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Closed Ideals in Dirichlet Spaces

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Abstract. It is shown that the closed lattice ideals of Dirichlet spaces and of the Sobolev spaces $W^{1,p}$ are those subspaces which consist of all functions which vanish on a prescribed set.

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1. Introduction

In function spaces it is sometimes possible to describe explicitly all closed ideals, by which we mean subspaces I with the property that $f \in I$, $|g| \leqslant |f|$ implies $g \in I$. E.g., every closed ideal in $C(K)$ can be written as

$$C_0(U) = \{f \in C(K); f = 0 \text{ on } U^c\},$$

with an open set U, and every closed ideal in $L_p(X)$, $1 \leqslant p < \infty$, is of the form

$$L_p(Y) = \{f \in L_p(X); f = 0 \text{ on } Y^c\}$$

for a suitable $Y \subset X$ (see [11], Examples, p. 157f).

In this note we present an analogous result for spaces $(D, \|\cdot\|_{\mathfrak{h}})$, where $(D, \mathfrak{h})$ is a regular Dirichlet form and for Sobolev spaces $W^{1,p}$. To be precise, we prove that the closed ideals of D are exactly the subspaces

$$D_o(M) = \{f \in D; f \text{ vanishes outside } M\}.$$

Despite the fact that Dirichlet forms are defined by their order properties, it seems that the structure of its closed ideals has not been studied so far. We close this gap in the following section by proving the above mentioned characterization.

In the rest of this section we shall describe the general framework of the present article. At the same time we recall some notions and fix the notation. Our standard references for Dirichlet forms and Banach lattices are [6] and [11], respectively. The necessary background concerning Sobolev spaces can be found in [1] as well as in the survey article [8].

In what follows, X will denote a locally compact, second countable Hausdorff space, $\mathfrak{B}$ the σ-field generated by the open subsets of X, and m a Radon measure on X with $\mathrm{supp}(m) = X$. We use $\|\cdot\|$ for the norm on $L_2(X, m)$. We denote the positive cone of a subspace L by $L_+ = \{f \in L; f \geqslant 0\}$. For $f, g \in L_2(X, m)$, $f \wedge g(x) := \min\{f(x), g(x)\}$, $f \vee g(x) := \max\{f(x), g(x)\}$, and $f^+ := f \vee 0$ denotes the positive part of f.

Recall that a *closed, regular Dirichlet form* in $L_2(X, m)$ is a pair $(D, \mathfrak{h})$ consisting of a dense subspace $D \subset L_2(X, m)$ and a symmetric, sesquilinear mapping $\mathfrak{h}: D \times D \to \mathbb{K}$ which satisfies the conditions given in [6] (see [10], Appendix 1 to section XIII.12 for the complex case). In particular, $\|f\|_{\mathfrak{h}} := (\mathfrak{h}[f, f] + \|f\|^2)^{1/2}$ defines a norm on D under which it is complete. The closed, regular Dirichlet form $\mathfrak{h}$ induces a monotone, countably sub-additive set-function, the *capacity*, by

$$\mathrm{cap}(U) := \inf\{\|f\|_{\mathfrak{h}}^2; f \in D, f \geqslant \chi_U\}$$

for $U \subset X$, U open,

$$\mathrm{cap}(A) := \inf\{\mathrm{cap}\,(U);\ U \text{ open},\ U \supset A\}$$

for arbitrary $A \subset X$ (see [6], p. 61f).

Sets of capacity zero are called *polar* and a property is said to hold q.e. (*quasi-everywhere*) if it holds outside some polar set.

It is a key fact that every element $f \in D$ admits a *quasi-continuous* version, i.e. there exists a function f in the equivalence class f with the following property: $\forall \epsilon > 0\, \exists U$ open: $\mathrm{cap}(U) < \epsilon$, $\tilde{f} \in C(X \setminus U)$ (cf [6], p. 64f) which is unique q.e.

2. Ideals in Dirichlet Spaces

To motivate the results of this section, let us start with the standard example, the classical Dirichlet form on $\mathbb{R}^n$: It is defined by

$$D = W^{1,2}(\mathbb{R}^n),\ \mathfrak{h}[f, g] = \sum_{i=1}^{n} \int \partial_i f(x) \partial_i \bar{g}(x)\, \mathrm{d}x.$$

Typical ideals are the subspaces $W_o^{1,2}(U)$, where U is an open subset of $\mathbb{R}^n$:

$$W_o^{1,2}(U) := \overline{C_c^\infty(U)}^{W^{1,2}} = \{f \in W^{1,2}; \tilde{f} = 0 \text{ q.e. on } U^c\};$$

the equality sign is proved in [8]. We shall see below that using the second description we already obtain all closed ideals in $W^{1,2}$, provided we drop the assumption that U is open. The corresponding result will hold true for general regular Dirichlet forms and for the spaces $W^{1,p}$, as we will show in Theorem 1 and Theorem 3. Speaking of ideals, we refer to the natural order on D which is induced by L_2; from [6], Lemma 3.1.4 we infer $f \leqslant g \Leftrightarrow \tilde{f} \leqslant \tilde{g}$ q.e. The main result of this note is the surjectivity of the mapping

$$\mathfrak{B} \to \{\text{closed ideals of } D\},\ M \mapsto D_o(M) := \{f \in D; \tilde{f} = 0 \text{ q.e. on } M^c\}.$$

(See [5], where the corresponding spaces $H_0^1(M)$ are introduced and studied.) This is the content of the following

THEOREM 1. *Let I be a closed ideal in D. Then there exists an $M \in \mathfrak{B}$ such that $I = D_o(M)$.*

It will be clear from the proof that the set M in the above theorem can be chosen *quasi-open* which means, that it is of the form $\{F > 0\}$ for some quasi-continuous function F. As a preparation we now show that every closed ideal is generated by a single element (see [4], especially Section IX, for related information).

LEMMA 2. *Let I be a closed ideal. Then there exists $F \in I$ such that $I = \langle F \rangle$, where $\langle F \rangle$ denotes the closed ideal generated by F.*

Proof. Fix a sequence (f_n) in I_+, such that the closure of its linear hull is dense; without restriction, we may assume f_n to be quasi-continuous. Define

$$F := \sum_{n \in \mathbb{N}} (2^n \|f_n\|)^{-1} \cdot f_n.$$

Then $F \in I$, which implies $\langle F \rangle \subset I$.

On the other hand, $I = \langle \{f_n; n \in \mathbb{N}\} \rangle \subset \langle F \rangle$. □

Proof of Theorem 1. We already have a candidate for M, namely

$$M := \{F > 0\},$$

where F is as in Lemma 2. From there we also deduce that $I \subset D_o(M)$. To prove the converse, let $f \in D_o(M)$. Without restriction f is quasi-continuous, $0 \leqslant f \leqslant 1$, and $f = 0$ q.e. outside some fixed compact set $K \subset X$.

Changing f on a polar set, if necessary, we may assume

$$\{f > 0\} \subset \{F > 0\}.$$

It remains to prove that this inclusion implies that f is in the closed ideal generated by F.

Since $(f - 1/n)^+ \to f$ in D, for $n \to \infty$, it suffices to show $g := (f - 1/n)^+ \in I$, for fixed $n \in \mathbb{N}$. This will be accomplished, if we verify the

CLAIM. $\forall \varepsilon > 0 \, \exists U_\varepsilon$ open, $\operatorname{cap}(U_\varepsilon) < \varepsilon \, \exists \alpha_\varepsilon > 0$:

$$\operatorname{supp} g = \left\{ f \geqslant \frac{1}{n} \right\} \subset \{F > \alpha_\varepsilon\} \cup U_\varepsilon.$$

In fact, for quasi-continuous $\phi_\varepsilon \in D$, $\phi_\varepsilon \geqslant \chi_{U_\varepsilon}$, $\|\phi_\varepsilon\|_{\mathfrak{h}} < 2\varepsilon$ it follows that $g(1 - \phi_\varepsilon)^+ \leqslant (1/\alpha_\varepsilon)F$, which implies $g_\varepsilon := g(1 - \phi_\varepsilon)^+ \in I$.

Note that

$$\sup_{\varepsilon > 0} \|g_\varepsilon\|_{\mathfrak{h}} < \infty, \; g_\varepsilon \to g \text{ in } L_2 \quad \text{for } \varepsilon \to 0.$$

(The boundedness follows since $g_\varepsilon = g(1 - 1 \wedge \phi_\varepsilon)$, so that $\|g_\varepsilon\|_{\mathfrak{h}} \leqslant \|g\|_{\mathfrak{h}} + \|g(1 \wedge \phi_\varepsilon)\|_{\mathfrak{h}} \leqslant \|g\|_{\mathfrak{h}} + \|g\|_{\mathfrak{h}} \|\phi_\varepsilon\|_\infty + \|g\|_\infty \|\phi_\varepsilon\|_{\mathfrak{h}}$, where we have used [6], Theorem 1.4.2(ii) for the second inequality sign.)

It now follows by standard arguments that $g_\varepsilon \to g$ weakly in D so that g is in the weak closure of I, which equals I.

To prove the claim, we use the existence of quasi-continuous representatives: There exist open sets V, W with capacity less than ε such that

$$f \in C(V^c), \quad F \in C(W^c).$$

Hence $\{f \geqslant 1/n\} \setminus V$ is a compact subset of $\{F > 0\}$.

Moreover,

$$\{F > 0\} \subset \bigcup_{a > 0} \{F > a\} \cup W,$$

and $\{F > a\} \cup W$ is open for all $a > 0$. Therefore we find an $a > 0$ such that

$$\left\{ f \geqslant \frac{1}{n} \right\} \setminus V \subset \{F > a\} \cup W,$$

which implies that

$$\left\{ f \geqslant \frac{1}{n} \right\} \subset \{F > a\} \cup V \cup W;$$

since $\operatorname{cap}(V \cup W) < 2\varepsilon$, the proof of the claim and hence the proof of the theorem is complete. □

As a first application, let us give a partial answer to a question of W. Arendt which was communicated to us by J. Voigt:

What do the closed ideals between $W_o^{1,2}(U)$ and $W^{1,2}(U)$ look like, where $U \subset \mathbb{R}^n$ is an open subset?

We are going to consider relatively compact U only. If the boundary of U is sufficiently smooth (cf. [1], Theorem 3.18), then $W^{1,2}(U)$ is the domain of a regular Dirichlet form on $\bar{U}$. Hence, the ideals $D_o(M) = \{f \in W^{1,2}(U); \tilde{f} = 0 \text{ q.e. on } M^c\}$, $U \subset M \subset \bar{U}$ are those we are looking for.

In case of a non-smooth boundary, one can use certain compactification U^*, see [7] for details, and obtain an analogous result.

We now turn to the spaces $W^{1,p}$, where $1 \leqslant p < \infty$. To prove the analog of Lemma 2 and Theorem 1 for these spaces, it suffices to check two basic facts:

– $W^{1,p}$ is closed under truncation, i.e.

$$f \in W^{1,p} \Rightarrow f^+, f \wedge 1 \in W^{1,p}.$$

– $f, g \in W^{1,p} \cap L_\infty \Rightarrow \|fg\|_{1,p} \leqslant \|f\|_{1,p}\|g\|_\infty + \|f\|_\infty\|g\|_{1,p}$.

Hence the same proof as for Dirichlet spaces yields:

THEOREM 3. *Let I be a closed ideal in $W^{1,p}$, $1 \leqslant p < \infty$. Then there exists an $M \in \mathfrak{B}$ such that $I = W_o^{1,p}(M)$.*

REMARK 4. (1) We have treated above regular Dirichlet forms within the framework of Fukushimas monograph [6]. With the notion of quasi-regular forms (see the forthcoming book [9]) there is now a well-established potential theory for certain forms, for which $D \cap C_c(X)$ does not need to be dense. Using the results of [2] it is not hard to obtain the analog of Theorem 1 for quasi-regular forms, as was noted by M. Röckner.

(2) In a previous version of this note we used Theorem 1 to obtain the following generalization of results of Baxter, DalMaso and Mosco [3] and Sturm [12]:

If $\mathfrak{h}$ is a regular Dirichlet form, and $\mu: \mathfrak{B} \to [0, \infty]$ a measure which is absolutely continuous with respect to cap, then there exists a measure $\nu \in D^*$ and $q: X \to [0, \infty]$ such that $\langle \mu, f \rangle = \langle q\nu, f \rangle$ for every quasi-continuous f.

A referee kindly communicated a nice direct proof (due to Ancona, unpublished) for this result.

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