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Integral Equations and Operator Theory

Eigenfunction Expansions for Schrödinger Operators on Metric Graphs

Daniel Lenz, Carsten Schubert and Peter Stollmann

Abstract. We construct an expansion in generalized eigenfunctions for Schrödinger operators on metric graphs. We require rather minimal assumptions concerning the graph structure and the boundary conditions at the vertices.

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1. Introduction

Expansion in generalized eigenfunctions is a topic that dates back to Fourier's work, at least. A classical reference is Berezanskii's monograph [2]. Motivated by examples from Mathematical Physics there has been a steady development involving new models. One trigger of more recent results is the importance of generalized eigenfunction expansions in the discussion of random models. See [4, 11, 22] and the references in there. This was also the background of the first paper that established eigenfunction expansions for quantum graphs, [10] (see [1, 10, 15, 16, 17, 7, 12, 13, 14] for recent results on quantum graphs). There the authors consider a rather special class of metric graphs, due to the random model they have in mind. We point out, however that part of their discussion is rather abstract and pretty much equivalent to what had been obtained in [4]. As was pointed out in [3], the Dirichlet form framework of the latter article applies to a class of quantum graphs with Kirchhoff boundary conditions.

The point of the present paper is to establish an expansion in generalized eigenfunctions under somewhat minimal conditions. This means we require just the usual conditions necessary to define the operators in question. These conditions essentially amount to providing a continuous embedding from the form domain of the operator to the Sobolev space $W^{1,2}$ of the graph. More concretely, we allow for general boundary conditions, unboundedness of the (locally finite) vertex degree function, loops, multiple edges and edges of infinite lengths. However, we require a uniform lower bound on the length of the edges. To the best of our knowledge, this framework contains all classes of models that have been considered so far. Our discussion is intrinsic and does not require an embedding of the metric graph into an ambient space.

As far as methods are concerned, we rely on the results from [18] rather than the approach of [2] that had been used in [10]. However, this is mostly a question of habit. In either approach a main point is to establish certain trace class properties of auxiliary functions. Here, we can rely upon one-dimensional techniques for quantum graphs. An extra asset is that we are able to establish pointwise properties of generalized eigenfunctions.

Our paper is structured as follows: In Section 2 we set up model and notation, define metric graphs and introduce the kind of boundary conditions we allow. Moreover, we check the necessary operator theoretic input for the Poerschke-Stolz-Weidmann method for constructing generalized eigenfunctions. In Section 3 we discuss the notion of generalized eigenfunctions and explore pointwise properties in the quantum graph case. It turns out that in this case generalized eigenfunctions have versions that satisfy the boundary conditions at the vertices. In Section 4 we present the necessary material from [18]. The application to the quantum graph case comes in Section 5 that contains our main results, Theorem 5.1 and Corollary 5.4. The former deals with quantum graphs and the latter includes additional perturbations by a potential that is uniformly locally square integrable.

2. Metric graphs and the associated operators

In this section we introduce metric graphs and the associated operators. The basic idea is that a metric graph consists of line segments – edges – that are glued together at vertices. In contrast to combinatorial graphs, these line segments are taken seriously as differential structures and in fact one is interested in the Laplacian on the union of the line segments. To get a self-adjoint operator one has to specify boundary conditions at the vertices. Our discussion of the unperturbed operator associated to a quantum graph in this section relies on the cited works of Kostrykin & Schrader, [12], Kuchment, [16], and the second named author, [21]. In particular, the subsequent discussion up to Lemma 2.3 can essentially be found in [16].

Definition 2.1. A *metric graph* is $\Gamma = (E, V, i, j)$ where

- E (edges) is a countable family of open intervals $(0, l(e))$ and V (vertices) is a countable set.
- $i : E \to V$ defines the initial point of an edge and $j : \{e \in E | l(e) < \infty\} \to V$ the end point for edges of finite length.

We let $X_e := \{e\} \times e$, $X = X_\Gamma = V \cup \bigcup_{e \in E} X_e$ and $\overline{X_e} := X_e \cup \{i(e), j(e)\}$.

Note that X_e is basically just the interval $(0, l(e))$, the first component is added to force the X_e's to be mutually disjoint. The topology on X will be such that the mapping $\pi_e : X_e \to (0, l(e)), (e, t) \mapsto t$ extends to a homeomorphism again denoted by $\pi_e : \overline{X_e} \to \overline{(0, l(e))}$ that satisfies $\pi_e(i(e)) = 0$ and $\pi_e(j(e)) = l(e)$ (the latter in case that $l(e) < \infty$). A piece of the form $I = \pi_e^{-1}(J)$ with an edge e and an interval $J \subset \overline{(0, l(e))}$ is called an *edge segment*. The length of the edge segment is the length of J. Edge segments will play a role, when we discuss local properties of functions.

While we allow multiple edges and loops, we will assume finiteness of each single vertex degree d_v, $v \in V$, i. e.

(F) $d_v := |\{(0, e) : v = i(e)\} \cup \{(l(e), e) : v = j(e)\}| < \infty$.

To define a metric structure on X we then proceed as follows: we say that $p = (x_1, x_2, \ldots, x_N) \in X^N$ is a *good polygon* if for every $k \in \{1, ..., N-1\}$ there is a unique edge $e \in E$ such that $\{x_k, x_{k+1}\} \subset \overline{X_e}$. Using the usual distance on $[0, l(e)]$ we get a distance d on $\overline{X_e}$ and use it do define

$$l(p) = \sum_{k=1}^{N-1} d(x_k, x_{k+1}).$$

Since multiple edges are allowed, we needed to restrict our attention to good polygons to exclude the case that $\{x_k, x_{k+1}\}$ are joined by edges of different length. Given connectedness of the graph and (F), a metric on X is given by

$$d(x, y) := \inf\{l(p) \mid p \text{ a good polygon with } x_1 = x \text{ and } x_N = y\}.$$

In fact, symmetry and triangle inequality are evident and the separation of points follows from the finiteness. Clearly, with the topology induced by that metric, X is a locally compact, separable metric space. If X is not connected, we can do the above procedure on any connected component.

We will assume a lower bound on the length of the edges:

(LB) There exists a $u > 0$ with $l(e) \geq u$ for all $e \in E$.

We will now turn to the relevant Hilbert spaces and operators. We define

$$L^2(X) := \bigoplus_{e \in E} L^2(e), \quad W^{1,2}(X) := \bigoplus_{e \in E} W^{1,2}(e), \quad W^{2,2}(X) := \bigoplus_{e \in E} W^{2,2}(e).$$

Here, of course, $L^2(e)$ ($W^{1,2}(e)$, $W^{2,2}(e)$) consists of functions u_e on $e = (0, l(e))$. In the sequel we will view those families $u = (u_e)_{e \in E} \in L^2(X)$ rather as functions defined on X. Note that $W^{1,2}(X)$ and $W^{2,2}(X)$ are sometimes referred to as decoupled or maximal Sobolev spaces, see e.g. [9, 19]. Other Sobolev spaces can also be found in the literature. For our purpose, the above definitions seem to be the most convenient ones.

Consider $a > 0$ and recall that $h \in W^{1,2}(0, a)$ is continuous and $h(0) := \lim_{x \to 0+} h(x)$ exists and satisfies

$$|h(0)|^2 \leq \frac{2}{a} \|h\|^2_{L^2(0,a)} + a \|h'\|^2_{L^2(0,a)} \tag{2.1}$$

by standard Sobolev type theorems. Consider now an edge e and $u \in W^{1,2}(e)$. Then the limit $u(0) := \lim_{t\to 0} u(t)$ exists, as well as $u(l(e)) := \lim_{t\to l(e)} u(t)$ and (2.1) holds (with the obvious modifications). Similarly, for an edge e and $u \in W^{2,2}(e)$ the limits $u'(0) := \lim_{t\to 0} u'(t)$ and $u'(l(e)) := -\lim_{t\to l(e)} u'(t)$ exist. Here, we have introduced a sign. This makes our definition of the derivative canonical, i. e. independent of the choice of orientation of the edge.

For $f \in W^{1,2}(X)$ and each vertex v we gather the boundary values of f_e over all edges e adjacent to v in a vector $f(v)$. More precisely, let $E_v := \{(0,e) : v = i(e)\} \cup \{(l(e),e) : v = j(e)\}$ denote the set of outgoing and incoming edges adjacent to v and define $f(v) := (f_e(t))_{(t,e)\in E_v} \in \mathbb{C}^{E_v}$. Similarly, for $f \in W^{2,2}(X)$ we further gather the boundary values of $f_e'(t)$ over all edges e adjacent to v in a vector $f'(v) \in \mathbb{C}^{E_v}$. Note that for each loop at a vertex v there are two entries in the vectors $f(v)$ and $f'(v)$. These boundary values of functions will play a crucial role when we discuss the concept of boundary condition.

Definition 2.2. A *boundary condition* is given by a pair (L, P) consisting of a family $L = (L_v)_{v\in V}$ of self-adjoint operators $L_v : \mathbb{C}^{E_v} \longrightarrow \mathbb{C}^{E_v}$ and a family $P = (P_v)$ of projections $P_v : \mathbb{C}^{E_v} \longrightarrow \mathbb{C}^{E_v}$.

We will assume the following upper bound on $(L_v)_{v\in V}$:

(UB) There exists an $S > 0$ with $\|L_v^+\| \leq S$ for any $v \in V$, where the $+$ denotes the positive part of a self-adjoint operator.

Given a metric graph satisfying (F) and(LB) and a boundary condition satisfying (UB), we obtain from (2.1) by a direct calculation that

$$\sum_{v\in V} \langle L_v f(v), f(v)\rangle \leq \frac{4S}{\varepsilon}\|f\|_{L^2(X)} + 2S\varepsilon\|f'\|_{L^2(X)} \tag{2.2}$$

for any $f \in W^{1,2}(X)$ and any $\varepsilon > 0$ with $\varepsilon \leq u$. Given a boundary condition (L, P) we define the form $s_0 := s_{L,P}$ by

$$\begin{aligned} D(s_0) &:= \{f \in W^{1,2}(X) : P_v f(v) = 0 \text{ for all } v \in V\ \},\\ s_0(f,g) &:= \sum_{e\in E}\int_0^{l(e)} f_e'(t)\overline{g}_e'(t)dt - \sum_{v\in V}\langle L_v f(v), \overline{g}(v)\rangle. \end{aligned}$$

By (2.2) we easily see that for $C > 0$ large enough

$$s_0(f,f) + C(f,f) \geq \frac{1}{2}\|f\|_{W^{1,2}(X)} \tag{2.3}$$

for any $f \in D(s_0)$. This shows that s_0 is bounded below and closed. Hence, there exists an associated self-adjoint operator. This operator is denoted by $H_0 := H_{L,P}$. It can be explicitly characterized by

$$\begin{aligned} D(H_0) := \{f \in W^{2,2}(X) :\ & P_v f(v) = 0 \text{ and}\\ & L_v f(v) + (1 - P_v)f'(v) = 0 \text{ for all } v \in V\ \},\\ (H_0 f)_e := -f_e'' &\text{ for all } e \in E. \end{aligned}$$

We will assume the following setting:

(S) Γ is a metric graph satisfying (F) and (LB) with associated space X. (L, P) is a boundary condition satisfying (UB). The induced form is denoted by s_0 and the corresponding operator by $H_0 = H_{L,P}$.

Lemma 2.3. *Assume* (S). *Then* $(H_0+C)^{-\frac{1}{2}}$ *provides a continuous map from* $L^2(X)$ *to* $L^\infty(X)$ *for sufficiently large* $C > 0$.

Proof. As H_0 is bounded below, $(H_0+C)^{-\frac{1}{2}}$ provides a bounded map from $L^2(X)$ to the form domain equipped with the form norm $\|\cdot\|_{s_0}$ for sufficiently large $C > 0$. By (2.3), the form domain (with the form norm) is continuously embedded into $W^{1,2}(X)$. By (2.1), $W^{1,2}(X)$ is continuously embedded in $L^\infty(X)$. Putting this together we obtain the statement. $\square$

Lemma 2.4. *Assume* (S). *Then*

$$\{f \in D(H_0) : \operatorname{supp} f \textit{ compact}\}$$

is a core for H_0.

Proof. Choose $f \in D(H_0)$. We have to find $f_n \in D(H_0)$ with compact support and $f_n \longrightarrow f$ and $H_0 f_n \longrightarrow H_0 f$. We will provide $f_n = \psi_n f$ with suitable cut-off functions ψ_n. We will assume without loss of generality that X is connected (otherwise we will have to perform the process simultaneously on each connected component).

Choose $x \in X$. For $n \in \mathbb{N}$ let $B_n = B(x, n)$ be the ball around x with radius n. Construct $\psi_n = (\psi_{n,e})_{e\in E}$ with

$$\psi_n|_{B(x;n-2u)} \equiv 1, \quad \psi_n|_{B(x;n+2u)^c} \equiv 0 \tag{2.4}$$

by distinguishing three cases: For edges e with both ends $i(e)$ and $j(e)$ contained in B_n set $\psi_{n,e} \equiv 1$. For edges e with both ends $i(e)$ and $j(e)$ contained in the complement of B_n set $\psi_{n,e} \equiv 0$. For edges e with one endpoint, say $i(e) \in B_n$ and $j(e) \in B_n^c$ we choose $\psi_{n,e}$ two times continuously differentiable on e, $\psi_{n,e} \equiv 1$ on a suitable neighborhood of $i(e)$, $\psi_{n,e} \equiv 0$ on a suitable neighborhood of $j(e)$ such that $\psi_{n,e}$ and its first two derivatives are bounded by $(1 + 4/u)^2$. This is possible, since the length of the edges is bounded below by u.

Since in this way ψ_n is constant in the neighborhood of any vertex, smooth and bounded the functions $f_n := \psi_n f$ belong to $D(H_0)$ for every $n \in \mathbb{N}$. By (2.4) we conclude $f_n \to f$ in $L^2(X)$ as $n \to \infty$. Similarly,

$$H_0(\psi_n f) = -\psi_n f'' - 2\psi_n' f' - \psi_n'' f \to H_0 f,$$

as ψ_n', ψ_n'' are uniformly bounded and supported on $B(x; n+2u) \setminus B(x; n-2u)$. $\square$

Remark 2.5. Let us shortly discuss the necessity of conditions of the form (LB) and (UB) in our context. Our aim is to show (2.3), i. e. that the identity is continuous as a map from the form domain with $W^{1,2}$ norm to the form domain with form norm.

As we allow for rather general boundary conditions and do not assume any connectedness, we need a pointwise estimate on the boundary values of a function on an edge in terms of the corresponding $W^{1,2}(e)$ norm. In this respect, the Sobolev estimate (2.1) is essentially optimal. More precisely, testing with the constant function on an interval of finite length shows that the factor $1/a$ can not be avoided. In particular, (2.3) fails for a graph consisting of countably infinite disjoint edges with lengths going to zero and a δ-boundary condition (corresponding to L_v being the identity) on one of the vertices of each edge. In this sense, a condition of the form (LB) seems unavoidable.

Similarly, given (LB), we need a bound of the form (UB) to bound the boundary terms. In particular, (2.3) fails for a graph consisting of countably infinite disjoint edges with lengths one and boundary conditions of the form $c_v L$ with c_v going to infinity.

3. A word on locality

Let a locally compact space X with a measure dx be given. Let $\mathrm{L}^2_{\mathrm{loc}}(X)$ be the space of functions on X whose restrictions to compact sets are square integrable. Let $\mathrm{L}^2_{\mathrm{comp}}(X)$ be the set of functions in $\mathrm{L}^2(X)$ which have compact support. The usual inner product can be "extended" to give a map (again denoted by $\langle\cdot,\cdot\rangle$)

$$\mathrm{L}^2_{\mathrm{comp}}(X) \times \mathrm{L}^2_{\mathrm{loc}}(X) \longrightarrow \mathbb{C},\ \langle f, g\rangle := \int f(x)\overline{g}(x)dx.$$

Definition 3.1. Let X be a topological space with a measure dx. Let H be an operator on X which is local i. e. Hf has compact support whenever f has and $D(H) \cap \mathrm{L}_{\mathrm{comp}}$ is a core for H. A nontrivial function ϕ on X is called a *generalized eigenfunction* for H corresponding to λ if it belongs to $\mathrm{L}^2_{\mathrm{loc}}(X)$ and satisfies

$$\langle Hf, \phi\rangle = \lambda\langle f, \phi\rangle \tag{3.1}$$

for any $f \in D(H)$ with compact support.

Remark 3.2. Here, $\langle Hf, \phi\rangle$ is defined in the sense discussed at the beginning of the section. The inner product $\langle f, \phi\rangle$ is defined in the same way. The condition on the core of H is not necessary to state the definition. However, it is only this condition that makes the definition a sensible one.

The question arises to which extent a generalized eigenfunction is locally a good function. We say that $\phi \in \mathrm{L}^2_{\mathrm{loc}}(X)$ *is locally in* $W^{2,2}$ if the restriction ϕ_I belongs to $W^{2,2}(I)$ for any compact edge segment. In particular, $\phi_e \in W^{2,2}(e)$ for every edge of finite length. Note that $\mathrm{L}^2_{\mathrm{loc}}(X)$-functions belong to L^2 of any edge of finite length.

Here is one answer in the case of quantum graphs:

Lemma 3.3. *Assume* (S). *If ϕ is a generalized eigenfunction for H_0, then ϕ is locally in $W^{2,2}$ and admits a version that satisfies the boundary condition at any vertex.*

Proof. To check that ϕ belongs locally to $W^{2,2}$ is suffices to consider $f \in D(H_0)$ with compact support contained in an edge and apply (3.1). This gives $-\phi'' = \lambda\phi$ so that ϕ belongs locally to $W^{2,2}$, since $\phi \in \mathrm{L}^2_{\mathrm{loc}}(X)$ by our definition of generalized eigenfunction.

To check that ϕ satisfies the boundary condition at a vertex v, it suffices to consider $f \in D(H_0)$ supported on a neighborhood of v and apply (3.1). In fact, let $f \in D(H_0)$ with $f_e \equiv 0$ for all edges e not adjacent to v. Then we get

$$\begin{aligned}\langle f, \lambda\phi\rangle &= \langle H_0 f, \phi\rangle \\ &= \langle -f'', \phi\rangle;\end{aligned}$$

integration by parts and the condition on the support of f give (with the evident notation for the inner product in $\mathbb{C}^{E_v}$)

$$\begin{aligned}\ldots &= \langle f, -\phi''\rangle + \langle f'(v), \phi(v)\rangle - \langle f(v), \phi'(v)\rangle \\ &= \langle f, \lambda\phi\rangle + \langle f'(v), \phi(v)\rangle - \langle f(v), \phi'(v)\rangle\end{aligned}$$

as the second weak derivative of $\overline{\phi}$ is $-\lambda\overline{\phi}$. Therefore,

$$\langle f'(v), \phi(v)\rangle = \langle f(v), \phi'(v)\rangle$$

for every choice of $f \in D(H_0)$. Splitting the scalar products in the parts living in the images of P_v and $1 - P_v$ gives

$$\begin{aligned}&\langle P_v f'(v), P_v\phi(v)\rangle + \langle (1-P_v)f'(v), (1-P_v)\phi(v)\rangle \\ &= \langle P_v f(v), P_v\phi'(v)\rangle + \langle (1-P_v)f(v), (1-P_v)\phi'(v)\rangle.\end{aligned}$$

Choosing $f \in D(H_0)$ with arbitrary $P_v f'(v)$ and $(1-P_v)f(v) = 0$ (granting $(1-P_v)f'(v) = 0$), we see that $P_v\phi(v)$ has to be equal to zero.

If we use the boundary condition for f, the last equation can be transformed to

$$\langle P_v f'(v), P_v\phi(v)\rangle = \langle (1-P_v)f(v), L_v\phi(v) + (1-P_v)\phi'(v)\rangle.$$

Taking an f with arbitrary $(1-P_v)f(v)$, we conclude that $L_v\phi(v) + (1-P_v)\phi'(v)$ also equals zero, thus giving the boundary condition for ϕ. □

4. Expansion in generalized eigenfunctions: general framework

In this section we discuss the expansion in generalized eigenfunctions of a self-adjoint operator. We follow the work of Poerschke, Stolz and Weidmann [18]. This will be used to provide an expansion for metric graphs in a spirit similar to the considerations of [4] for Dirichlet forms. Note that in [10] a different approach has been used. However, an important point in both the different methods is to establish suitable trace class properties for operators constructed from H. In that respect, the analysis of [4, 10] is similar. Actually, the case of quantum graphs is rather easy as far as trace class properties are concerned, as we have a locally one-dimensional situation at hand.

Let a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and a self-adjoint operator $T \geq 1$ in $\mathcal{H}$ be given. We will define the following two auxiliary Hilbert spaces: $\mathcal{H}_+ := \mathcal{H}_+(T) := D(T)$, $\langle x, y\rangle_+ := \langle Tx, Ty\rangle$ and $\mathcal{H}_-$ as completion of $\mathcal{H}$ with respect to the scalar product $\langle x, y\rangle_- := \langle T^{-1}x, T^{-1}y\rangle$. Thus, the inner product on $\mathcal{H}$ can be naturally extended to give a map

$$\langle\cdot,\cdot\rangle : \mathcal{H}_+ \times \mathcal{H}_- \longrightarrow \mathbb{C}.$$

Let N be a positive integer or infinity, H a self-adjoint operator in $\mathcal{H}$ and μ a spectral measure for H.

A sequence of subsets $M_j \subset \mathbb{R}$, such that $M_j \supset M_{j+1}$ together with a unitary map U

$$U = (U_j) : \mathcal{H} \to \bigoplus_{j=1}^{N} L^2(M_j, d\mu)$$

is said to be an *ordered spectral representation* of H if

$$U\phi(H) = M_\phi U,$$

for every measurable function ϕ on $\mathbb{R}$.

Theorem 4.1 (Theorem 1 of Section 3 in [18]**).** *Let H, T, $\mathcal{H}_+$, $\mathcal{H}_-$ be as above. Let μ be a spectral measure for H and U an ordered spectral representation. Let $\gamma : \mathbb{R} \longrightarrow \mathbb{C}$ be continuous and bounded with $|\gamma| > 0$ on $\sigma(H)$ such that $\gamma(H)T^{-1}$ is a Hilbert-Schmidt operator. Then there are measurable functions $\phi_j : M_j \to \mathcal{H}_-$, $\lambda \mapsto \phi_{j,\lambda}$ for $j = 1, \ldots, N$ such that the following properties hold:*

(i) *$U_j f(\lambda) = \langle f, \phi_j(\lambda)\rangle$ for $f \in \mathcal{H}_+$ and μ-a. e. $\lambda \in M_j$.*

(ii) *For every $g = (g_j) \in \bigoplus_j L^2(M_j, d\mu)$*

$$U^{-1}g = \lim_{n\to N, E\to\infty} \sum_{j=1}^{N} \int\limits_{M_j\cap[-E,E]} g_j(\lambda)\phi_{j,\lambda} d\mu(\lambda)$$

and, for every $f \in \mathcal{H}$,

$$f = \lim_{n\to N, E\to\infty} \sum_{j=1}^{N} \int\limits_{M_j\cap[-E,E]} (U_j f)(\lambda) d\mu(\lambda).$$

(iii) *For $f \in \{g \in D(H) \cap \mathcal{H}_+ |\ Hg \in \mathcal{H}_+\}$ and μ-a. e. $\lambda \in M_j$*

$$\langle Hf, \phi_{j,\lambda}\rangle = \lambda\langle f, \phi_{j,\lambda}\rangle. \tag{4.1}$$

If the functions $\phi_{j,\lambda}$ fulfill (i) and (ii) of the theorem, we will speak of a Fourier type expansion. If the set $\{g \in D(H) \cap \mathcal{H}_+ |\ Hg \in \mathcal{H}_+\}$ is a core for H, we speak of an expansion in generalized eigenfunctions.

We will apply the previous theorem to the Hilbert Space $\mathcal{H} = L^2(X)$ and the operator H_0, where X is a quantum graph satisfying (F), (LB) and (UB) as discussed in Section 2. As T we then use the operator $T := M_w$ of multiplication with a suitable weight function w, i. e. a continuous map $w : X \longrightarrow [1, \infty)$.

5. The main theorem

Theorem 5.1. *Assume* (S). *Let μ be a spectral measure for H_0. Let $w : X \to [1, \infty)$ be continuous with $w^{-1} \in L^2(X)$. Then there exists a Fourier type expansion (ϕ_j) for H_0, such that for μ-a. e. $\lambda \in \sigma(H_0)$ the function $\phi_{j,\lambda}$ is a generalized eigenfunction of H_0 for λ with $w^{-1}\phi_{j,\lambda} \in L^2$.*

Proof. We will apply the abstract result of the previous section. Let γ be the function $\gamma(t) = (C + t)^{-1/2}$. As T choose multiplication with w. Then, $\gamma(H_0)$ is a bounded map from $\mathrm{L}^2(X)$ to $\mathrm{L}^\infty(X)$ by Lemma 2.3. This, together with the assumption on w easily shows that the operator $T^{-1}\gamma(H_0)$ has an L^2 kernel and is therefore a Hilbert-Schmidt operator. Thus, its adjoint operator $\gamma(H_0)T^{-1}$ is a Hilbert-Schmidt operator as well. We can therefore apply the result of the previous section. This gives a Fourier type expansion. By definition of T any function in $\mathcal{H}_-$ is locally in L^2. Moreover,

$$\langle H_0 f, \phi_{j,\lambda}\rangle = \lambda\langle f, \phi_{j,\lambda}\rangle$$

holds μ-a. e. (in λ) for $f \in D_w := \{g \in D(H_0)|\ wg, wH_0 g \in L^2(X)\}$. As w is continuous and $H_0 f$ has compact support whenever f has compact support by definition of H_0, the set D_w obviously contains $D(H_0) \cap \mathrm{L}^2_{\mathrm{comp}}(X)$. Thus, the functions $\phi_{j,\lambda}$ are generalized eigenfunctions in the sense of Section 3. This finishes the proof. □

We denote by m the measure induced on X by the Lebesgue measure on the edges X_e, pulled back via π_e.

Remark 5.2 (A weight function). Assume that X is connected and define, for $\epsilon > 0$,

$$w(x) = m\left(B_{d(x,x_0)+1}(x_0)\right)^{1+\epsilon}.$$

Clearly, w is continuous and $w \geq 1$. To see that $w^{-1} \in L^2(X)$, it suffices to consider the case that Γ is infinite. In this case, $m(B_r(x_0)) \geq r$ for every $x_0 \in X$ and $r > 0$ by construction of the metric. We consider the volume of the annuli $B_n(x_0) \setminus B_{n-1}(x_0)$. For x in this annulus we obviously have that $w(x) \geq m(B_n(x_0))^{1+\epsilon}$. Hence, suppressing the x_0 in the notation of the balls,

$$\begin{aligned}\int_X |w^{-1}|^2 dx &\leq \int_{B_1\setminus B_0} w^{-2} dx + \int_{B_2\setminus B_1} w^{-2} dx + \dots \\ &\leq \int_{B_1\setminus B_0} m(B_1)^{-2-2\epsilon} dx + \int_{B_2\setminus B_1} m(B_2)^{-2-2\epsilon} dx + \dots \\ &\leq \sum_{i=1}^{\infty} i^{-1-2\epsilon} < \infty,\end{aligned}$$

where we used $m(B_n) \geq n$.

Schrödinger operators

Now we show that our main result can be extended to Schrödinger operators on metric graphs. Here, we treat a rather simple case. More singular perturbations will be considered elsewhere. In the following proposition we gather some operator theoretic results for potential perturbations of the operators $H_0 = H_{L,P}$ for a quantum graph satisfying assumption (S). For a general background, we refer the reader to [20], Section X. 2 as well as [8], §5 and §6.

We are going to consider the class of potentials $V \in \prod_e \mathrm{L}^2(e)$ with

$$M := M_V := \sup\{\|V_I\|_2 : I \text{ edge segment with length between } u \text{ and } 2u\} < \infty.$$

This class will be denoted by $\mathrm{L}^2_{\mathrm{loc,u}}(X)$.

Proposition 5.3. *Assume* (S) *and let* $V \in L^2_{\mathrm{loc,u}}(X)$. *Then we have:*

(i) V *is infinitesimally small with respect to* H_0. *In particular,* $H = H_0 + V$ *is self-adjoint on* $D(H_0)$.

(ii) $(H+C)^{-\frac{1}{2}}$ *provides a continuous map from* $L^2(X)$ *to* $L^\infty(X)$ *for sufficiently large* $C > 0$.

(iii)
$$\{f \in D(H_0) : \operatorname{supp} f \ \textit{compact}\}$$
is a core for H.

(iv) *If* ϕ *is a generalized eigenfunction for* H, *then* ϕ *is locally in* $W^{2,2}$ *and admits a version that satisfies the boundary condition at any vertex.*

Proof. (i) Let $a > 0$ be arbitrary. Assume w.l.o.g. that $a \leq u$. We now decompose the edges of the graph into edge segments, which are disjoint up to their boundary and have length between a and $2a$. Then any point of the graph belongs to such an edge segment I. Accordingly, our usual Sobolev estimate (2.1) gives

$$\|f|_I\|_\infty^2 \leq \frac{a}{2}\|f'|_I\|_2^2 + \frac{4}{a}\|f|_I\|_2^2. \tag{5.1}$$

Note that we pick up an extra factor of 2 compared to estimate (2.1) as the point may not lie at the boundary of I (in which case we only have an interval of length $a/2$ at our disposal). Recall the estimate

$$\|f\|_{W^{1,2}}^2 \leq 2s_0(f,f) + C\|f\|_2^2. \tag{5.2}$$

Summing over all I of our decomposition we obtain

$$\begin{aligned}
\|Vf\|_2^2 &= \sum_I \|(Vf)|_I\|_2^2 \\
&\leq \sum_I \|V|_I\|_2^2 \|f|_I\|_\infty^2 \\
&\leq M^2 \sum_I \|f|_I\|_\infty^2
\end{aligned}$$

$$
\begin{aligned}
(5.1)\quad &\le M^2 \sum_I \Big(\frac{a}{2} \|f'|_I\|_2^2 + \frac{4}{a}\|f|_I\|_2^2 \Big) \\
&\le M^2 \frac{a}{2}\|f\|_{W^{1,2}}^2 + M^2 \frac{4}{a}\|f\|_2^2 \\
(5.2)\quad &\le M^2 a s_0(f,f) + M^2 \frac{Ca}{2}\|f\|_2^2 + M^2\frac{4}{a}\|f\|_2^2 \\
&= M^2 a s_0(f,f) + C(a)\|f\|_2^2,
\end{aligned}
$$

where

$$C(a) = M^2\Big(\frac{Ca}{2} + \frac{4}{a}\Big).$$

As $s_0(f,f) \le \|H_0 f\|\|f\| \le \|H_0 f\|^2 + \|f\|^2$, we obtain

$$\|Vf\|^2 \le M^2 a \|H_0 f\|^2 + (C(a) + M^2 a)\|f\|_2^2.$$

As $a > 0$ is arbitrary, self-adjointness of H and (iii) both follow from the Kato-Rellich theorem, cf [20], Theorem X. 12.

(ii) It follows from (i) that V is also form small with respect to H_0, see [8] and [20] so that the form norm of H_0 and H are equivalent. Hence (ii) follows from Lemma 2.3 above.

(iv) For every compact edge segment I we get that the restriction ϕ_I of ϕ to I satisfies

$$\phi_I'' = V_I \phi_I - \lambda \phi_I$$

in the weak sense. Since $\phi_I \in \mathrm{L}^2(I)$ for every compact I and $V_I \in \mathrm{L}^2(I)$, we get that $\phi_I \in L^1$. In particular, ϕ_I' admits a continuous version so that $\phi_I \in C(I)$. Since $V_I \in \mathrm{L}^2(I)$ this gives that ϕ is locally in $W^{2,2}$. The rest of the argument can be taken from the proof of Lemma 3.3 with the obvious rewording. □

This gives the following analog of Theorem 5.1 for Schrödinger operators:

Corollary 5.4. *Assume* (S) *and let $V \in L^2_{\mathrm{loc,u}}(X)$. Let μ be a spectral measure for H. Let $w : X \to [1,\infty)$ be continuous with $w^{-1} \in L^2(X)$. Then there exists a Fourier type expansion (ϕ_j) for H, such that for μ-a. e. $\lambda \in \sigma(H_0)$ the function $\phi_{j,\lambda}$ is a generalized eigenfunction of H for λ with $w^{-1}\phi_{j,\lambda} \in L^2$.*

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Daniel Lenz
Fakultät für Mathematik
Technische Universität Chemnitz
09107 Chemnitz
Germany
Current address:
Department of Mathematics
Rice University
P. O. Box 1892
Houston, TX 77251
USA
e-mail: dlenz@mathematik.tu-chemnitz.de

Carsten Schubert
Fakultät für Mathematik
Technische Universität Chemnitz
09107 Chemnitz
Germany
e-mail: carsten.schubert@mathematik.tu-chemnitz.de

Peter Stollmann
Fakultät für Mathematik
Technische Universität Chemnitz
09107 Chemnitz
Germany
e-mail: P.Stollmann@mathematik.tu-chemnitz.de

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