}_{\boldsymbol{k}}| < \infty.$$

Proof. Applying the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\sum_{\boldsymbol{k}\in\mathbb{Z}^d}|\hat{f}_{\boldsymbol{k}}| = \sum_{\boldsymbol{k}\in\mathbb{Z}^d}\frac{\omega^{\alpha,\beta}(\boldsymbol{k})}{\omega^{\alpha,\beta}(\boldsymbol{k})}|\hat{f}_{\boldsymbol{k}}| &\leq \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\frac{1}{\omega^{\alpha,\beta}(\boldsymbol{k})^2}}\sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\omega^{\alpha,\beta}(\boldsymbol{k})^2|\hat{f}_{\boldsymbol{k}}|^2} \\
&= \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\frac{1}{\max(1,\|\boldsymbol{k}\|_1)^{2\alpha}\,\prod_{s=1}^{d}\max(1,|k_s|)^{2\beta}}}\,\|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|.
\end{aligned}$$

Due to $\prod_{s=1}^d \max(1,|k_s|) \le \max(1,\|\boldsymbol{k}\|_1)^d$ for $\boldsymbol{k}\in\mathbb{Z}^d$, we infer

$$\begin{aligned}\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}| &\le \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\prod_{s=1}^d \frac{1}{\max(1,|k_s|)^{2(\beta+\frac{\alpha}{d})}}}\ \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\| \\ &= \left(1+2\,\zeta\left(2\left(\beta+\frac{\alpha}{d}\right)\right)\right)^{\frac{d}{2}}\ \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|,\end{aligned}$$

where ζ is the Riemann zeta function. Since $\beta\ge 0$ and $\alpha > d(\frac{1}{2}-\beta)$, we obtain $2\left(\beta+\frac{\alpha}{d}\right) > 2\left(\beta+\frac{1}{2}-\beta\right)=1$. Due to this and since $f\in\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$, we infer $\sum_{\boldsymbol{k}\in\mathbb{Z}^d}|\hat{f}_{\boldsymbol{k}}|<\infty$. ■

Theorem 4.7. *Let the dimension $d\in\mathbb{N}$, $d\ge 2$, a function $f\in C(\mathbb{T}^d)\cap\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$ and a refinement $N\in\mathbb{R}$, $N\ge 2$ be given, where $\beta\ge 0$, $\alpha\ge 0$,*

$$\alpha+\beta > \left(1+\frac{d}{d-T}\right)\frac{T-1}{T/d-1}\frac{1}{2} \tag{4.4}$$

and the parameter $T:=-\alpha/\beta$. Additionally, let a prime number $M\in\mathbb{N}$,

$$M > \frac{d\left|I^{d,T}_{2^{\frac{d-T}{1-T}}N^{1+\frac{d}{d-T}}}\right|}{1-2^{-\kappa}}+1, \tag{4.5}$$

be given, where we set the parameter $\kappa:=\alpha+\beta-(1+\frac{d}{d-T})\frac{T-1}{T/d-1}\frac{1}{2}$. Then, there exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z},M,I_N^{d,T})$ with generating vector $\boldsymbol{z}:=(1,a,\ldots,a^{d-1})^\top\in\mathbb{Z}^d$ of Korobov form and nodes $\boldsymbol{x}_j:=\frac{j}{M}\boldsymbol{z}\bmod \boldsymbol{1}$, $j=0,\ldots,M-1$, such that the aliasing error is bounded by

$$\|S_{I_N^{d,T}}f-\tilde{S}_{I_N^{d,T}}f|L^2(\mathbb{T}^d)\| \le C(d,\alpha,\beta)\ N^{-(\alpha+\beta)}\ \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|,$$

where $C(d,\alpha,\beta)>0$ is a constant which only depends on d, α, β.

Proof. We are going to apply Theorem 3.4. Therefore, we set $\omega(\boldsymbol{k}):=\omega^{\alpha,\beta}(\boldsymbol{k})$, $\nu:=\alpha+\beta$ and $I_N:=I_N^{d,T}$. Due to $d\ge 2$, $\alpha\ge 0$ and $\beta\ge 0$, we have $\left(1+\frac{d}{d-T}\right)\frac{T-1}{T/d-1}\frac{1}{2}=\frac{d}{2}\frac{2d\beta+\alpha}{d\beta+\alpha}\frac{\alpha+\beta}{d\beta+\alpha}>0$ and consequently, $\nu=\alpha+\beta>0$ follows from condition (4.4). From Lemma 4.3, we obtain $\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\subset L^2(\mathbb{T}^d)$. Furthermore, we obtain $\mathcal{D}(I_N^{d,T})\subset I^{d,T}_{2^{\frac{d-T}{1-T}}N^{1+\frac{d}{d-T}}}$ from Lemma 4.2. Thus, we set $\mathcal{I}_N:=I^{d,T}_{2^{\frac{d-T}{1-T}}N^{1+\frac{d}{d-T}}}$ for all $N\in\mathbb{R}$, $N\ge 1$. Applying Lemma 4.5, we infer

$$\begin{aligned}&|\{\boldsymbol{m}\in\mathcal{I}_{N2^l}\colon \exists \boldsymbol{m}'\in\mathbb{Z}^d \text{ such that } \boldsymbol{m}=M\boldsymbol{m}'\}|\\ =\ &|\{\boldsymbol{m}\in I^{d,T}_{2^{\frac{d-T}{1-T}}N^{1+\frac{d}{d-T}}2^{l(1+\frac{d}{d-T})}}\colon \exists\boldsymbol{m}'\in\mathbb{Z}^d \text{ such that } \boldsymbol{m}=M\boldsymbol{m}'\}|\\ \le\ & C_{\mathrm{A}}(d,T)\frac{\left|I^{d,T}_{2^{\frac{d-T}{1-T}}N^{1+\frac{d}{d-T}}2^{l(1+\frac{d}{d-T})}}\right|}{M}+1 \quad \text{for all } l\in\mathbb{N}.\end{aligned}$$

In order to apply Lemma 4.6, we first show $\alpha>\frac{d}{2}-d\beta$. Due to (4.4), we have $\alpha+\beta>$

$\frac{d}{2}\frac{2d\beta+\alpha}{d\beta+\alpha}\frac{\alpha+\beta}{d\beta+\alpha}$. This is equivalent to the condition $2(d\beta+\alpha)^2 > d(2d\beta+\alpha)$ since $d\beta+\alpha \geq \alpha+\beta > 0$. Due to $2d\beta \geq d\beta$, we obtain $2(d\beta+\alpha)^2 > d(d\beta+\alpha)$. Consequently, we have $\alpha > \frac{d}{2}-d\beta$ such that we can apply Lemma 4.6 and we obtain that f has an absolutely converging Fourier series. Next, we apply Theorem 3.4 with $\psi \equiv 1$ and we obtain that there exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N)$ with generating vector $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top \in \mathbb{Z}^d$ of Korobov form, such that the aliasing error is bounded by

$$\begin{aligned}\|S_{I_N} f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\| \;&\leq\; 2^{\alpha+\beta}\, N^{-(\alpha+\beta)}\, \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\| \\ &\quad \cdot \sum_{l=0}^{\infty} 2^{(l+1)(\frac{\kappa}{2}-(\alpha+\beta))} \sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}} \sqrt{2\,(2+(1-2^{-\kappa})\,C_{\mathrm{A}}(d,T))}.\end{aligned}$$

- Case $T = -\infty$, i.e., $\beta = 0$ and $\alpha > \frac{d}{2}$. Due to $\sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}} = \sqrt{\frac{|I^{d,-\infty}_{2N2^{l+1}}|}{|I^{d,-\infty}_{2N}|}} \leq \sqrt{\frac{C_1(d)}{c_1(d)}}\sqrt{\frac{2^d\,N^d\,2^{(l+1)d}}{2^d\,N^d}} = \sqrt{\frac{C_1(d)}{c_1(d)}}\,2^{(l+1)\frac{d}{2}}$ by Lemma 4.1, where $c_1(d) = d^{-d}$ and $C_1(d) = 3^d$, we obtain

$$\begin{aligned}\sum_{l=0}^{\infty} 2^{(l+1)(\frac{\kappa}{2}-\alpha)} \sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}} \;&\leq\; \sqrt{\frac{C_1(d)}{c_1(d)}} \sum_{l=0}^{\infty} 2^{(l+1)(-\frac{\alpha}{2}+\frac{d}{4})} \\ &=\; \sqrt{\frac{C_1(d)}{c_1(d)}}\,\frac{2^{-\frac{\alpha}{2}+\frac{d}{4}}}{1-2^{-\frac{\alpha}{2}+\frac{d}{4}}} =: \widetilde{C}(d,\alpha,0).\end{aligned}$$

- Case $-\infty < T < 0$, i.e., $\beta > 0$, $\alpha > d\left(\frac{1}{4}+\frac{1}{4}\sqrt{8\beta+1}-\beta\right)$. Due to

$$\begin{aligned}\sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}} = \sqrt{\frac{\left|I^{d,T}_{2^{\frac{d-T}{1-T}}\left(N2^{l+1}\right)^{1+\frac{d}{d-T}}}\right|}{\left|I^{d,T}_{2^{\frac{d-T}{1-T}}N^{1+\frac{d}{d-T}}}\right|}} \;&\leq\; \sqrt{\frac{C_2(d,T)}{c_1(d)}} \sqrt{\frac{\left(2^{\frac{d-T}{1-T}}\,N^{1+\frac{d}{d-T}}\,2^{(l+1)(1+\frac{d}{d-T})}\right)^{\frac{T-1}{T/d-1}}}{\left(2^{\frac{d-T}{1-T}}\,N^{1+\frac{d}{d-T}}\right)^{\frac{T-1}{T/d-1}}}} \\ &=\; \sqrt{\frac{C_2(d,T)}{c_1(d)}}\,2^{(l+1)(1+\frac{d}{d-T})\frac{T-1}{T/d-1}\frac{1}{2}}\end{aligned}$$

by Lemma 4.1, where $C_2(d,T)$ is a constant which only depends on d and T, and since we have $(-\frac{\alpha+\beta}{2}+\frac{1}{4}(1+\frac{d}{d-T})\frac{T-1}{T/d-1}) < 0$ by property (4.4), we obtain

$$\begin{aligned}\sum_{l=0}^{\infty} 2^{(l+1)(\frac{\kappa}{2}-(\alpha+\beta))} \sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}} \;&\leq\; \sqrt{\frac{C_2(d,T)}{c_1(d)}} \sum_{l=0}^{\infty} 2^{(l+1)(-\frac{\alpha+\beta}{2}+\frac{1}{4}(1+\frac{d}{d-T})\frac{T-1}{T/d-1})} \\ &=\; \sqrt{\frac{C_2(d,T)}{c_1(d)}}\,\frac{2^{-\frac{\alpha+\beta}{2}+\frac{1}{4}(1+\frac{d}{d-T})\frac{T-1}{T/d-1}}}{1-2^{-\frac{\alpha+\beta}{2}+\frac{1}{4}(1+\frac{d}{d-T})\frac{T-1}{T/d-1}}} =: \widetilde{C}(d,\alpha,\beta).\end{aligned}$$

- Case $T=0$, i.e., $\beta>1$ and $\alpha=0$. Due to

$$\begin{aligned}\sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}} &\le \sqrt{\frac{C_3(d)}{c_3(d)}}\sqrt{\frac{2^d\,N^2\,2^{2(l+1)}\left(\log(2^d\,N^2\,2^{2(l+1)})\right)^{d-1}}{2^d\,N^2\left(\log(2^d\,N^2)\right)^{d-1}}}\\ &= \sqrt{\frac{C_3(d)}{c_3(d)}}\,2^{l+1}\left(\frac{\log(2^d\,N^2)+\log(2^{2(l+1)})}{\log(2^d\,N^2)}\right)^{\frac{d-1}{2}}\\ &\le \sqrt{\frac{C_3(d)}{c_3(d)}}\,2^{l+1}\left(2\log(2^{2(l+1)})\right)^{\frac{d-1}{2}}=\sqrt{\frac{C_3(d)}{c_3(d)}}\,(2\log 2)^{\frac{d-1}{2}}\,2^{l+1}\,(2l+2)^{\frac{d-1}{2}}\end{aligned}$$

by Lemma 4.1, where $c_3(d)$ and $C_3(d)$ are constants which only depend on d, we have

$$\sum_{l=0}^{\infty}2^{(l+1)(\frac{\kappa}{2}-(\alpha+\beta))}\sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}}\le\sqrt{\frac{C_3(d)}{c_3(d)}}\,(2\log 2)^{\frac{d-1}{2}}\sum_{l=0}^{\infty}\frac{(2l+2)^{\frac{d-1}{2}}}{2^{(l+1)\frac{\beta-1}{2}}}.$$

Since $\beta>1$, the term $\sum_{l=0}^{\infty}\frac{(2l+2)^{\frac{d-1}{2}}}{2^{(l+1)\frac{\beta-1}{2}}}<\infty$ and we are going to estimate this sum. The function $g:[0,\infty)\to\mathbb{R}$, $g(l):=\frac{(2l+2)^{\frac{d-1}{2}}}{2^{(l+1)\frac{\beta-1}{2}}}$, has its only maximum at

$$l_{\max}:=\max(0,\frac{d-1}{(\beta-1)\log_{\mathrm{e}}2}-1)$$

and we estimate

$$\begin{aligned}\sum_{l=0}^{\infty}\frac{(2l+2)^{\frac{d-1}{2}}}{2^{(l+1)\frac{\beta-1}{2}}}&=\sum_{l=0}^{\infty}g(l)\le\sum_{l=0}^{\lfloor l_{\max}\rfloor}g(l)+\sum_{l=\lceil l_{\max}\rceil}^{\infty}g(l)\\ \le\;& g(l_{\max})+\int_0^{\lfloor l_{\max}\rfloor}g(l)\mathrm{d}l+g(l_{\max})+\int_{\lceil l_{\max}\rceil}^{\infty}g(l)\mathrm{d}l\le 2\,g(l_{\max})+\int_0^{\infty}g(l)\mathrm{d}l\\ \le\;& 2\max\left(\frac{2^{\frac{d-1}{2}}}{2^{\frac{\beta-1}{2}}},\left(\frac{2(d-1)}{(\beta-1)\,\mathrm{e}\log_{\mathrm{e}}2}\right)^{\frac{d-1}{2}}\right)+\frac{(d-1)\left(\frac{4}{(\beta-1)\log_{\mathrm{e}}2}\right)^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})+2^{\frac{d+2-\beta}{2}}}{(\beta-1)\log_{\mathrm{e}}2}\\ =:\;&\widetilde{C}(d,0,\beta).\end{aligned}$$

These estimates yield

$$\|S_{I_N}f-\tilde{S}_{I_N}f|L^2(\mathbb{T}^d)\|\le\underbrace{\sqrt{2\,(2+(1-2^{-\kappa})\,C_{\mathrm{A}}(d,T))}\,\widetilde{C}(d,\alpha,\beta)\,2^{\alpha+\beta}}_{:=C(d,\alpha,\beta)}\,N^{-(\alpha+\beta)}\,\|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|.$$

■

4.2.2 Cases $0<T<1$

Lemma 4.8. *Let the dimension $d \in \mathbb{N}$, $d \geq 2$, a parameter T, $0 < T < 1$, a parameter $\kappa > 0$, and a number $M \in \mathbb{N}$, $M > \frac{d\,|\mathcal{D}(I_N^{d,T})|}{1-2^{-\kappa}} + 1$ be given. Then, we have*

$$|\mathcal{D}(I_{N2^{l+1}}^{d,T}) \cap M\mathbb{Z}^d| \leq C_{\mathrm{A}}(d,T)\, \frac{|\mathcal{D}(I_{N2^{l+1}}^{d,T})|}{M} (l+1)^{d-1} + 1$$

for all refinements $N \in \mathbb{R}$, $N \geq 1$, and levels $l \in \mathbb{N} \cup \{0\}$, where $C_{\mathrm{A}}(d,T) \geq 1$ is a constant which only depends on d and T.

Proof. For $0 \leq T < 1$, we denote

$$\mathcal{A}_{N,t}^{d,T} := \{\boldsymbol{m} \in \mathcal{D}(I_N^{d,T}) \cap M\mathbb{Z}^d \colon \text{exactly } t \text{ components of } \boldsymbol{m} \text{ are non-zero}\}, \quad t = 0, \ldots, d.$$

Then, we have $\mathcal{D}(I_N^{d,T}) \cap M\mathbb{Z}^d = \bigcup_{t=0}^d \mathcal{A}_{N,t}^{d,T}$ and $|\mathcal{D}(I_N^{d,T}) \cap M\mathbb{Z}^d| = \sum_{t=0}^d |\mathcal{A}_{N,t}^{d,T}|$. Next, we estimate $|\mathcal{A}_{N,t}^{d,T}|$ for $t = 0, \ldots, d$.

- Case $t = 0$. Obviously, we have $\mathcal{A}_{N2^{l+1},0}^{d,T} = \{\boldsymbol{0}\}$ and $|\mathcal{A}_{N2^{l+1},0}^{d,T}| = 1$.
- Case $t = 1$. $|\mathcal{A}_{N2^{l+1},1}^{d,T}| \leq d\, \frac{2N2^{l+1}}{M} < 2d \frac{|I_{N2^{l+1}}^{d,T}|}{M} < 2d|\mathcal{D}(I_{N2^{l+1}}^{d,T})|/M$.
- Case $2 \leq t \leq d$. Due to [16, Lemma 2.4] with $\widetilde{T} := 0$, we have $I_N^{d,T} \subset I^{d,0}_{d^{\frac{T}{1-T}} N}$, $N \in \mathbb{R}$, $N \geq 1$, and consequently, we infer

$$\left(\mathcal{D}\left(I_{N2^{l+1}}^{d,T}\right) \cap M\mathbb{Z}^d\right) \subset \left(\mathcal{D}\left(I^{d,0}_{d^{\frac{T}{1-T}} N2^{l+1}}\right) \cap M\mathbb{Z}^d\right)$$

as well as

$$\mathcal{A}_{N2^{l+1},t}^{d,T} \subset \mathcal{A}^{d,0}_{d^{\frac{T}{1-T}} N2^{l+1},t} \subset A^{d,0}_{2^d(d^{\frac{T}{1-T}} N2^{l+1})^2,t} = A^{d,0}_{2^d d^{\frac{2T}{1-T}} N^2\, 2^{2(l+1)},t},$$

where $A_{N,t}^{d,0} := \left\{\boldsymbol{m} \in I_N^{d,0} \cap M\mathbb{Z}^d \colon \text{exactly } t \text{ components of } \boldsymbol{m} \text{ are non-zero}\right\}$.
From the proof of Lemma 4.5 and since $|\mathcal{D}(I_N^{d,T})| \geq (2N+1)^2 > N^2$, we obtain

$$\begin{aligned}
&\left|\mathcal{A}_{N2^{l+1},t}^{d,T}\right| \leq \left|A^{d,0}_{2^d d^{\frac{2T}{1-T}} N^2\, 2^{2(l+1)},t}\right| \\
\leq\ & C_3(d) \binom{d}{t} \frac{2^d d^{\frac{2T}{1-T}} N^2\, 2^{(l+1)2}}{M^t} \log_2^{t-1}\left(\frac{2^d d^{\frac{2T}{1-T}} N^2\, 2^{(l+1)2}}{M^t}\right) \\
\leq\ & C_3(d) \binom{d}{t} 2^d d^{\frac{2T}{1-T}}\, \frac{|\mathcal{D}(I_{N2^{l+1}}^{d,T})|}{M\, M^{t-1}} \log_2^{t-1}\left(2^d d^{\frac{2T}{1-T}}\, \frac{2^{(l+1)2}}{M^{t-1}}\right) \\
\leq\ & C_3(d) \binom{d}{t} 2^d d^{\frac{2T}{1-T}}\, \frac{|\mathcal{D}(I_{N2^{l+1}}^{d,T})|}{M\, M^{t-1}} \left(\log_2\left(2^d d^{\frac{2T}{1-T}}\right) + \log_2 2^{(l+1)2}\right)^{t-1} \\
\leq\ & C_3(d) \binom{d}{t} 2^{d+t-1} d^{\frac{2T}{1-T}}\, \log_2^{t-1}\left(2^d d^{\frac{2T}{1-T}}\right) \frac{|\mathcal{D}(I_{N2^{l+1}}^{d,T})|}{M} \left(\frac{2(l+1)}{M}\right)^{t-1} \\
\leq\ & C_3(d)\, 2^{2d-1} d^{\frac{2T}{1-T}} \log_2^{d-1}\left(2^d d^{\frac{2T}{1-T}}\right) \frac{|\mathcal{D}(I_{N2^{l+1}}^{d,T})|}{M} (l+1)^{d-1} \binom{d}{t}.
\end{aligned}$$

Consequently, this yields

$$|\mathcal{D}(I_{N2^{l+1}}^{d,T}) \cap M\mathbb{Z}^d| \leq C_3(d)\, 2^{2d-1} d^{\frac{2T}{1-T}} \log_2^{d-1}\left(2^d d^{\frac{2T}{1-T}}\right) \frac{|\mathcal{D}(I_{N2^{l+1}}^{d,T})|}{M} (l+1)^{d-1} + 1.$$

■

Lemma 4.9. *Let the dimension $d \in \mathbb{N}$, $d \geq 2$ and a function $f \in \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$, where $0 > \alpha > \frac{1}{2} - \beta$. Then, the function f has an absolutely converging Fourier series,*

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}| < \infty.$$

Proof. As in the proof of Lemma 4.6, we apply the Cauchy-Schwarz inequality and obtain

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}| \quad \leq \quad \sqrt{\sum_{\boldsymbol{k} \in \mathbb{Z}^d} \frac{1}{\max(1, \|\boldsymbol{k}\|_1)^{2\alpha} \prod_{s=1}^d \max(1, |k_s|)^{2\beta}}}\ \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|.$$

Due to $\max(1, \|\boldsymbol{k}\|_1) \leq 2^d \prod_{s=1}^d \max(1, |k_s|)$ for $\boldsymbol{k} \in \mathbb{Z}^d$, we infer

$$\begin{aligned}
\sum_{\boldsymbol{k} \in \mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}| \quad &\leq \quad \sqrt{\sum_{\boldsymbol{k} \in \mathbb{Z}^d} 2^{-d\alpha} \prod_{s=1}^d \frac{1}{\max(1, |k_s|)^{2(\beta+\alpha)}}}\ \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\| \\
&= \quad 2^{-\frac{d\alpha}{2}}\ (1 + 2\,\zeta\,(2(\alpha+\beta)))^{\frac{d}{2}}\ \ \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|.
\end{aligned}$$

Since we have $2(\alpha + \beta) > 1$ and $f \in \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$, we obtain $\sum_{\boldsymbol{k} \in \mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}| < \infty$. ■

Theorem 4.10. *Let the dimension $d \in \mathbb{N}$, $d \geq 2$, a function $f \in C(\mathbb{T}^d) \cap \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$ and a refinement $N \in \mathbb{R}$, $N \geq 2$ be given, where $\alpha < 0$ and $\beta > 1 - \alpha$. Additionally, let a prime number $M \in \mathbb{N}$,*

$$M > \frac{d\,|\mathcal{D}(I_N^{d,T})|}{1 - 2^{-\kappa}} + 1, \tag{4.6}$$

be given, where the parameter $T := -\alpha/\beta$ and the parameter $\kappa := \alpha + \beta - 1$. Then, there exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N^{d,T})$ with generating vector $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top \in \mathbb{Z}^d$ of Korobov form and nodes $\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1}$, $j = 0, \ldots, M-1$, such that the aliasing error is bounded by

$$\|S_{I_N^{d,T}} f - \tilde{S}_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\| \leq C(d, \alpha, \beta)\ N^{-(\alpha+\beta)}\ \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|,$$

where $C(d, \alpha, \beta) > 0$ is a constant which only depends on d, α, β.

Proof. We are going to apply Theorem 3.4. Therefore, we set $\omega(\boldsymbol{k}) := \omega^{\alpha,\beta}(\boldsymbol{k})$, $\nu := \alpha + \beta$, $I_N := I_N^{d,T}$ and $\mathcal{I}_N := \mathcal{D}(I_N^{d,T})$. From Lemma 4.3, we obtain $\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$. We apply Lemma 4.8 and this yields

$$\begin{aligned}
&|\{\boldsymbol{m} \in \mathcal{I}_{N2^l} : \exists \boldsymbol{m}' \in \mathbb{Z}^d \text{ such that } \boldsymbol{m} = M\boldsymbol{m}'\}| = |\mathcal{D}(I_{N2^l}^{d,T}) \cap M\mathbb{Z}^d| \\
\leq\ & C_{\mathrm{A}}(d, T) \frac{|\mathcal{D}(I_{N2^l}^{d,T})|}{M} l^{d-1} + 1 \quad \text{for all } l \in \mathbb{N}.
\end{aligned}$$

Furthermore, we need the property that f has a absolutely convergent Fourier series. Since $\alpha > 1-\beta > \frac{1}{2}-\beta$, we can apply Lemma 4.9 and obtain this property.
Next, we apply Theorem 3.4 with $\psi(l) := l^d$ and we obtain

$$\begin{aligned} & \|S_{I_N} f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\| \\ \leq \; & 2^{\alpha+\beta} \, N^{-(\alpha+\beta)} \, \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\| \\ & \cdot \sum_{l=0}^{\infty} 2^{(l+1)(\frac{\kappa}{2}-(\alpha+\beta))} \sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}} \sqrt{2\,(2+(1-2^{-\kappa})\,C_{\mathrm{A}}(d,T)\,(l+1)^{d-1})}. \end{aligned}$$

Due to $|\mathcal{I}_{N2^{l+1}}| = |\mathcal{D}(I^{d,T}_{N2^{l+1}})| \leq \left(C_4(d,T) N\, 2^{l+1}\right)^2$ and $|\mathcal{I}_N| = |\mathcal{D}(I^{d,T}_N)| \geq (2N)^2 > N^2$, where $C_4(d,T)$ is a constant which only depends on d and T, we infer $\sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}} \leq C_4(d,T)\, 2^{l+1}$. Then, we obtain

$$\begin{aligned} & \sum_{l=0}^{\infty} 2^{(l+1)(\frac{\kappa}{2}-(\alpha+\beta))} \sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}} \sqrt{2\,(2+(1-2^{-\kappa})\,C_{\mathrm{A}}(d,T)\,(l+1)^{d-1})} \\ < \; & \sum_{l=0}^{\infty} 2^{(l+1)(\frac{\kappa}{2}-(\alpha+\beta))}\, C_4(d,T)\, 2^{l+1} \sqrt{8\, C_{\mathrm{A}}(d,T)\,(l+1)^{d-1}} \\ = \; & C_4(d,T) \sqrt{8\, C_{\mathrm{A}}(d,T)} \sum_{l=0}^{\infty} 2^{(l+1)(\frac{\alpha+\beta-1}{2}-(\alpha+\beta)+1)}\, (l+1)^{\frac{d-1}{2}} \\ = \; & C_4(d,T) \sqrt{8\, C_{\mathrm{A}}(d,T)} 2^{-\frac{d-1}{2}} \sum_{l=0}^{\infty} \frac{(2l+2)^{\frac{d-1}{2}}}{2^{(l+1)(\frac{\alpha+\beta-1}{2})}} \end{aligned}$$

and the term $\sum_{l=0}^{\infty} \frac{(2l+2)^{\frac{d-1}{2}}}{2^{(l+1)(\frac{\alpha+\beta-1}{2})}} < \infty$ since $\alpha+\beta > 1$. As in the proof of Theorem 4.7 for the case $T=0$ replacing β by $\alpha+\beta$, we infer

$$\begin{aligned} & \frac{C_4(d,T) \sqrt{8\, C_{\mathrm{A}}(d,T)}\, 2^{-\frac{d-1}{2}}}{\sqrt{2\,(2+(1-2^{-\kappa})\,C_{\mathrm{A}}(d,T))}} \sum_{l=0}^{\infty} \frac{(2l+2)^{\frac{d-1}{2}}}{2^{(l+1)\frac{\alpha+\beta-1}{2}}} \\ \leq \; & \frac{C_4(d,T) \sqrt{8\, C_{\mathrm{A}}(d,T)}\, 2^{-\frac{d-1}{2}}}{\sqrt{2\,(2+(1-2^{-\kappa})\,C_{\mathrm{A}}(d,T))}} 2 \left[\max\left(\frac{2^{\frac{d-1}{2}}}{2^{\frac{\alpha+\beta-1}{2}}}, \left(\frac{2(d-1)}{(\alpha+\beta-1)\,\mathrm{e} \log_{\mathrm{e}} 2} \right)^{\frac{d-1}{2}} \right) \right. \\ & \left. + \frac{(d-1)\,(\frac{4}{(\alpha+\beta-1)\log_{\mathrm{e}} 2})^{\frac{d-1}{2}}\, \Gamma(\frac{d-1}{2}) + 2^{\frac{d+2-\alpha+\beta}{2}}}{(\alpha+\beta-1)\log_{\mathrm{e}} 2} \right] \\ =: \; & \widetilde{C}(d,\alpha,\beta). \end{aligned}$$

These estimates yield

$$\|S_{I_N} f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\| \leq \underbrace{\sqrt{2\,(2+(1-2^{-\kappa})\,C_{\mathrm{A}}(d,T))}\, \widetilde{C}(d,\alpha,\beta)\, 2^{\alpha+\beta}}_{:=C(d,\alpha,\beta)} \, N^{-(\alpha+\beta)} \, \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|.$$

■

4.3 Comparison with previous results

In [16], the truncation error $\|f - S_{I_N^{d,T}} f|\mathcal{H}^{r,t}(\mathbb{T}^d)\|$ and aliasing error $\|S_{I_N^{d,T}} f - \tilde{S}_{I_N^{d,T}} f|\mathcal{H}^{r,t}(\mathbb{T}^d)\|$ were considered for arbitrarily chosen reconstructing rank-1 lattices $\Lambda(\boldsymbol{z}, M, I_N^{d,T})$ and functions $f \in \mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)$, where $r, t \in \mathbb{R}$, $t \geq 0$, $r > -t$, $\beta \geq 0$, $\alpha > -\beta$, $r + t < \alpha + \beta$, $\lambda > 1/2$, and $T := -\frac{\alpha - r}{\beta - t}$. The truncation error was estimated by

$$\|f - S_{I_N^{d,T}} f|\mathcal{H}^{r,t}(\mathbb{T}^d)\| \leq N^{-(\alpha - r + \beta - t)} \, \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|$$

in the proof of [16, Theorem 3.4] and for functions f with absolutely convergent Fourier series, the aliasing error was estimated by

$$\|S_{I_N^{d,T}} f - \tilde{S}_{I_N^{d,T}} f|\mathcal{H}^{r,t}(\mathbb{T}^d)\| \leq (1 + 2\zeta(2\lambda))^{\frac{d}{2}} \, N^{-(\alpha - r + \beta - t)} \, \|f|\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)\| \tag{4.7}$$

in [16, Section 3.2], which yields

$$\|f - \tilde{S}_{I_N^{d,T}} f|\mathcal{H}^{r,t}(\mathbb{T}^d)\| \leq \left(1 + (1 + 2\zeta(2\lambda))^{\frac{d}{2}}\right) N^{-(\alpha - r + \beta - t)} \, \|f|\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)\| \tag{4.8}$$

for the approximation error. We remark that a constructive method for obtaining a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I)$ for a given frequency $I \subset \mathbb{Z}^d$ of finite cardinality is described in [12]. In the present paper, we were able to improve the estimates (4.7) and (4.8). We showed that there exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N^{d,T})$ with generating vector $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top$ of Korobov form such that we do not have the dependence on λ for the special cases $r = t = 0$, $\alpha + \beta > (1 + \frac{d}{d - (T)_-}) \frac{(T)_- - 1}{(T)_-/d - 1} \frac{1}{2}$, where $(T)_- := \min(0, T)$, see Theorem 4.7 and 4.10. However, we do not know a constructive method for obtaining such a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N^{d,T})$.

In [7], functions from the spaces of generalized mixed Sobolev smoothness

$$\mathcal{H}^{t,r}_{\text{mix}}(\mathbb{T}^d) := \left\{ f \colon \sqrt{\sum\nolimits_{\boldsymbol{k} \in \mathbb{Z}^d} \prod\nolimits_{s=1}^{d} (1 + |k_s|)^{2t} (1 + \|\boldsymbol{k}|_\infty)^{2r} |\hat{f}_{\boldsymbol{k}}|^2} < \infty \right\}.$$

and generalized hyperbolic cross frequency index sets $I = \Gamma_N^T := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \prod_{s=1}^d (1 + |k_d|) \cdot (1 + \|\boldsymbol{k}\|_\infty)^{-T} \leq N^{1-T}\}$ were considered. As sampling nodes $\boldsymbol{x}_j$, the nodes of a (generalized) sparse grid with size $M = |\Gamma_N^T|$ were used. We remark that the inclusions $\mathcal{I}^{d,T}_{(N+1)2^{(T-d)/(1-T)}} \subset \Gamma_N^T \subset \mathcal{I}^{d,T}_{(N+1)d^{-T/(1-T)}}$ are valid in the cases $-\infty \leq T \leq 0$ and $\mathcal{I}^{d,T}_{(N+1)d^{-T/(1-T)} 2^{-d/(1-T)}} \subset \Gamma_N^T \subset \mathcal{I}^{d,T}_{(N+1)2^{T/(1-T)}}$ in the cases $0 < T < 1$ for $d \in \mathbb{N}$ and arbitrary refinement $N \in \mathbb{R}$, $N \geq 1$, cf. [16, Lemma 2.6]. Furthermore, we obtain from the proof of [16, Lemma 2.6] that $c(d,r,t)\|f|\mathcal{H}^{r,t}(\mathbb{T}^d)\| \leq \|f|\mathcal{H}^{t,r}_{\text{mix}}(\mathbb{T}^d)\| \leq C(d,r,t)\|f|\mathcal{H}^{r,t}(\mathbb{T}^d)\|$, where

$$c(d,r,t) := \begin{cases} d^{-r} & \text{for } r \geq 0, t \geq 0, \\ 2^r & \text{for } 0 > r > -t, t > 0, \end{cases} \qquad C(d,r,t) := \begin{cases} 2^r 2^{dt} & \text{for } r \geq 0, t \geq 0, \\ d^{-r} 2^{dt} & \text{for } 0 > r > -t, t > 0. \end{cases}$$

For the approximation error (and the aliasing error), it was shown, cf. [7, Lemma 8], that

$$\|f - \mathcal{L}_{\Gamma_N^T} f|\mathcal{H}^{0,r}_{\text{mix}}(\mathbb{T}^d)\| \lesssim N^{-(t-r)} \, (\log N)^{d-1} \, \|f|\mathcal{H}^{t,0}_{\text{mix}}(\mathbb{T}^d)\|,$$

where $\mathcal{L}_{\Gamma_N^T}$ is the interpolation operator on the (generalized) sparse grid, $0 \le r < t$, $t > \frac{1}{2}$, $f \in \mathcal{H}^{t,0}_{\text{mix}}(\mathbb{T}^d)$ and $T := \frac{r}{t}$. In particular in the case $r = 0$, the frequency index sets Γ_N^0 are hyperbolic crosses and the above estimate yields

$$\|f - \mathcal{L}_{\Gamma_N^0} f | L^2(\mathbb{T}^d)\| \lesssim N^{-t} \, (\log N)^{d-1} \, \|f|\mathcal{H}^{t,0}_{\text{mix}}(\mathbb{T}^d)\|,$$

i.e., there is an additional factor of $(\log N)^{d-1}$ compared to [25, Theorem 2] and (1.4). Similarly in [29, 22], where the case $r = 0$ and sparse grids sampling nodes were considered, it was proven that the approximation error

$$\|f - \mathcal{L}_{\Gamma_N^0} f | L^2(\mathbb{T}^d)\| \le C(d) \, N^{-\beta} \, (\log N)^{\frac{d-1}{2}} \, \|f|\mathcal{H}^{0,\beta}(\mathbb{T}^d)\|,$$

where $C(d) > 0$ is a constant which only depends on d, see [22, Theorem 1]. This means, there is an additional factor of $(\log N)^{\frac{d-1}{2}}$ compared to [25, Theorem 2] and (1.4). However, the sampling schemes in [7, 29, 22] only use $M = |I| = \Theta(N \log^{d-1} N)$ many samples, whereas we require $M = \Theta(N^2 \log^{d-1} N)$ many samples, see (1.3). The advantage of our approach is that the computation of the approximated Fourier coefficients $\tilde{\hat{f}}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, using the sampling method (1.1) is numerically perfectly stable whereas the computation using the sampling schemes from [7, 29, 22] may be numerically unstable, cf. [14].

5 Numerical results

In practice, we do not know a method for verifying if a generating vector $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top \in \mathbb{Z}^d$ of Korobov form fulfills property (2.8) in Lemma 2.1 for a given reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I)$. Furthermore, we also do not know how to construct a generating vector $\boldsymbol{z}$ fulfilling property (2.8). However, this special property is crucial for obtaining the estimate (1.4) by Theorem 4.7 and Theorem 4.10. Consequently, we have only the upper bounds from Section 4.3 available. Nevertheless, numerical tests performed in [16, Section 6], which use reconstructing rank-1 lattices $\Lambda(\boldsymbol{z}, M, I)$ obtained from a constructive method described in [12], showed that the approximation error $\|f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ is in $\mathcal{O}(N^{-\beta}) \, \|f|\mathcal{H}^{0,\beta}(\mathbb{T}^d)\|$ for the functions considered there, which is of optimal order, cf. Lemma 4.4. This suggests that the aliasing error can also be

$$\|S_{I_N^{d,0}} f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\| \lesssim N^{-\beta} \, \|f|\mathcal{H}^{0,\beta}(\mathbb{T}^d)\|$$

for reconstructing rank-1 lattices $\Lambda(\boldsymbol{z}, M, I)$ with generating vectors $\boldsymbol{z}$ which are not necessarily of Korobov form.

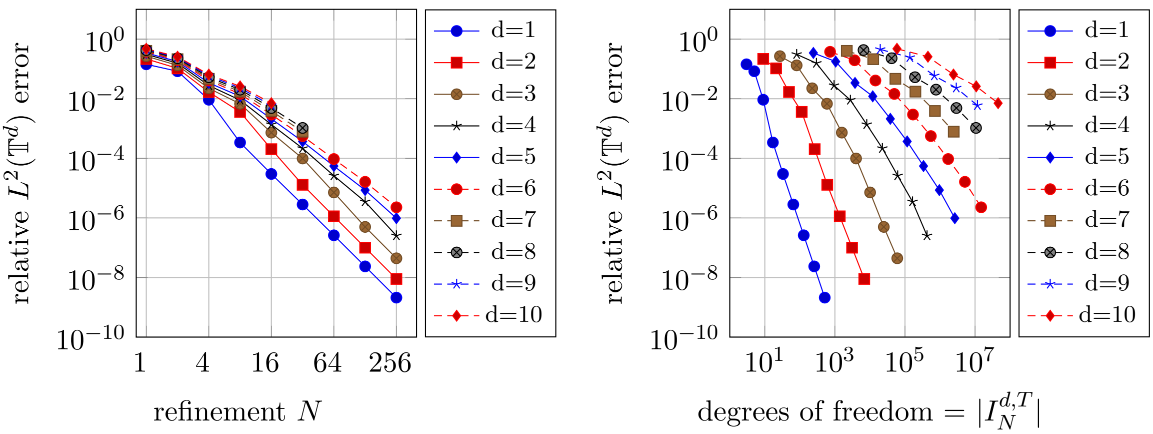
Here, we investigate the approximation error more closely and consider the truncation error and the aliasing error. As in [7] and in [16, Example 6.1], we consider the function

$$f(\boldsymbol{x}) = \prod_{s=1}^{d} \frac{8\sqrt{6}\sqrt{\pi}}{\sqrt{6369\pi - 4096}} \left[4 + \operatorname{sgn}(x_s - \frac{1}{2}) \left(\sin(2\pi x_s)^3 + \sin(2\pi x_s)^4\right)\right], \tag{5.1}$$

where $\|f|L^2(\mathbb{T}^d)\| = 1$, $f \in \mathcal{H}^{0,\frac{7}{2}-\epsilon}(\mathbb{T}^d)$, $\epsilon > 0$, $f \notin \mathcal{H}^{0,\frac{7}{2}}(\mathbb{T}^d)$, and the Fourier coefficients

$$\hat{f}_{\boldsymbol{k}} = \prod_{s=1}^{d} \frac{8\sqrt{6}\sqrt{\pi}}{\sqrt{6369\pi - 4096}} \begin{cases} \frac{-12}{(k_s-3)(k_s-1)(k_s+1)(k_s+3)\pi} & \text{for } k_s \in 2\mathbb{Z} \setminus \{0\}, \\ \frac{48\mathrm{i}}{(k_s-4)(k_s-2)k_s(k_s+2)(k_s+4)\pi} & \text{for } k_s \text{ odd}, \\ 4 - \frac{4}{3\pi} & \text{for } k_s = 0. \end{cases}$$

As frequency index sets I, we use the symmetric hyperbolic cross index sets $I = I_N^{d,0}$ with different refinements N and as sampling nodes $\boldsymbol{x}_j$, we use the nodes of the reconstructing rank-1 lattices $\Lambda(\boldsymbol{z}, M, I_N^{d,0})$ with generating vectors $\boldsymbol{z}$ of Korobov form. In Table 5.1, the used generating vectors $\boldsymbol{z}$, rank-1 lattice sizes M and the resulting oversampling factors $M/|I_N^{d,0}|$ are listed for the three largest refinements N of each dimension d. We observe that these oversampling factors $M/|I_N^{d,0}|$ grow for increasing refinements N and fixed dimension d. Moreover, the obtained rank-1 lattices sizes are up to about 4 times larger compared to the ones in [16, Table 6.2]. We remark that the reconstructing rank-1 lattices of Korobov form used in this section fulfill the requirement (2.7) of Lemma 2.1 but do not necessarily fulfill the condition (2.8). Nevertheless, we observe that the truncation errors dominate the aliasing errors, i.e., $\|S_{I_N^{d,0}} f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\| \leq \|f - S_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$. Plots of the $L^2(\mathbb{T}^d)$ approximation error $\|f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ are depicted in Figure 5.1. We observe that the approximation error decreases like $\sim N^{-3.45}$ in the one-dimensional case and slightly slower in the multi-dimensional cases.



(a) $G_{3,4}^d$, $T = 0$ (b) $G_{3,4}^d$, $T = 0$

Figure 5.1: Relative $L^2(\mathbb{T}^d)$ error for the approximation of the function $G_{3,4}^d$.

In Figure 5.2 the truncation errors $\|f - S_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ and aliasing errors $\|S_{I_N^{d,0}} f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ of the function f from (5.1) are shown for the cases $d = 2, \ldots, 10$. We stress the fact that the truncation errors only depend on the frequency index set $I_N^{d,0}$ and do not depend on the sampling sets. The truncation errors should be asymptotically of optimal order, cf. Lemma 4.4. For the cases $d = 2, 3, 4$, we observe this optimal order, whereas the truncation errors seem to decrease slower for the cases $d = 5, \ldots, 10$. We suspect that the used values of N are still too small for the cases $d \geq 5$ to see the asymptotic behavior. In Figure 5.2, we observe that the aliasing errors $\|S_{I_N^{d,0}} f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ are smaller than the truncation errors $\|f - S_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ and that the aliasing errors $\|S_{I_N^{d,0}} f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ decrease approximately as stated in the theoretical results, i.e., with an order of about $N^{-3.5+\epsilon}$.

Additionally, we investigate the truncation errors $\|f - S_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\|$ and aliasing errors $\|S_{I_N^{d,T}} f - \tilde{S}_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\|$ of the function f from (5.1) for further function classes $\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$ and corresponding frequency index sets $I_N^{d,T}$, $T := -\alpha/\beta$. Due to the inequalities from the proof of

d	N	$\lvert I_N^{d,0}\rvert$	a	M	$\frac{M}{\lvert I_N^{d,0}\rvert}$	$\lVert f - \tilde{S}_{I_N^{d,0}} f \vert L^2(\mathbb{T}^d)\rVert$
2	64	1 377	129	8 451	6.1	1.1e-06
2	128	3 093	257	33 283	10.8	1.0e-07
2	256	6 889	513	132 099	19.2	9.0e-09
3	64	10 113	129	54 745	5.4	7.2e-06
3	128	24 869	257	216 318	8.7	5.0e-07
3	256	60 217	513	860 146	14.3	4.4e-08
4	64	61 889	129	658 768	10.6	2.6e-05
4	128	164 137	257	2 899 974	17.7	3.5e-06
4	256	426 193	513	12 402 996	29.1	2.5e-07
5	64	338 305	129	7 012 279	20.7	5.5e-05
5	128	958 345	257	33 509 650	35.0	8.6e-06
5	256	264 4977	513	186 198 186	70.4	9.8e-07
6	64	1 709 857	129	64 329 589	37.6	9.5e-05
6	128	5 137 789	257	418 596 194	81.5	1.6e-05
6	256	14 977 209	523	2 356 403 754	157.3	2.3e-06
7	8	198 369	17	2 450 453	12.4	1.7e-02
7	16	716 985	33	16 405 121	22.9	3.9e-03
7	32	2 465 613	65	98 758 658	40.1	7.8e-04
8	8	768 609	17	14 004 649	18.2	2.0e-02
8	16	2 935 521	33	109 592 068	37.3	4.9e-03
8	32	10 665 297	65	893 885 429	83.8	1.0e-03
9	4	688 905	9	12 792 805	18.6	5.9e-02
9	8	2 910 897	17	101 881 573	35.0	2.3e-02
9	16	11 693 889	43	937 909 924	80.2	5.9e-03
10	4	2 421 009	9	64 679 873	26.7	6.5e-02
10	8	10 819 089	17	682 254 539	63.1	2.6e-02
10	16	45 548 649	41	6 537 062 011	143.5	7.1e-03

Table 5.1: Cardinalities $|I_N^{d,0}|$, numbers a used for generating vector $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top$, rank-1 lattice sizes M, oversampling factors $M/|I_N^{d,0}|$ and approximation errors $\|f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ of the function f from (5.1) for various values of d and N.

[16, Lemma 2.3] and due to [16, Lemma 2.4], we have $f \in \mathcal{H}^{-7d/(4d-2), 7/2+7/(4d-2)-\epsilon}(\mathbb{T}^d)$ and the corresponding parameter $T := -\alpha/\beta = 1/2$, i.e., the truncation and aliasing errors may asymptotically decrease slower like $\sim N^{-7/2-7(1-d)/(4d-2)+\epsilon}$ when using energy-norm based hyperbolic crosses $I_N^{d,1/2}$ for dimension $d \geq 2$ compared to $\sim N^{-7/2+\epsilon}$ when using hyperbolic crosses $I_N^{d,0}$. Furthermore, we have $f \in \mathcal{H}^{\alpha,\beta-\epsilon}(\mathbb{T}^d)$, $\alpha \geq 0$ and $\beta = 3.5 - \alpha$ with the corresponding parameter $T := -\alpha/\beta \leq 0$ for the frequency index set $I_N^{d,T}$, i.e., the expected order of decrease for the truncation and aliasing errors is the same compared to when using hyperbolic crosses $I_N^{d,0}$. In Figure 5.3, the numerical results are depicted for the parameter $T = 1/2, 0, -5, -\infty$ and dimensions $d = 2, 3, 4$. We observe that for parameters $T = -5, -\infty$ and dimensions $d = 2, 3, 4$, the truncation errors and aliasing errors almost coincide with each other in the Figures 5.3j to 5.3l. The truncation errors for the symmetric hyperbolic cross

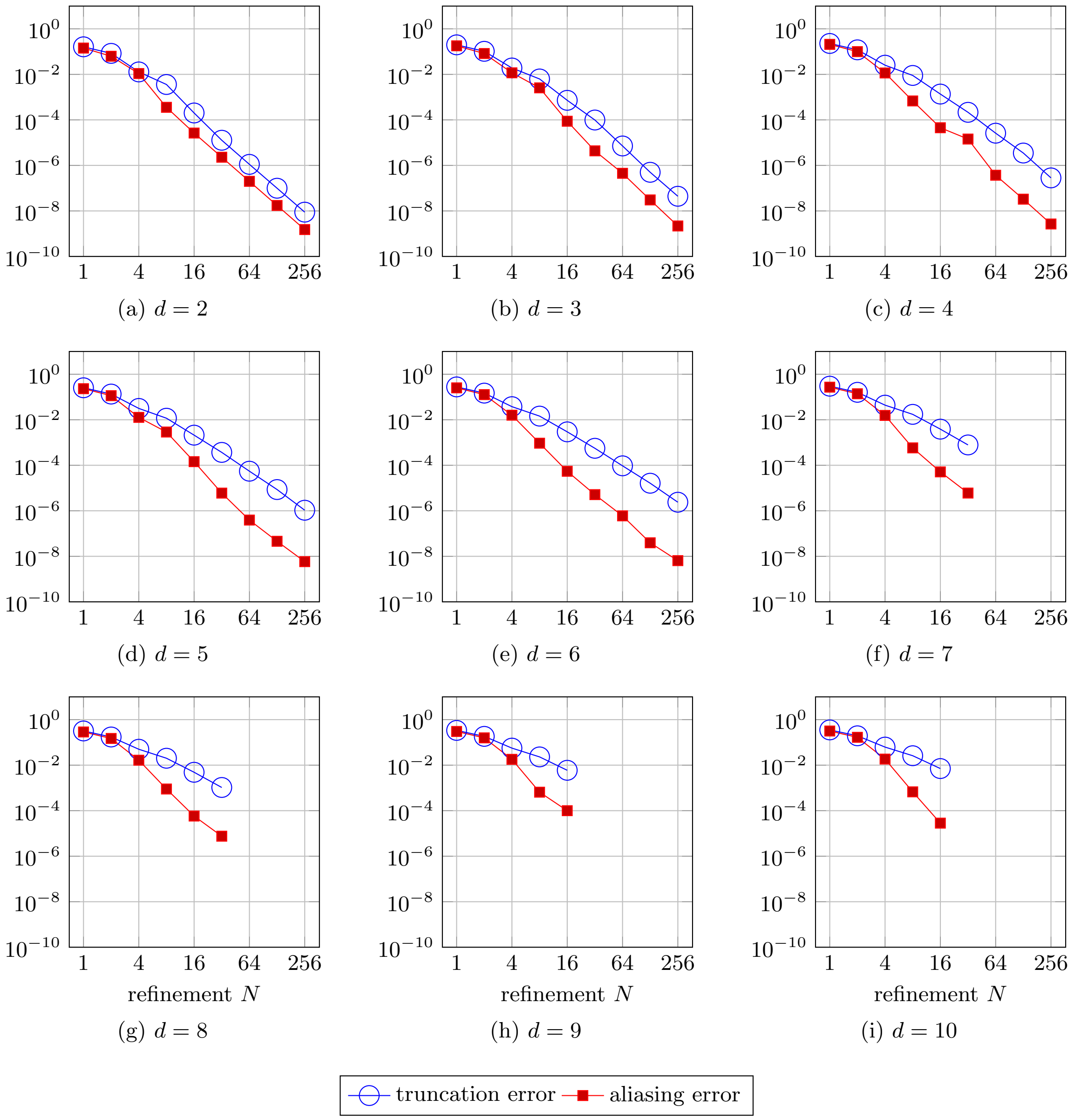
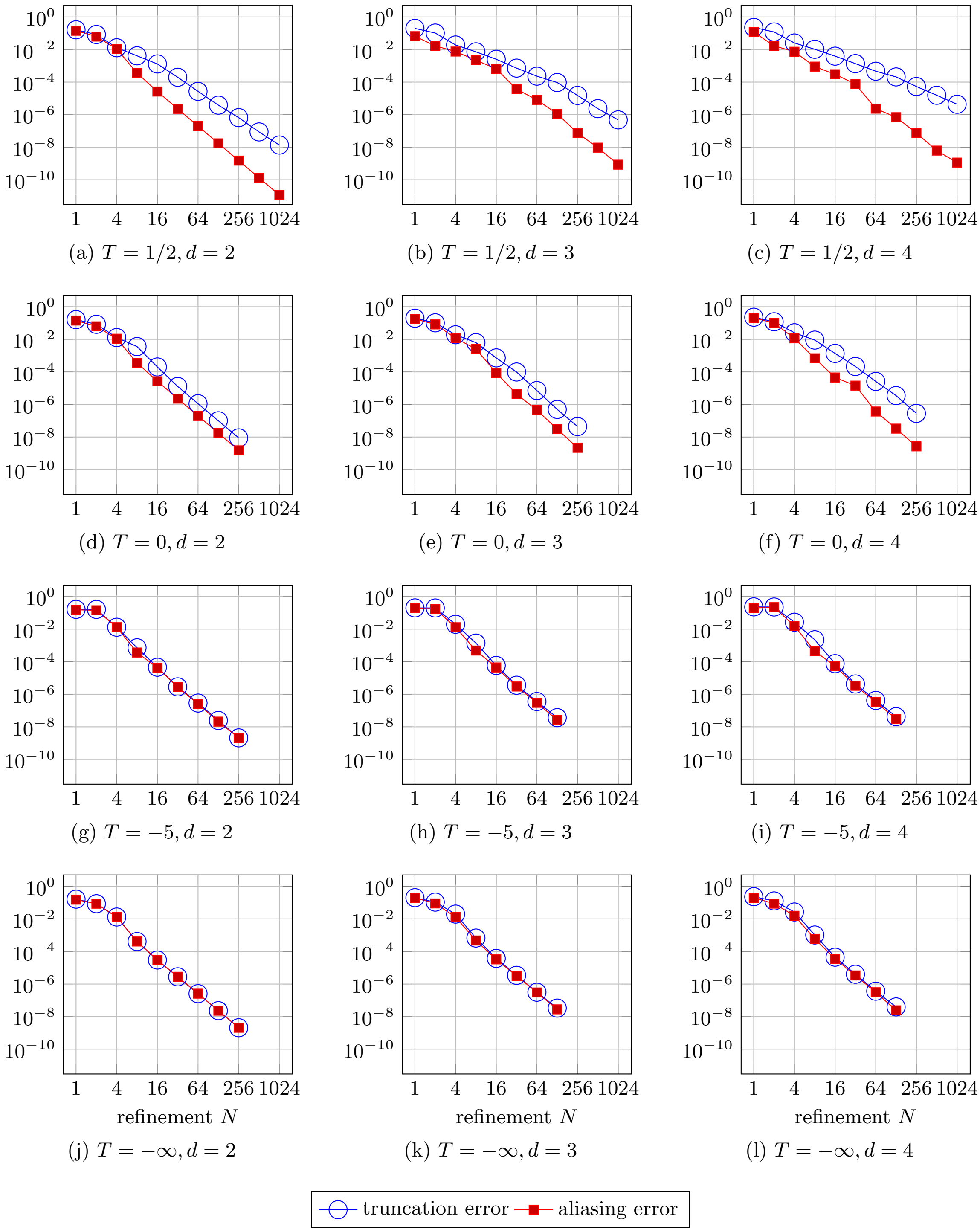


Figure 5.2: Truncation errors $\|f - S_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ and aliasing errors $\|S_{I_N^{d,0}} f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ of the function f from (5.1) as a function of the refinement N.

case $T = 0$ in the Figures 5.3d to 5.3f decrease similarly like the aliasing errors but are higher. Moreover, the aliasing errors for $T = 0$ are similar to ones of the cases $T = -5, -\infty$. In Figure 5.4, we present the truncation errors multiplied by $N^{-3.45}$ and the aliasing errors multiplied by $N^{-3.45}$ for the cases $T = 0, -5, -\infty$ and dimensions $d = 2, 3, 4$. In most cases, the shown error plots behave approximately like horizontal lines for refinements $N \geq 16$. This means that the observed errors decrease approximately like $N^{-3.45}$.

In the following, we construct an example also for the case $T \geq 0$ where the aliasing error is identical to the truncation error and both errors are in the order of the upper error bounds



(a) $T = 1/2, d = 2$

(b) $T = 1/2, d = 3$

(c) $T = 1/2, d = 4$

(d) $T = 0, d = 2$

(e) $T = 0, d = 3$

(f) $T = 0, d = 4$

(g) $T = -5, d = 2$

(h) $T = -5, d = 3$

(i) $T = -5, d = 4$

(j) $T = -\infty, d = 2$

(k) $T = -\infty, d = 3$

(l) $T = -\infty, d = 4$

Figure 5.3: Truncation errors $\|f - S_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\|$ and aliasing errors $\|S_{I_N^{d,T}} f - \tilde{S}_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\|$ of the function f from (5.1) as a function of the refinement N for $T \in \{1/2, 0, -5, -\infty\}$.

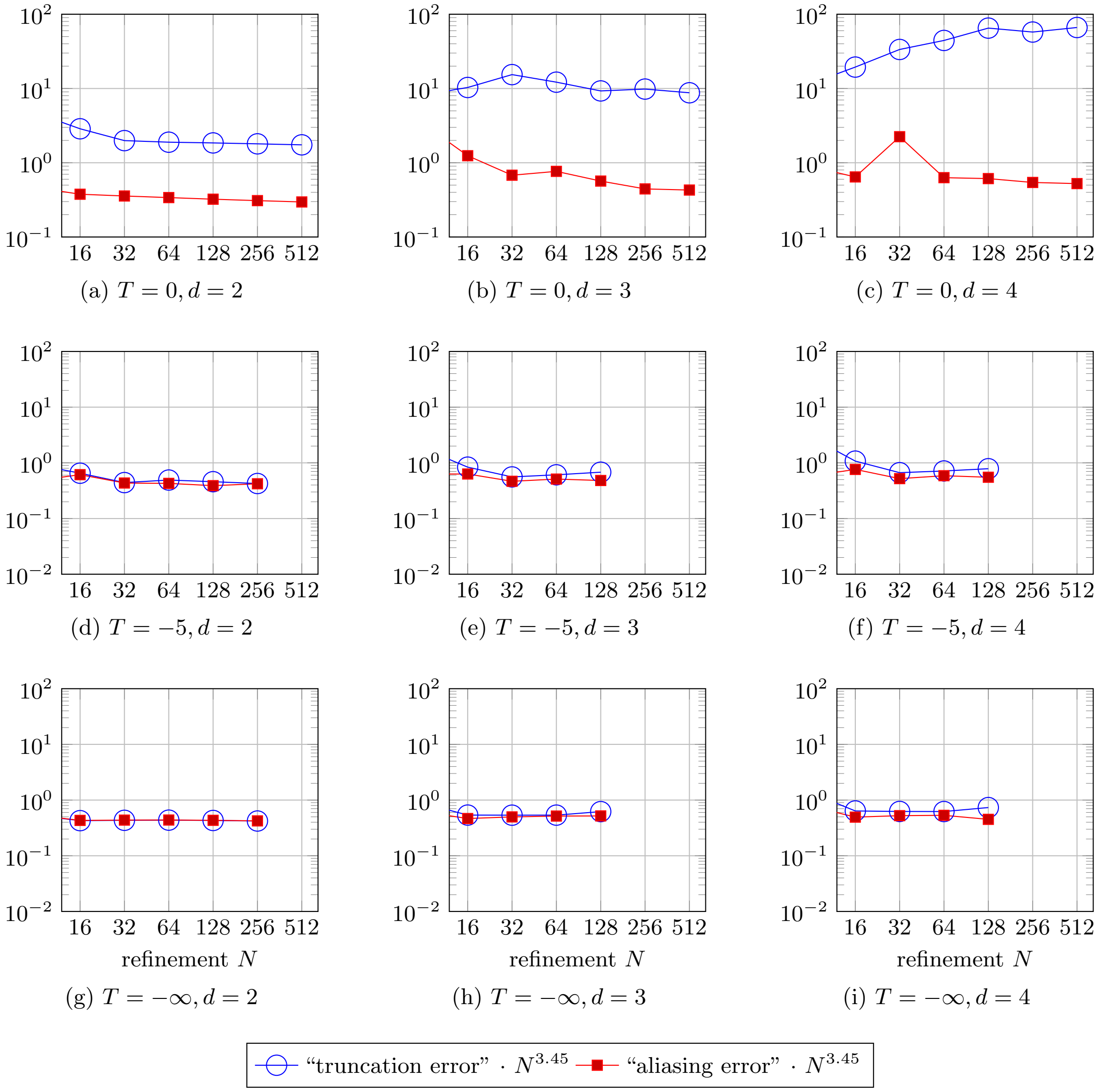
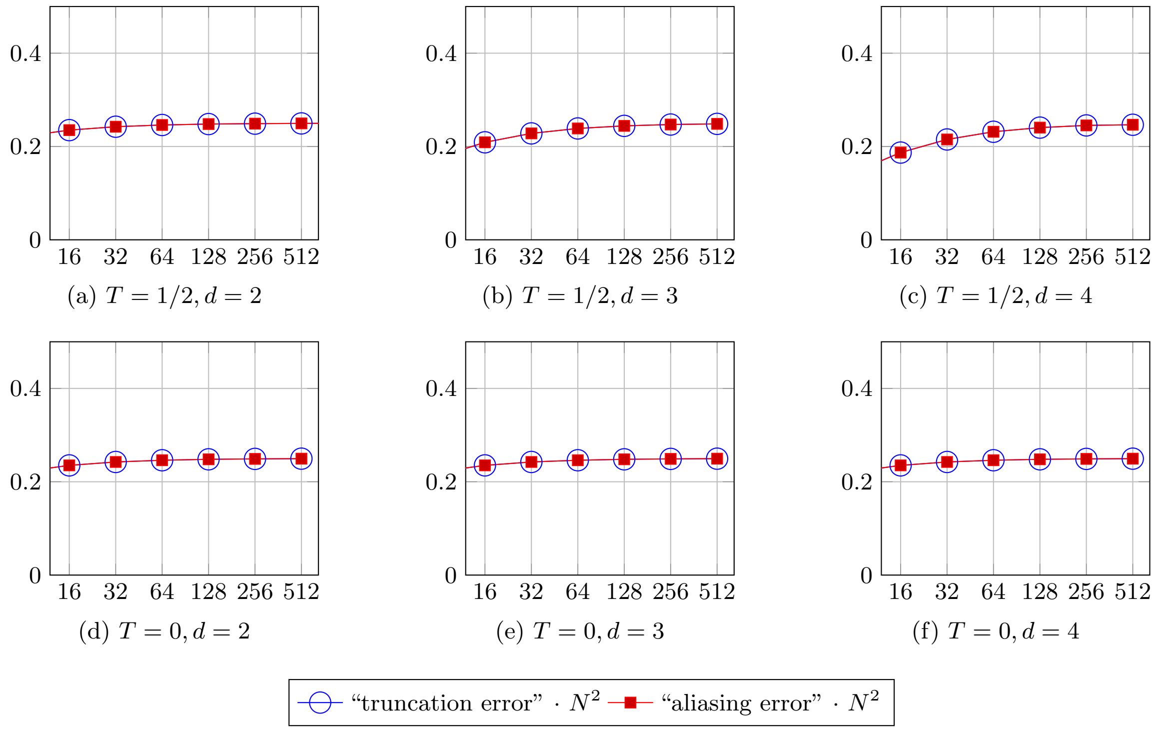


Figure 5.4: Truncation errors $\|f - S_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\|$ and aliasing errors $\|S_{I_N^{d,T}} f - \tilde{S}_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\|$ of the function f from (5.1) multiplied by $N^{3.45}$ as a function of the refinement N for $T \in \{0, -5, -\infty\}$.

in N, cf. Lemma 4.4, Theorem 4.7 and Theorem 4.10. We illustrate the construction of such test functions for $T = 1/2$ and $T = 0$. For each reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N^{d,T})$ for the frequency index set $I_N^{d,T}$, $T := -\alpha/\beta$, we determine one frequency

$$\boldsymbol{k}' = \mathop{\arg\min}_{\substack{\boldsymbol{k} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\} \\ \boldsymbol{k}\boldsymbol{z} \equiv 0 \pmod{M}}} \omega^{\alpha,\beta}(\boldsymbol{k}),$$

which aliases to the origin $\boldsymbol{0}$ and has smallest weight. Due to the reconstruction property (2.2) of each reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N^{d,T})$, we have $\boldsymbol{k}' \in \mathbb{Z}^d \setminus I_N^{d,T}$. Then, we



(a) $T = 1/2, d = 2$ (b) $T = 1/2, d = 3$ (c) $T = 1/2, d = 4$

(d) $T = 0, d = 2$ (e) $T = 0, d = 3$ (f) $T = 0, d = 4$

Figure 5.5: Truncation errors $\|f - S_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\|$ and aliasing errors $\|S_{I_N^{d,T}} f - \tilde{S}_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\|$ of the sequence of trigonometric polynomials $p_N^{d,T}$ multiplied by N^2 as a function of the refinement N for $T \in \{1/2, 0\}$.

define the sequence of test functions $p_N^{d,T} := 1/\omega^{\alpha,\beta}(\boldsymbol{k}')\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}'\circ}$ for $N \in \mathbb{N}$, $T := -\alpha/\beta$, which are (scaled) trigonometric monomials such that $\|p_N^{d,T} | \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\| = 1$. The truncation errors $\|p_N^{d,T} - S_{I_N^{d,T}} p_N^{d,T} | L^2(\mathbb{T}^d)\|$ and aliasing errors $\|S_{I_N^{d,T}} p_N^{d,T} - \tilde{S}_{I_N^{d,T}} p_N^{d,T} | L^2(\mathbb{T}^d)\|$ coincide and are equal to $1/\omega^{\alpha,\beta}(\boldsymbol{k}')$. Moreover, both errors should approximately decrease like $N^{-(\alpha+\beta)}$. The actual decrease rate depends only on $\omega^{\alpha,\beta}(\boldsymbol{k}')$, where the frequency $\boldsymbol{k}'$ depends only on the reconstructing rank-1 lattices $\Lambda(\boldsymbol{z}, M, I_N^{d,T})$. In Figure 5.5, we fixed $\alpha + \beta = 2$ and considered the cases $T = 1/2$ and $T = 0$ for dimensions $d = 2, 3, 4$. We observe that the truncation and aliasing errors coincide as expected as well as that both errors decrease nearly like N^{-2}.

6 Conclusion

In this paper, we generalized the ideas from [25] in order to improve the estimates for the aliasing error $\|S_{I_N^{d,T}} f - \tilde{S}_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\|$ from [16] for functions f from the Hilbert spaces $\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$ of isotropic and dominating mixed smoothness when using the lattice rule (1.1). We proved the existence of special reconstructing rank-1 lattices $\Lambda(\boldsymbol{z}, M, I_N^{d,T})$ with generating vectors $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top \in \mathbb{Z}^d$ of Korobov form which yield that the order of the aliasing error $\|S_{I_N^{d,T}} f - \tilde{S}_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\|$ is bounded by the order of the truncation error $\|f - S_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\|$.

The central statement of this paper is Theorem 3.4, which is a generalization of the ideas of V. N. Temlyakov, see [25]. We stress the fact that our theorem is quite general and applicable to a wide range of frequency index sets I_N. In order to apply Theorem 3.4 to a given sequence of frequency index sets I_N, $N \in \mathbb{R}$, $N \geq 1$, we need to choose a nested sequence of index sets $\mathcal{I}_N$, see (3.2), such that the inclusion $\mathcal{I}_N \supset \mathcal{D}(I_N)$ is valid, where $\mathcal{D}(I_N)$ is the difference set of I_N, cf. Section 2.1. Thereby, $\mathcal{I}_N$ has to fulfill the following properties:

- The cardinalities $|\mathcal{I}_N|$ should be close to the cardinalities $|\mathcal{D}(I_N)|$. This is crucial for a small size M of the reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N)$ used as sampling set, see (3.3).
- The upper and lower bound of the cardinalities $|\mathcal{I}_N|$ need to be known and should be almost of the same order, e.g., gaps of logarithmic order between the upper and lower bound are manageable as demonstrated in Section 4.2.2.

Then, the strategy to bound the aliasing error is analog to the approach in Section 4.2. We remark that we dealt with the difference sets themselves in Section 4.2.2 and set $\mathcal{I}_N := \mathcal{D}(I_N^{d,T})$, whereas we covered the difference sets $\mathcal{D}(I_N^{d,T})$ with larger index sets $\mathcal{I}_N := I^{d,T}_{2^{\frac{d-T}{1-T}} N^{1+\frac{d}{d-T}}}$ in Section 4.2.1.

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References

[1] G. Baszenski and F.-J. Delvos. A discrete Fourier transform scheme for Boolean sums of trigonometric operators. In C. K. Chui, W. Schempp, and K. Zeller, editors, *Multivariate Approximation Theory IV*, ISNM 90, pages 15 – 24. Birkhäuser, Basel, 1989.

[2] H.-J. Bungartz and M. Griebel. A note on the complexity of solving Poisson's equation for spaces of bounded mixed derivatives. *J. Complexity*, 15:167 – 199, 1999.

[3] H.-J. Bungartz and M. Griebel. Sparse grids. *Acta Numer.*, 13:147 – 269, 2004.

[4] R. Cools, F. Y. Kuo, and D. Nuyens. Constructing lattice rules based on weighted degree of exactness and worst case error. *Computing*, 87:63 – 89, 2010.

[5] J. Dick, F. Y. Kuo, and I. H. Sloan. High-dimensional integration: The quasi-Monte Carlo way. *Acta Numer.*, 22:133 – 288, 2013.

[6] V. Gradinaru. Fourier transform on sparse grids: Code design and the time dependent Schrödinger equation. *Computing*, 80:1 – 22, 2007.

[7] M. Griebel and J. Hamaekers. Fast discrete Fourier transform on generalized sparse grids. In J. Garcke and D. Pflüger, editors, *Sparse Grids and Applications - Munich 2012*, volume 97 of *Lect. Notes Comput. Sci. Eng.*, pages 75 – 107. Springer International Publishing, 2014.

[8] M. Griebel and S. Knapek. Optimized Tensor-Product Approximation Spaces. *Constructive Approximation*, 16:525–540, 2000.

[9] M. Griebel and S. Knapek. Optimized general sparse grid approximation spaces for operator equations. *Mathematics of Computation*, 78:2223–2257, 2009.

[10] K. Hallatschek. Fouriertransformation auf dünnen Gittern mit hierarchischen Basen. *Numer. Math.*, 63:83 – 97, 1992.

[11] Y. Jiang and Y. Xu. Fast discrete algorithms for sparse Fourier expansions of high dimensional functions. *J. Complexity*, 26:51 – 81, 2010.

[12] L. Kämmerer. Reconstructing hyperbolic cross trigonometric polynomials by sampling along rank-1 lattices. *SIAM J. Numer. Anal.*, 51:2773 – 2796, 2013.

[13] L. Kämmerer. Reconstructing multivariate trigonometric polynomials from samples along rank-1 lattices. In G. E. Fasshauer and L. L. Schumaker, editors, *Approximation Theory XIV: San Antonio 2013*, pages 255 – 271. Springer International Publishing, 2014.

[14] L. Kämmerer and S. Kunis. On the stability of the hyperbolic cross discrete Fourier transform. *Numer. Math.*, 117:581 – 600, 2011.

[15] L. Kämmerer, S. Kunis, and D. Potts. Interpolation lattices for hyperbolic cross trigonometric polynomials. *J. Complexity*, 28:76 – 92, 2012.

[16] L. Kämmerer, D. Potts, and T. Volkmer. Approximation of multivariate functions by trigonometric polynomials based on rank-1 lattice sampling. *Preprint 145, DFG Priority Program 1324*, 2013.

[17] T. Kühn, W. Sickel, and T. Ullrich. Approximation numbers of Sobolev embeddings—sharp constants and tractability. *J. Complexity*, 30:95 – 116, 2014.

[18] F. Y. Kuo, I. H. Sloan, and H. Woźniakowski. Lattice rules for multivariate approximation in the worst case setting. In H. Niederreiter and D. Talay, editors, *Monte Carlo and Quasi-Monte Carlo Methods 2004*, pages 289 – 330. Springer Berlin Heidelberg, Berlin, 2006.

[19] D. Li and F. J. Hickernell. Trigonometric spectral collocation methods on lattices. In S. Y. Cheng, C.-W. Shu, and T. Tang, editors, *Recent Advances in Scientific Computing and Partial Differential Equations*, volume 330 of *Contemp. Math.*, pages 121 – 132. AMS, 2003.

[20] H. Munthe-Kaas and T. Sørevik. Multidimensional pseudo-spectral methods on lattice grids. *Appl. Numer. Math.*, 62:155 – 165, 2012.

[21] E. Novak and H. Woźniakowski. *Tractability of Multivariate Problems Volume II: Standard Information for Functionals.* Eur. Math. Society, EMS Tracts in Mathematics Vol 12, 2010.

[22] W. Sickel and T. Ullrich. The Smolyak algorithm, sampling on sparse grids and function spaces of dominating mixed smoothness. *East J. Approx.*, 13:387 – 425, 2007.

[23] I. H. Sloan and S. Joe. *Lattice methods for multiple integration.* Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1994.

[24] F. Sprengel. A class of function spaces and interpolation on sparse grids. *Numer. Funct. Anal. Optim.*, 21:273 – 293, 2000.

[25] V. N. Temlyakov. Reconstruction of periodic functions of several variables from the values at the nodes of number-theoretic nets. *Anal. Math.*, 12:287 – 305, 1986. In Russian.

[26] V. N. Temlyakov. Approximation of functions with bounded mixed derivative. *Proc. Steklov Inst. Math.*, 1989. A translation of Trudy Mat. Inst. Steklov 178 (1986).

[27] V. N. Temlyakov. *Approximation of periodic functions.* Computational Mathematics and Analysis Series. Nova Science Publishers Inc., Commack, NY, 1993.

[28] V. N. Temlyakov. Cubature formulas, discrepancy, and nonlinear approximation. *J. Complexity*, 19(3):352 – 391, 2003.

[29] T. Ullrich. Smolyak's algorithm, sampling on sparse grids and Sobolev spaces of dominating mixed smoothness. *East J. Approx.*, 14:1 – 38, 2008.

[30] C. Zenger. Sparse grids. In *Parallel algorithms for partial differential equations (Kiel, 1990)*, volume 31 of *Notes Numer. Fluid Mech.*, pages 241 – 251. Vieweg, Braunschweig, Germany, 1991.