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Stability of the essential spectrum of second-order complex elliptic operators

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Abstract. We prove compactness of the resolvent difference for second order divergence form operators whose matrix functions are close enough at infinity. Our theorems include certain subelliptic cases as well as cases with unbounded coefficients.

Introduction

In this paper we consider the stability of the essential spectrum of second order divergence operators with complex coefficients.

More precisely, we give L^p-conditions on the difference of b and a under which the operators formally given by $B = D^*bD$ and $A = D^*aD$ have the same essential spectrum by proving the compactness of the resolvent difference. Here $b, a \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^{d+1,d+1})$ are matrix-valued functions and D^*aD is shorthand for

$$- \sum_{k,j=1}^{d} D_j(a_{jk} D_k + a_{j,d+1}) + a_{d+1,k} D_k + a_{d+1,d+1}$$

which formally reduces to D^*aD by setting $D = (D_1, \ldots, D_d, I)^t$.

We think of A as being a comparison operator, whose spectrum is understood and of B as a perturbation which has less restrictive properties.

Questions of this type have already been considered in [6], and more recently in [11] and [15]. However, in these references, the assumptions on A are quite restrictive (in [11] only the case $A = -\Delta$ is treated).

The present paper reports on considerable progress which is stated in our two main results, Theorems 2.1 and 3.1. A common feature of these theorems is that we allow for complex measurable coefficients. Divergence operators with complex coefficients have attracted considerable attention recently, as it turned out that they exhibit properties quite different from those of their real-valued relatives. This is mainly due to the fact that the latter are associated with Dirichlet forms, which implies that their semigroups are bounded operators on the whole scale of L^p-spaces, while the former will not share this property in general. This fact is illustrated by an example given in [1]. So part of the proofs used both in [11] and [15] don't carry over to the present context, for instance the use of ultracontractivity.

In Theorem 2.1 we treat the case that both a, b have bounded coefficients. Then if A is elliptic and B satisfies a certain subellipticity, the difference of their resolvents is compact if $(a-b) \in c_0(L^1)$. Note that we do not assume smoothness of either a or b. In Theorem 3.1 we consider unbounded b. Here $(a-b) \in c_0(L^p)$ suffices for the compactness of the resolvent difference, if p is bigger than some $p(A)$. Roughly speaking, $p(A)$ decreases with increasing smoothness of a, the limiting case being $p(A) = 2$, which is achieved, e.g. for Hölder-continuous a.

Summarizing, we give conditions on $(a-b)$ which imply that this difference vanishes near infinity in a certain weak sense, and under which B has the same essential spectrum as A. Related work by Barbatis [3], [4] can be viewed as complementary. He considers the case of symmetric operators on bounded domains and compact manifolds and proves Schatten p-norm estimates in terms of p-norms of the coefficients.

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1. Preliminaries

As explained in the introduction, we are going to consider two operators A, B in divergence form with complex coefficients. While we assume A to be a known comparison operator with some regularity, the assumptions on B are rather weak. More precisely, consider

(ACC) Let $b = (b_{jk})_{1 \leqq j,k \leqq d+1}$ be a matrix, whose entries $b_{jk} \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C})$; we assume that b is *accretive*, i.e., $\operatorname{Re} \sum\limits_{j,k=1}^{d+1} b_{jk} \xi_j \overline{\xi_k} \geqq 0$ and such that the accretive form given by

$$\mathfrak{b}(u,v) = \int \left(b(x) \begin{pmatrix} \nabla u \\ u \end{pmatrix} \middle| \begin{pmatrix} \nabla v \\ v \end{pmatrix} \right)_{\mathbb{C}^{d+1}} dx \tag{1.1}$$

with domain C_c^∞ is *closable* and *continuous*, where the latter means that there exists a constant c such that

$$|\mathfrak{b}(u,v)| \leqq c\left(\operatorname{Re}\mathfrak{b}(u,u) + \|u\|_2^2\right)^{\frac{1}{2}} \left(\operatorname{Re}\mathfrak{b}(v,v) + \|v\|_2^2\right)^{\frac{1}{2}}. \quad \square$$

Denote also by $\mathfrak{b}$ the closure of the above form $\mathfrak{b}$ and let B be the operator associated with $\mathfrak{b}$.

Note that the accretivity of the form follows from the accretivity of the matrix. As a standard reference for accretive forms we refer to Kato's book [8] (note however, that the term accretive form doesn't appear in that book). For symmetric forms with real coefficients, a lot of information can be found in Davies's monograph [5]. The state of the art concerning closability can be found in [10]. Now to see what B looks like, let us write down the right hand side of (1.1), using the summation convention and $D_j = \dfrac{\partial}{\partial x_j}$

$$\mathfrak{b}(u,v) = \int [b_{jk}(x)\, D_k u \overline{D_j v} + b_{j,d+1}(x)\, u \overline{D_j v} + b_{d+1,k}(x)\, D_k u \bar{v} + b_{d+1,d+1}(x)\, u\bar{v}]\, dx\,.$$

Hence, after formal integration by parts,

$$B = -D_j(b_{jk} D_k + b_{j,d+1}) + b_{d+1,k} D_k + b_{d+1,d+1}\,. \tag{1.2}$$

For the comparison operator A, we assume stronger conditions, namely

(Ell) $a = (a_{jk})_{j,k} \in L^\infty(\mathbb{R}^d, \mathbb{C}^{d+1,d+1})$, and a is accretive with *elliptic* principal part, i.e., there exists a constant $\eta > 0$ such that

$$\mathrm{Re} \sum_{j,k=1}^{d} a_{jk}\, \xi_j \overline{\xi_k} \geqq \eta\, |\xi|^2 \quad \forall \xi \in \mathbb{C}^d \text{ a.e.} \tag{1.3}$$

Under these assumptions, closability and continuity of the form are guaranteed and we get a form $\mathfrak{a}$ with domain $D(\mathfrak{a}) = W^{1,2}$ and with an associated operator A on $L^2(\mathbb{R}^d)$. The maximal possible $\eta(a)$ in (1.3) is called the ellipticity constant. □

Note that the accretivity of the matrix a in (Ell) is just assumed for convenience. If the principal part is elliptic and the coefficients are bounded, we could achieve it by adding a suitable constant to $a_{d+1,d+1}$. To abbreviate notation, let $a \in L^\infty$ mean that all the coefficients are bounded and $\|a\|_\infty := \operatorname{esssup}_{x \in \mathbb{R}^d} \|a(x)\|$ (here the matrix norm of $a(x)$ is meant) which is finite in that case.

The next preparatory results are stated for a satisfying (Ell). It is, however, easy to see that the assumptions on the first and zeroth order coefficients could be weakened substantially.

Lemma 1.1. *Let a be as in* (Ell). *Then there exist constants c, γ, depending only upon k, $\eta(a)$, $\|a\|_\infty$ such that for all $\chi, \tilde{\chi} \in L^\infty(\mathbb{R}^d)$ with* $\operatorname{dist}(\operatorname{supp}\chi, \operatorname{supp}\tilde{\chi}) = \delta > 1$ *and $\|\chi\|_\infty, \|\tilde{\chi}\|_\infty \leqq 1$ the following estimates hold*:

$$\|\chi(A+1)^{-k}\tilde{\chi}\| \leqq c.\exp(-\gamma.\delta)\,, \tag{1.4}$$

$$\|\chi D_j(A+1)^{-k}\tilde{\chi}\| \leqq c.\exp(-\gamma.\delta)\,. \tag{1.5}$$

Proof. Let w be a real-valued L^∞_{loc}-function with $\|\nabla w\|_\infty \leqq 1$ and define $\mathfrak{a}_w$, which is formally associated to $e^w A e^{-w}$ by

$$\mathfrak{a}_w(u,v) = \mathfrak{a}(e^{-w}u, e^w v) \quad (u, v \in C_c^\infty)\,.$$

Note that under the assumptions on w, both $e^w u$ and $e^{-w} u$ belong to $W^{1,2} = D(\mathfrak{a})$.

Using the expression

$$\mathfrak{a}_w(u,v) = \int \left(a(x) \begin{pmatrix} \nabla(e^{-w}u) \\ e^{-w}u \end{pmatrix} \middle| \begin{pmatrix} \nabla(e^w v) \\ e^w u \end{pmatrix} \right) dx$$

we find that

$$\mathfrak{a}_w(u,v) = \mathfrak{a}(u,v) + \int \left(a(x) \begin{pmatrix} \nabla u \\ u \end{pmatrix} \middle| \begin{pmatrix} v\nabla w \\ 0 \end{pmatrix} \right) - \left(a(x) \begin{pmatrix} u\nabla w \\ 0 \end{pmatrix} \middle| \begin{pmatrix} \nabla v \\ v \end{pmatrix} \right)$$
$$- \left(a\,(x) \begin{pmatrix} u\nabla w \\ 0 \end{pmatrix} \middle| \begin{pmatrix} v\nabla w \\ 0 \end{pmatrix} \right) dx$$

which implies

$$|(\mathfrak{a}_w - \mathfrak{a})(u,v)| \leqq 3\|a\|_\infty \|\nabla w\|_\infty \|u\|_{1,2} \|v\|_{1,2} \,.$$

Using now the fact that $D(\mathfrak{a}) = W^{1,2}$, which implies that the norm $\|u\|_{1,2}$ of $W^{1,2}$ is equivalent to $\|u\|_{\mathfrak{a}} := \big(\operatorname{Re}\mathfrak{a}(u,u) + \|u\|_2^2\big)^{\frac{1}{2}}$ we obtain that (K is a constant)

$$(1.6) \qquad |(\mathfrak{a}_w - \mathfrak{a})(u,v)| \leqq K\|\nabla w\|_\infty \|u\|_{\mathfrak{a}} \|v\|_{\mathfrak{a}} \,.$$

By standard theory of sectorial forms ([8], chap. VI), $\mathfrak{a}_w$ has a sectorial closure with domain $D(\mathfrak{a}) = W^{1,2}$, whose operator we denote by A_w.

In the next step, we want to prove that $(A_w + 1)^{-1}$ exists, and find a formula for this perturbed resolvent. To this end, let

$$\mathfrak{h}(u,v) := \frac{1}{2}\big(\mathfrak{a}(u,v) + \overline{\mathfrak{a}(v,u)}\big)$$

be the real part of $\mathfrak{a}$ which is non-negative, since $\mathfrak{a}$ is accretive, and let H be the associated self-adjoint operator.

From (1.6) and [8], p. 336, with $\beta = 2K\|\nabla w\|_\infty$, it follows that

$$(\mathfrak{a}_w - \mathfrak{a})(u,v) = \big(C(H+1)^{\frac{1}{2}}u, (H+1)^{\frac{1}{2}}v\big)\,,$$

for a bounded operator $C \in \mathscr{L}(L^2)$ with $\|C\| \leqq \beta$. Moreover

$$(\mathfrak{a}+1)(u,v) = \big((\tilde{A}+1)(H+1)^{\frac{1}{2}}u, (H+1)^{\frac{1}{2}}v\big)$$

with invertible (recall that $\mathfrak{a}$ is accretive) $\tilde{A}+1 \in \mathscr{L}(L^2)$. Hence

$$(\mathfrak{a}_w+1)(u,v) = \big((\tilde{A}+1+C)(H+1)^{\frac{1}{2}}u, (H+1)^{\frac{1}{2}}v\big)$$

and, if β is small enough,

(1.7) $$(A_w+1)^{-1} = (H+1)^{-\frac{1}{2}}(\tilde{A}+1+C)^{-1}(H+1)^{-\frac{1}{2}}$$

by the Neumann sum. Moreover, the norm of $(A_w+1)^{-1}$ only depends on β, since $\|(A+1)^{-1}\| \leqq 1$ and $\|(H+1)^{-\frac{1}{2}}\| \leqq 1$. Let us now proceed to a proof of (1.4), (1.5) for $k=1$.

Suppose that $\operatorname{supp}\tilde{\chi}$ is compact, whithout restriction. Take

$$w(x) = \varepsilon.\operatorname{dist}(x, \operatorname{supp}\tilde{\chi}) .$$

Then $w \in W^{1,2}_{\text{loc}}$ satisfies $\|\nabla w\|_\infty \leqq \varepsilon$ (this is clear if B contains only one point; for the general case, note that w is an infimum of $W^{1,2}_{\text{loc}}$-functions). If ε is so small that $\beta < 1$, we get

$$(A+1)^{-1}f = e^{-w}(A_w+1)^{-1}e^w f \quad \text{for } f \in C_c^\infty$$

by a simple calculation, so that

$$\chi(A+1)^{-1}\tilde{\chi}f = \chi e^{-w}(A_w+1)^{-1}e^w\tilde{\chi}f = \chi e^{-w}(A_w+1)^{-1}\tilde{\chi}f$$

and hence

$$\|\chi(A+1)^{-1}\tilde{\chi}\| \leqq \max_{\{x \in \operatorname{supp}\chi\}} e^{-w}\|(A_w+1)^{-1}\| \leqq e^{-\varepsilon\delta}\|(A_w+1)^{-1}\| .$$

To prove (1.5), we use (1.7) again:

$$\begin{aligned} \chi D_j(A+1)^{-1}\tilde{\chi} &= \chi D_j e^{-w}(A_w+1)^{-1}e^w\tilde{\chi} \\ &= \chi e^{-w}D_j(A_w+1)^{-1}e^w\tilde{\chi} - \chi(D_j w)e^{-w}(A_w+1)^{-1}e^w\tilde{\chi} . \end{aligned}$$

The second term can be estimated by $\varepsilon e^{-\varepsilon.\delta}\|(A_w+1)^{-1}\|$ which can be estimated by $e^{-\gamma.\delta}$ for some $\gamma < \varepsilon$. The first term on the right hand side of (1.8) can be estimated, using (1.7) and the fact that $D(H^{\frac{1}{2}}) = W^{1,2}$.

To prove the results for arbitrary $k \in \mathbb{N}$ we proceed by induction and write that

$$\chi(A+1)^{-k-1}\tilde{\chi} = \chi(A+1)^{-k}\hat{\chi}(A+1)^{-1}\tilde{\chi} + \chi(A+1)^{-k}(1-\hat{\chi})(A+1)^{-1}\tilde{\chi}$$

with $\hat{\chi} = 1_{\{x, \operatorname{dist}(x, \operatorname{supp}\tilde{\chi}) \leqq \frac{\delta}{2}\}}$ (the indicator function of $\{x,\ \operatorname{dist}(x, \operatorname{supp}\tilde{\chi}) \leqq \frac{\delta}{2}\}$) which implies that $\operatorname{dist}(\operatorname{supp}\hat{\chi}, \operatorname{supp}\chi) \geqq \frac{\delta}{2}$ and $\operatorname{dist}\big(\operatorname{supp}(1-\hat{\chi}), \operatorname{supp}\tilde{\chi}\big) \geqq \frac{\delta}{2}$. □

Off-diagonal decay estimates like (1.4), (1.5) have been used quite heavily in the context of Schrödinger operators (see [14] for references), Dirichlet forms [16] and second order divergence operators with real coefficients [15]. The next proposition can be regarded as a corollary to the above lemma and was stated in [15]. We use the notation

$$c_0(L^p) = \{ f \in L^1_{\text{loc}}, \int_{|x-y| \leqq 1} |f(y)|^p dy \to 0 \text{ as } x \to \infty \} ,$$

and moreover write $\|\,.\,\|_{p,q}$ for the norm of an operator from L^p to L^q. We also denote by $\mathscr{K}(E)$ the space of compact operators on a Banach space E.

Proposition 1.2. *Suppose that* $h \in c_0(L^p)$ *and* $D_j(A+1)^{-k}: L^2 \to L^q$ *for some* p, q *such that* $\frac{2}{p} + \frac{2}{q} < 1$, $j = 1, \ldots, d$ *and* $k \in \mathbb{N}$. *Then*

$$hD_j(A+1)^{-k-1} \in \mathscr{K}(L^2)\,.$$

Similarly, if $h \in c_0(L^p)$ *and* $(A+1)^{-k}: L^2 \to L^q$ *such that* $\frac{2}{p} + \frac{2}{q} < 1$ *then*

$$h(A+1)^{-k-1} \in \mathscr{K}(L^2)\,.$$

Proof. Since the arguments for $h(A+1)^{-k-1}$ are even easier, we restrict attention to the operator $hD_j(A+1)^{-k-1}$.

Denote the ball of radius R by $B(0,R)$. Using Rellich's compactness theorem and $(A+1)^{-1}: L^2 \to W^{1,2}$ one can easily deduce that

$$(A+1)^{-1}1_{B(0,R)} \in \mathscr{K}(L^2)\,. \tag{1.9}$$

The idea of the proof is to show that

$$hD_j(A+1)^{-k}(A+1)^{-1}1_{B(0,R)} \to hD_j(A+1)^{-k-1} \tag{1.10}$$

in norm (as $R \to \infty$), which gives the desired result, once we have established that

$$hD_j(A+1)^{-k} \in \mathscr{L}(L^2)\,. \tag{1.11}$$

Both of these assertions will be deduced by applying the so-called Schur-test (cf. [17], Theorem 6.23) to an auxiliary operator on $l^2(\mathbb{Z}^d)$. To this end, denote by 1_n the characteristic function of the unit cube centered at $n \in \mathbb{Z}^d$, and let

$$K_{n,m} := 1_n hD_j(A+1)^{-k}1_m, \quad c_{n,m} := \|K_{n,m}\|_{2,2}\,.$$

To see that this is finite, let s be such that $\frac{2}{p} + \frac{2}{s} = 1$, in particular $s < q$; then

$$\|1_n D_j(A+1)^{-k}1_m\|_{2,s} < \infty\,,$$

and

$$c_{n,m} \leqq \|1_n h\|_p \|1_n D_j(A+1)^{-k}1_m\|_{2,s}\,. \tag{1.12}$$

It is clear that (1.11) follows, once we have established that $(c_{n,m})$ defines a bounded operator on $l^2(\mathbb{Z}^d)$.

According to the Schur-test we have to show that the rows and colums of $(c_{n,m})$ are uniformly bounded in $l^1(\mathbb{Z}^d)$. To this end, let

$$R_{n,m} := 1_n D_j (A+1)^{-k} 1_m .$$

From Lemma 1.1 we know that $\|R_{n,m}\|_{2,2} \leqq c.\exp(-\gamma|n-m|)$ (note that if the cubes are neighbors then $\|R_{n,m}\|_{2,2} \leqq \|D_j(A+1)^{-k}\|_{2,2} \leqq c$ for some constant c). We now want to show a similar estimate for $\|R_{n,m}\|_{2,s}$, $2 \leqq s < q$. This in turn follows easily with the help of the Riesz-Thorin interpolation theorem, since $\|R_{n,m}\|_{2,q} \leqq \|D_j(A+1)^{-k}\|_{2,q}$. Now $\|R_{n,m}\|_{2,s} \leqq c'\exp(-\gamma'|n-m|)$ inserted in (1.12) gives the desired l^1-bound for the rows and colums of $(c_{n,m})$ and thus (1.11).

Consequently, $hD_j(A+1)^{-k-1}1_{B(0,R)} \in \mathscr{K}(L^2)$ for every $R>0$. We now finish the proof by showing (1.10). Write

$$\begin{aligned} (1.13) \qquad & \|hD_j(A+1)^{-k-1}1_{B(0,R)} - hD_j(A+1)^{-k-1}\| \\ &= \Big\|\sum_{n,m} 1_n\big(hD_j(A+1)^{-k-1}1_{B(0,R)} - hD_j(A+1)^{-k-1}\big)1_m\Big\| \\ &\leqq \Big\|\sum_{n,|m|>N(R)} 1_n hD_j(A+1)^{-k-1}1_m\Big\| \end{aligned}$$

for suitable $N(R) \in \mathbb{N}$, satisfying $N(R) \to \infty$ as $R \to \infty$. We estimate the RHS of (1.13) by

$$(1.14) \qquad \Big\|\sum_{|n|\leqq N(R)/2,\,|m|>N(R)} K_{n,m}\Big\| + \Big\|\sum_{|n|>N(R)/2,\,|m|>N(R)} K_{n,m}\Big\|$$

with $K_{n,m}$ as above. Another application of the Schur-test gives the convergence to zero of the RHS of (1.14), the reason for the first summand is the exponential off-diagonal estimate proven above, for the second summand one uses that $\|h.1_n\|_p \to 0$ as $|n| \to \infty$. □

2. The case of bounded coefficients

In this section we consider the case where the operators A and B have bounded coefficients $a = (a_{jk})$ and $b = (b_{jk})$, $1 \leqq j, k \leqq d+1$. The notation $a - b \in c_0(L^1)$ means that $a_{jk} - b_{jk} \in c_0(L^1)$, $1 \leqq j, k \leqq d+1$.

We prove the following theorem.

Theorem 2.1. *Let a be as in* (Ell), *b as in* (ACC) *and $b \in L^\infty$. Suppose that*

$$(B+1)^{-1},\ (B^*+1)^{-1}L^2 \subset W^{1,2} .$$

If $a - b \in c_0(L^1)$ then $(A+1)^{-1} - (B+1)^{-1} \in \mathscr{K}(L^2)$ and in particular $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(B)$.

Remark. Since there are various definitions of the essential spectrum σ_{ess}, we note that the definition used here is

$$\sigma_{\mathrm{ess}}(A) = \{\lambda \in \mathbb{C},\ \lambda - A \text{ is not a Fredholm operator}\} .$$

With this notation, if $(A+1)^{-1} - (B+1)^{-1} \in \mathscr{K}(L^2)$ then $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(B)$ (see [8], p. 244).

The statement of the above theorem takes a rather smooth form if we assume instead of (Ell) and (ACC) that the principal parts $(a_{jk})_{1 \leqq j,k \leqq d}$ and $(b_{jk})_{1 \leqq j,k \leqq d}$ are elliptic: we have the same conclusion $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(B)$. To see this it suffices to add a suitable constant c to $a_{d+1,d+1}$ and $b_{d+1,d+1}$ and apply the above result to $A + c$ and $B + c$. The assumption $(B+1)^{-1}, (B^*+1)^{-1} L^2 \subset W^{1,2}$ will be automatically satisfied in that case.

Corollary 2.2. *Let a, b be bounded with elliptic principal part. If $a - b \in c_0(L^1)$ then $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(B)$.*

Having in mind the general Lie group case, where subelliptic operators play an important role (see [12]) it seems worthwhile to settle the result in the generality presented in Theorem 2.1.

The following "regularization" result will be used in the proofs of Theorems 2.1 and 3.1.

Proposition 2.3. *Let C, D be two densely defined operators on a Banach space E such that $(-\infty, 0) \subset \varrho(C) \cap \varrho(D)$. Suppose in addition that*

(i) $\sup\limits_{\lambda > 0} \| \lambda(\lambda + C)^{-1} \|_{\mathscr{L}(E)}, \ \sup\limits_{\lambda > 0} \| \lambda(\lambda + D)^{-1} \|_{\mathscr{L}(E)} < \infty$

and

(ii) *there exists an $m \in \mathbb{N}$ and $\alpha \in (0,1)$ such that $D(D^m) \subset D(C^\alpha)$.*

Then $(D+\lambda)^{-1} - (C+\lambda)^{-1} \in \mathscr{K}(E)$ if and only if $(\lambda + C)^{-k}\big((D+\lambda)^{-1} - (C+\lambda)^{-1}\big)$ and $\big((D+\lambda)^{-1} - (C+\lambda)^{-1}\big)(\lambda + C)^{-k} \in \mathscr{K}(E)$ for some $k \in \mathbb{N}$.

Proof. Only the "if" part needs a proof. Whithout restriction we can assume $k \geqq m$.

We write

$$(\lambda + C)^{-1} = (2k-1) \int_0^\infty t^{2k-2} (\lambda + t + C)^{-2k} \, dt \tag{2.1}$$

and the same formula for D (this formula can be shown easily by integration by parts). According to (i), the integral in (2.1) converges in norm, therefore it suffices to check that

$$(t + \lambda + C)^{-2k} - (t + \lambda + D)^{-2k} \in \mathscr{K}(E) .$$

Now write, for arbitrary $\lambda > 0$,

$$\begin{aligned} &(\lambda + C)^{-2k} - (\lambda + D)^{-2k} \\ &= (\lambda + C)^{-2k+1}\big((\lambda + C)^{-1} - (\lambda + D)^{-1}\big) \\ &+ \cdots + \big((\lambda + C)^{-1} - (\lambda + D)^{-1}\big)(\lambda + D)^{-2k+1} \end{aligned}$$

which implies that it suffices to check that

$$(\lambda + C)^{-k}\big((\lambda + C)^{-1} - (\lambda + D)^{-1}\big)$$

and

$$\big((\lambda + C)^{-1} - (\lambda + D)^{-1}\big)(\lambda + D)^{-k}$$

are compact. The first operator is compact by assumption. It remains to prove

$$\big((\lambda + C)^{-1} - (\lambda + D)^{-1}\big)(\lambda + D)^{-k} \in \mathscr{K}(E)\,.$$

To this end, write

$$\begin{aligned}&\big((\lambda + C)^{-1} - (\lambda + D)^{-1}\big)(\lambda + D)^{-k}\\ &= \big((\lambda + C)^{-1} - (\lambda + D)^{-1}\big)(\lambda + C)^{-\alpha}(\lambda + C)^{\alpha}(\lambda + D)^{k},\end{aligned}$$

which is possible by (ii). It remains to check

$$\big((\lambda + C)^{-1} - (\lambda + D)^{-1}\big)(\lambda + C)^{-\alpha} \in \mathscr{K}(E)\,. \tag{2.2}$$

We know already that this is true for k instead of α. Using (2.1) it holds for 1 instead of k. Now the formula (cf. [8], p. 286)

$$(\lambda + C)^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha}(\lambda + t + C)^{-1}\,dt$$

and (ii) imply (2.2). □

Proof of Theorem 2.1. Since $a - b \in L^\infty \cap c_0(L^1)$ it follows that $a - b \in c_0(L^p)$ for all $p < \infty$. We want to apply Proposition 2.3 with $A = C$ and $B = D$. Note that (i) is satisfied, since both operators are accretive. Moreover, (ii) is valid in this situation, for $m = 1$, since $D(B) = R(B+1)^{-1} \subset W^{1,2} = D(\mathfrak{a}) \subset D\big((\lambda + A)^{\alpha}\big)$ for any $\alpha \in \left(0, \frac{1}{2}\right)$ (see [9]).

It remains to check that

$$\big((\lambda + B)^{-1} - (\lambda + A)^{-1}\big)(\lambda + A)^{-1} \in \mathscr{K}(L^2)\,, \tag{2.3}$$

$$(\lambda + A)^{-1}\big((\lambda + B)^{-1} - (\lambda + A)^{-1}\big) \in \mathscr{K}(L^2)\,. \tag{2.4}$$

By the fact that $a, b \in L^\infty$, $D(\mathfrak{a}) = W^{1,2} \subset D(\mathfrak{b})$, so we can write

$$\big(\big((\lambda + B)^{-1} - (\lambda + A)^{-1}\big) f, g\big) = (\mathfrak{a} - \mathfrak{b})\big((\lambda + A)^{-1} f, (\lambda + B^*)^{-1} g\big)\,. \tag{2.5}$$

Using the notation

$$Df = \begin{pmatrix} \nabla f \\ f \end{pmatrix},$$

we see from the definition of the forms in (ACC) that the RHS of (2.5) is given by

$$\begin{aligned}&\int \big((a - b)\, D(\lambda + A)^{-1} f, D(\lambda + B^*)^{-1} g\big)\, dx\\ &= \big((\lambda + B)^{-1} D^*(a - b)\, D(\lambda + A)^{-1} f, g\big)\,.\end{aligned} \tag{2.6}$$

Hence it suffices to show that

$$(2.7) \qquad (\lambda + B)^{-1} D^* (a - b) D (\lambda + A)^{-2} \in \mathscr{K}(L^2) .$$

In order to see this, first note that $(\lambda + B)^{-1} D^* : (L^2)^{d+1} \to L^2$ is bounded, since $(\lambda + B^*)^{-1} : L^2 \to W^{1,2}$. Now the components of $(a - b) D (\lambda + A)^{-2}$ are of the form

$$(2.8) \qquad \big(a_{jk}(x) - b_{jk}(x)\big) D_k (\lambda + A)^{-2}$$

where $D_{d+1} = I$. We want to apply Proposition 1.2 for $h = a_{jk} - b_{jk}$. First note that for $k = d + 1$ this is trivial, since $(\lambda + A)^{-1} : L^2 \to W^{1,2} \subset L^q$ for some $q > 2$ and by $h \in c_0(L^p)$ for all $p < \infty$, we have compactness of $(a_{jk} - b_{jk})(\lambda + A)^{-2}$.

To prove the compactness for $k = 1, \ldots, d$ we use Proposition 1.2 once again. To this end, we infer the following fact from [2], Proposition 3.1:

$$\exists \eta > 0 \text{ such that } (\lambda + A)^{-1} : L^q \to W^{1,q} \text{ for all } q \in (2 - \eta, 2 + \eta) .$$

Using the Sobolev embedding once again, we find that

$$\nabla (\lambda + A)^{-2} : L^2 \to (L^q)^d$$

for some $q > 2$, which implies the desired compactness. Note finally that (2.4) is obtained by taking the adjoint and using the same arguments as above. □

Remark 2.4. In the above proof we have used the fact that there exists $q > 2$ such that

$$(\lambda + A)^{-2} : L^2 \to W^{1,q} \text{ is bounded} .$$

Denote by

$$q(A) := \sup \{ q > 2, \exists k \in \mathbb{N} \text{ s.t. } (\lambda + A)^{-k} : L^2 \to W^{1,q} \text{ is bounded} \} .$$

If the coefficients of A are Hölder continuous, then $q(A) = \infty$ ([2], Remark 4.10). More precisely, in this case $(\lambda + A)^{-k} : L^2 \to W^{1,\infty}$ for some $k \geqq 1$. For $d = 1$ the latter holds without any regularity condition on A ([2], Lemma 2.10).

Finally, let us remark that Theorem 2.1 provides an extension of the results in [11], [15] in two important respects: we assume no regularity of the coefficients and the coefficients are allowed to be complex-valued.

3. The case of unbounded coefficients for B

In this section we prove a theorem which covers cases of unbounded coefficients for the operator B. To this end, we have to assume, however, ellipticity and sectoriality (recall that Theorem 2.1 is valid for certain subelliptic B of divergence form):

(Sec) Assume that $b \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^{d+1,d+1})$ satisfies the assumptions of (ACC) and that for some constants $\eta_0 > 0$, $k \geqq 0$

$$\operatorname{Re} \mathfrak{b}(u,u) \geqq \eta_0 (\|\nabla u\|_2^2 + \|u\|_2^2) \tag{3.1}$$

and

$$|\operatorname{Im} \mathfrak{b}(u,u)| \leqq K \operatorname{Re} \mathfrak{b}(u,u) \tag{3.2}$$

for all $u \in D(\mathfrak{b}) \subset W^{1,2}$. $\square$

Note that (3.1) is stronger than assuming the ellipticity of the principal part of b. However, in most cases one can achieve (3.1) for matrices, whose principal part is elliptic:

Remark. Let b satisfy (ACC) and assume that the principal part $(b_{jk})_{1 \leqq j,k \leqq d}$ is elliptic. If $|b_{j,d+1}|^2$, $|b_{d+1,j}|^2$, $j = 1, \ldots, d$ and $b_{d+1,d+1}$ are form bounded with respect to $-\Delta$ with relative bounds sufficiently small then for a suitable $c > 0$, the form $\mathfrak{b} + c$ satisfies (3.1). To see this, it suffices to write

$$\operatorname{Re} \int b_{j,d+1} u \overline{D_j u} \geqq -\frac{1}{2\varepsilon} \int |b_{j,d+1}|^2 |u|^2 - \frac{\varepsilon}{2} \int |D_j u|^2$$

which is valid for any $\varepsilon > 0$. The $-\Delta$-form boundedness of $|b_{j,d+1}|^2$ implies that for some $\alpha_j, \beta_j \geqq 0$,

$$-\frac{1}{2\varepsilon} \int |b_{j,d+1}|^2 |u|^2 \geqq -\frac{\alpha_j}{2\varepsilon} \int |D_j u|^2 - \frac{\beta_j}{2\varepsilon} \int |u|^2 .$$

Writing now the same for all terms we see that we obtain the above claim if the bounds α_j are small enough.

Note that the assumptions in this remark are satisfied in the particular case

$$b_{j,d+1}, b_{d+1,j}, b_{d+1,d+1} \in L^\infty .$$

Let now $q(A)$ be as in Remark 2.4. We have

Theorem 3.1. *Let b satisfy* (Sec), *a satisfy* (Ell) *and assume that $b - a \in c_0(L^p)$ for some $p > \max\left(\frac{2q(A)}{q(A)-2}, \frac{2q(A^*)}{q(A^*)-2}\right)$. Then $(A+1)^{-1} - (B+1)^{-1} \in \mathscr{K}(L^2)$ and, in particular, $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(B)$.*

Proof. Note that the condition on p guarantees that $\frac{2}{p} + \frac{2}{q(A)} < 1$.

By the definition of $q(A)$ we find $k \in \mathbb{N}$ and q satisfying $\frac{2}{p} + \frac{2}{q} < 1$ and

$$(A+1)^{-k} : L^2 \to W^{1,q} .$$

Assume, for the moment, that

$$\text{(3.3)}\quad \big((B+1)^{-1}-(A+1)^{-1}\big)(A+1)^{-k}=(B+1)^{-1}D^*(a-b)\,D(A+1)^{-k-1}\,.$$

Then the compactness of the operator in (3.3) follows from Proposition 1.2. The same arguments imply the compactness of

$$\big((B^*+1)^{-1}-(A^*+1)^{-1}\big)(A^*+1)^{-k}\,.$$

Proposition 2.3 then gives the compactness of the resolvent difference.

The rest of the proof will be devoted to proving equation (3.3) by a suitable approximation. Recall that (3.3) is valid, if the coefficients of B are bounded. By (3.1) and the boundedness of a, we find an $\alpha>0$ such that

$$\text{(3.4)}\qquad \operatorname{Re}\big(\mathfrak{b}(u,u)-\alpha\,\mathfrak{a}(u,u)\big)\geqq 0\quad (u\in D(\mathfrak{b})\subset D(\mathfrak{a})=W^{1,2})\,.$$

Using (3.2) and the fact that $a\in L^\infty$, we find $\varepsilon>0$ so small that

$$\text{(3.5)}\qquad \varepsilon\,|\operatorname{Im}\mathfrak{b}(u,u)|\leqq\operatorname{Re}\mathfrak{b}(u,u),\quad u\in D(\mathfrak{b})$$

and

$$\varepsilon\,|\operatorname{Im}\mathfrak{a}(u,u)|\leqq\operatorname{Re}\mathfrak{a}(u,u),\quad u\in D(\mathfrak{a})\,.$$

Consider

$$\mathfrak{b}(z):=\operatorname{Re}\mathfrak{b}+z\operatorname{Im}\mathfrak{b},\quad \text{for } z,\ \text{s.t. } |\operatorname{Re}z|<\varepsilon\,.$$

By (3.5) this is a family of sectorial forms, holomorphic in z and

$$\text{(3.6)}\qquad \mathfrak{b}(t)\geqq 0\quad \text{for } t\in(0,\varepsilon)\,.$$

Define $\Omega_n:=\{x\in\mathbb{R}^d,\ |b_{jk}(x)|\leqq n,\ 1\leqq j,k\leqq d+1\}$ and the matrix $b_n(x)$ by

$$b_n(x):=\big(b(x)-\alpha a(x)\big)1_{\Omega_n}(x)+\alpha a(x)\,.$$

These matrices satisfy a uniform ellipticity assumption and for the associated operators B_n we have that

$$\text{(3.7)}\qquad \sup_n\|D(B_n^*+1)^{-1}\|<\infty\,.$$

Define the forms $\mathfrak{b}_n(z)$. We want to prove that $\mathfrak{b}_n=\mathfrak{b}_n(i)$ converges strongly in the resolvent sense to $\mathfrak{b}=\mathfrak{b}(i)$. Since the resolvents of $B_n(z)$, $B(z)$ depend holomorphically upon z, it suffices (by Vitali's theorem, cf. [7], Theorem 3.14.1) to prove this convergence for $z=t\in(0,\varepsilon)$. But this is clear, since $\mathfrak{b}_n(t)$ is a monotonically increasing sequence of symmetric forms with limit $\mathfrak{b}(t)$. So we know already, that

$$\text{(3.8)}\qquad (B_n+1)^{-1}\to(B+1)^{-1}\ \text{strongly}\,.$$

Regard (3.3) and the analog for B_n,

$$(3.3_n) \qquad \big((B_n+1)^{-1}-(A+1)^{-1}\big)(A+1)^{-k} = (B_n+1)^{-1}D^*(a-b_n)\,D(A+1)^{-k-1}$$

which is valid, since $b_n \in L^\infty$. If we can prove that the RHS of (3.3_n) converges weakly to the RHS of (3.3), the latter equation holds and the proof of Theorem 3.1 is complete.

Firstly, by (3.7) we have $\sup_n \|(B_n+1)^{-1}D^*\| < \infty$. It follows that there exists a subsequence which converges weakly (apply the Alaoglu-Bourbaki theorem to $\mathscr{L}(L^2)$, which is the dual of the nuclear operators $\mathscr{N}(L^2)$).

Since $(B_n+1)^{-1}$ converges strongly to $(B+1)^{-1}$ the same holds for the adjoints and by weak-closedness of D, D^*, the weak limit of the subsequence equals $(B+1)^{-1}D^*$.

It remains to prove that

$$(3.9) \qquad (b_n-a)\,D(A+1)^{-k-1} \to (b-a)\,D(A+1)^{-k-1} \text{ strongly}.$$

So let $f \in L^2$. Then

$$|\big((b-a)+\alpha a\big)\,D(A+1)^{-k-1}f| \in L^2$$

by Proposition 1.2. Moreover,

$$b_n(x)-a(x) = \big(b(x)-a(x)\big)1_{\Omega_n}(x) + (\alpha-1)\,a(x)(1-1_{\Omega_n})$$

converges pointwise to $b(x)-a(x)$, so (3.9) follows by the dominated convergence theorem. □

Remark that in order to get theorems like Satz 10.4 in [15] we can apply the above technique, if we assume that both $(A+1)^{-k}: L^2 \to W^{1,q}$ and $(B+1)^{-1}: L^2 \to W^{1,r}$. We have not persued this further, since we were mainly interested in results which hold without smoothness conditions on the coefficients.

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