A GENERAL FORM OF INTEGRAL.

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Introduction. The idea of an integral has been extended by Radon, Young, Riesz* and others so as to include integration with respect to a function of bounded variation. These theories are based on the fundamental properties of sets of points in a space of a finite number of dimensions. In this paper a theory is developed which is independent of the nature of the elements. They may be points in a space of a denumerable number of dimensions or curves in general or classes of events so far as the theory is concerned. It follows that, although many of the proofs given are mere translations into other language of methods already classical (particularly those due to Young), here and there, where previous proofs rested on the theory of sets of points, new methods have been devised (see, for example, theorems 3(3), 3(4), 5(1)).

Moore† has developed a theory of a similarly general nature, but restricts himself to the use of relatively uniform sequences. This concept is not used nor is it necessary in the following paper. We consider a group of elements p, which may be whatever we choose, and certain classes of functions $f(p)$ of those elements so that to every element p of the group there exists a real number $f(p)$ (which may be infinite in certain cases). To each function of a certain class there corresponds a real number $S(f)$ or $I(f)$ which is defined so as to satisfy certain conditions. $S(f)$ is a generalized Stieltjes integral, while $I(f)$ may be called the positive integral and the latter possesses correlates of nearly all the properties of the Lebesgue integral. It is shown that any I-integral is an S-integral while any S-integral is expressible as the difference of two I-integrals.

Two symbols have been taken over from symbolic logic, namely those for logical sum and logical product. The concepts involved are used extensively by Young and by the author and the symbols have been introduced to save space and to clarify the reasoning.

The reader is referred also to Hildebrandt‡ for references to an exten-

* J. Radon, Sitzungsberichte der Akademie der Wissenschaften, Wien (1913), p. 1295. W. H. Young, Proceedings of the London Mathematical Society (1914), 13, p. 109. F. Riesz, Annales de l'Ecole Normale Superieure (1914), 31, p. 9.

† E. H. Moore, Bulletin of the American Mathematical Society (1912), 18, p. 334.

‡ T. H. Hildebrandt, Bulletin of the American Mathematical Society (1917), 24, p. 117 and p. 177.

sive literature on generalized integrals. Fréchet considers a general integral but does not discuss existence theorems so completely.*

1. **Logical addition.**—$f(p)$ is said to be the logical sum of $f_1(p)$ and $f_2(p)$ if, for each p, the value of $f(p)$ is the greater of the values of $f_1(p)$, $f_2(p)$ for that p. Symbolically

$$f(p) = f_1(p) \vee f_2(p).$$

Logical product.—$f(p)$ is said to be the logical product of $f_1(p)$ and $f_2(p)$ if, for each p, the value of $f(p)$ is the less of the values of $f_1(p)$, $f_2(p)$ for that p. Symbolically

$$f(p) = f(_1p) \wedge f_2(p).$$

Then

1(1) $$f_1 \vee f_2 + f_1 \wedge f_2 = f_1 + f_2,$$

1(2) $$(-f_1) \vee (-f_2) = -(f_1 \wedge f_2).$$

It is assumed that there is an initial class T_0 of numerically valued functions $f(p)$ of the elements p where T_0 is closed with respect to the operations:—multiplication by a constant (C), addition (A), logical addition (G) and logical multiplication (G'). It is further assumed that the functions of T_0 are limited, that is corresponding to any $f(p)$, a finite number $K(f)$ can be found such that $|f(p)| < K(f)$ for all p. The following properties C, A, L, P, M are essential in what follows. $f, f_1, f_2, \cdots f_n, \cdots$ represent functions of the elements p belonging to the class T_0. Let $U(f)$ be a functional operation on f, then each of the properties is as follows:

(C) $$U(cf) = cU(f),$$

where c is constant;

(A) $$U(f_1 + f_2) = U(f_1) + U(f_2);$$

(L) If $f_1(p) \geqq f_2(p) \geqq \cdots$ and if $\lim f_n(p) = 0$ for all p, $\lim U(f_n) = 0$.

(P) If $f(p) \geqq 0$ for all p, $U(f) \geqq 0$.

(M) A functional operation $M(\varphi)$ exists for all functions of the type $|f|$, where f is of class T_0, such that if $\varphi \leqq \psi$, $M(\varphi) \leqq M(\psi)$, and such that $|U(f)| \leqq M(|f|)$. The I-integral, or $I(f)$, is a functional on functions of T_0 satisfying $(C)(A)(L)(P)$, while the S-integral satisfies $(C)(A)(L)(M)$.

The class T_0 is also restricted so as to include only such functions f that $I(f)$, $M(|f|)$ are finite for each f, though these integrals will not be bounded in their class. A few instances of the theory are as follows:

(a) The element p is a real number x in an interval (a, b). The

* M. Fréchet, Bulletin de la Société Mathématique de France (1915), 43, p. 249.

class T_0 is the class of continuous functions. Then the Riemann integral is an I-integral and the Stieltjes integral an S-integral. The extension to the class of summable functions leads to Lebesgue or Young integrals in the former case and to Radon or Young-Stieltjes integrals in the latter.

(*b*) p is a complex of numbers $(x_1, x_2, \cdots, x_n)$ or a point in a finite number of dimensions. Again T_0 is the class of continuous functions in a fundamental interval which may have infinite bounds. Again we obtain the Radon or Young-Stieltjes integral as an S-integral. The modular integral $\int f(p) \mid dv(p) \mid$ is an I-integral.

In place of continuous functions we may take T_0 to be the class of step-functions (functions constant over each of a finite number of sub-intervals), or else polygonal functions, or the class of polynomials together with their combinations by logical addition and multiplication. The resulting integral is the same. The Fréchet and Moore integrals (see references above) are also special instances.

(*c*) To show that our analysis applies also to integrals of a really new kind we may consider a particular example, namely

$$\int_0^1 f(x) d \log x.$$

Here $\log x$ is not a function of limited variation. Take T_0 as the class of functions $f(x)$ such that $f(x)/x$ is continuous. Substitute $t = -\log x$, and the integral may be defined as

$$\int_0^\infty [f(e^{-t})e^t]e^{-t}dt.$$

Since $f(x)/x$ is continuous,

$$|f(e^{-t})e^t| \leqq \text{some } K,$$

or the integral is absolutely convergent.

An example is interesting where a linear functional operation satisfies (C) (A) (P) but not (L), and therefore is *not* an instance. Suppose T_0 is the class of step functions $f(x)$ defined in the interval $0 \leqq x \leqq 1$, and let c be a number between 0 and 1, then we can define

$$U(f) = \lim_{\substack{\epsilon \doteq 0 \\ \epsilon > 0}} f(c - \epsilon).$$

This $U(f)$ satisfies (C) (A) (P), but consider

$$\begin{aligned} f_n(x) &= 0, \qquad 0 \leqq x \leqq c - \frac{1}{n}, \\ &= 1, \qquad c - \frac{1}{n} < x < c, \\ &= 0, \qquad c \leqq x \leqq 1. \end{aligned}$$

Evidently $f_1 \geqq f_2 \geqq \cdots$, and $\lim f_n(x) = 0$ for all x. But $U(f_n) = 1$ for all n, or, $\lim U(f_n) = 1$ instead of 0.

2. **Relations between $S(f)$ and $I(f)$.**—In this and the next two paragraphs all functions mentioned will be of class T_0.

2(1). If $f(p) \leqq g(p)$, $I(f) \leqq I(g)$. For $g - f$ is of class T_0 and $g(p) - f(p) \geqq 0$.

$$\therefore \quad I(g) - I(f) = I(g-f) \qquad \text{[By } (C), (A)$$

$$\geqq 0 \qquad \text{[By } (P).$$

$|f| = f \vee (-f)$ must be of class T_0, then in condition (M), $I(|f|)$ satisfies the conditions to be satisfied by $M(|f|)$.

2(2). $$|I(f)| \leqq I(|f|).$$
For
$$-|f| \leqq f \leqq +|f|.$$
$$\therefore \quad -I(|f|) \leqq I(f) \leqq I(|f|) \qquad \text{By 2(1).}$$

Then $I(f)$ satisfies the condition (M) and it already satisfies (C) (A) (L), hence any I-integral is an S-integral.

3. If $f(p) \geqq 0$ for all p we define $I_1(f)$ *as the upper bound of* $S(\varphi)$ *for all functions* φ *of class* T_0 *such that* $0 \leqq \varphi \leqq f$.

This upper bound exists for

$$S(\varphi) \leqq M(|\varphi|) \qquad \text{by } (M).$$

$$M(|\varphi|) = M(\varphi) \qquad \text{for } 0 \leqq \varphi,$$

$$\leqq M(f) \qquad \text{for } \varphi \leqq f.$$

$$\therefore \quad S(\varphi) \leqq M(f).$$

$I_1(f)$ is called the positive integral associated with S.

3(1). If $f(p) \geqq 0$ for all p, $I_1(f) \geqq 0$. $0 = 0 \cdot f$ is a function of class T_0 and $S(0) = 0$. But 0 is one of the functions φ by which $I_1(f)$ is defined.

$$\therefore \quad I_1(f) \geqq 0.$$

Thus $I_1(f)$ satisfies condition (P).

3(2). If $f(p) \geqq 0$ for all p and c is a positive constant,

$$I_1(cf) = cI_1(f).$$

For if $0 \leqq \varphi \leqq f$, $0 \leqq c\varphi \leqq cf$, and vice versa. Also $S(c\varphi) = cS(\varphi)$.

3(3). If $f_1(p) \geqq 0$, $f_2(p) \geqq 0$ for all p, $I_1(f_1 + f_2) = I_1(f_1) + I_1(f_2)$. Firstly, $I_1(f_1 + f_2) \geqq I_1(f_1) + I_1(f_2)$. For if $0 \leqq \varphi_1 \leqq f_1$, $0 \leqq \varphi_2 \leqq f_2$, $0 \leqq \varphi_1 + \varphi_2 \leqq f_1 + f_2$.

$$\therefore \quad I_1(f_1 + f_2) \geqq S(\varphi_1 + \varphi_2), \quad \text{or} \quad S(\varphi_1) + S(\varphi_2).$$

But this is true however we vary φ_1, φ_2, hence

$$I_1(f_1 + f_2) \geqq I_1(f_1) + I_1(f_2).$$

Secondly,

$$I_1(f_1 + f_2) \leqq I_1(f_1) + I_1(f_2).$$

Let φ be any function such that $0 \leqq \varphi \leqq f_1 + f_2$, then $\varphi - f_1 \leqq f_2$,

$$\therefore \quad (\varphi - f_1) \vee 0 \leqq f_2,$$

$$\varphi = \varphi \wedge f_1 + (\varphi - f_1) \vee 0.$$

For if $\varphi(p) \geqq f_1(p)$ for a particular p, on the right-hand side the first term is f_1 and the second $\varphi - f_1$, while if $\varphi(p) \leqq f_1(p)$ the first term is φ and the second 0. Now

$$0 \leqq \varphi \wedge f_1 \leqq f_1 \qquad 0 \leqq (\varphi - f_1) \vee 0 \leqq f_2.$$

$$\therefore \quad S(\varphi \wedge f_1) \leqq I_1(f_1), \qquad S[(\varphi - f_1) \vee 0] \leqq I_1(f_2).$$

$$\therefore \quad S(\varphi) \leqq I_1(f_1) + I_1(f_2).$$

But we may vary φ in any way so long as $0 \leqq \varphi \leqq f_1 + f_2$.

$$\therefore \quad I_1(f_1 + f_2) \leqq I_1(f_1) + I_1(f_2).$$

From what has been proved firstly and secondly it follows that

$$I_1(f_1 + f_2) = I_1(f_1) + I_1(f_2).$$

3(4). If $f_1 \geqq f_2 \geqq \cdots$ and $\lim f_n = 0$ for all p, $\lim \mathrm{I}_1(f_n) = 0$. $I_1(f_n)$ is defined as the upper bound of $S(\varphi_n)$, where $0 \leqq \varphi_n \leqq f_n$. Hence given any positive e we can find φ_n $(0 \leqq \varphi_n \leqq f_n)$ such that

$$I_1(f_n) < S(\varphi_n) + 2^{-n}e.$$

Lemma. Given that

$$I_1(f_{n-1}) < S(\psi_{n-1}) + e_{n-1}, \qquad I_1(f_n) < S(\varphi_n) + 2^{-n}e,$$

and that

$$f_n \leqq f_{n-1}, \qquad 0 \leqq \psi_{n-1} \leqq f_{n-1}, \qquad 0 \leqq \varphi_n \leqq f_n.$$

Then

$$I_1(f_n) < S(\psi_{n-1} \wedge \varphi_n) + e_{n-1} + 2^{-n}e.$$

Since

$$0 \leqq \psi_{n-1} \leqq f_{n-1}, \qquad 0 \leqq \varphi_n \leqq f_n \leqq f_{n-1}, \qquad 0 \leqq (\psi_{n-1} \vee \varphi_n) \leqq f_{n-1}.$$

$$\therefore \quad S(\psi_{n-1} \vee \varphi_n) \leqq I_1(f_{n-1}) < S(\psi_{n-1}) + e_{n-1},$$

$$\psi_{n-1} \wedge \varphi_n = \psi_{n-1} + \varphi_n - (\psi_{n-1} \vee \varphi_n) \qquad \text{[By 1(1).}$$

$$\therefore \quad S(\psi_{n-1} \wedge \varphi_n) = S(\psi_{n-1}) + S(\varphi_n) - S(\psi_{n-1} \vee \varphi_n) > S(\varphi_n) - e_{n-1}.$$

$$\therefore \quad I_1(f_n) < S(\varphi_n) + 2^{-n}e < S(\psi_{n-1} \wedge \varphi_n) + e_{n-1} + 2^{-n}e,$$

which proves the lemma.

For each f_n in the sequence choose the appropriate φ_n so that

$$I_1(f_n) < S(\varphi_n) + 2^{-n}e.$$

Let

$$\psi_1 = \varphi_1, \qquad \psi_n = \psi_{n-1} \wedge \varphi_n \quad (n = 2, 3, \cdots),$$

$$e_1 = e/2, \qquad e_n = e_{n-1} + 2^{-n}e \quad (n = 2, 3, \cdots).$$

Then, using the Lemma successively with $n = 2, 3, \cdots$, we get

$$I_1(f_n) < S(\psi_n) + e_n, \quad e_n = e(1/2 + 1/4 + \cdots + 2^{-n}) < e.$$

$$\therefore \quad I_1(f_n) < S(\psi_n) + e.$$

Moreover $0 \leqq \psi_n \leqq \varphi_n \leqq f_n$, and $\lim f_n = 0$, therefore $\lim \psi_n = 0$.

$$\psi_1 \geqq \psi_2 \geqq \cdots \text{ and } \lim \psi_n = 0.$$

$$\therefore \quad \lim S(\psi_n) = 0 \qquad \text{[By } (L).$$

$$\therefore \quad \lim I_1(f_n) \leqq e.$$

But e was any positive quantity.

$$\therefore \quad \lim I_1(f_n) = 0.$$

Hence $I_1(f)$ satisfies the condition (L).

If $f = \varphi - \psi$, where $\varphi \geqq 0$, $\psi \geqq 0$, by definition

$$I_1(f) = I_1(\varphi) - I_1(\psi).$$

This definition is self-consistent, for if

$$f = \varphi_1 - \psi_1 = \varphi_2 - \psi_2, \qquad \varphi_1 + \psi_2 = \varphi_2 + \psi_1,$$

$$I_1(\varphi_1) + I_1(\psi_2) = I_1(\varphi_2) + I_1(\psi_1) \qquad \text{[By 3(3).}$$

$$\therefore \quad I_1(\varphi_1) - I_1(\psi_1) = I_1(\varphi_2) - I_1(\psi_2).$$

$I_1(f)$ satisfies condition (C). For if c is positive

$$I_1(cf) = I_1(c\varphi) - I_1(c\psi) = cI_1(\varphi) - cI_1(\psi) \qquad \text{[By 3(2)}$$

$$= cI_1(f).$$

If c is negative,

$$I_1(cf) = I_1(-c\psi) - I_1(-c\varphi)$$

$$= -cI_1(\psi) - [-cI_1(\varphi)] \qquad \text{[By 3(2)}$$

$$= cI_1(f).$$

If $c = 0$, $I_1(cf) = I_1(0) = 0 = 0 \cdot I_1(f)$.

$I_1(f)$ satisfies condition (A). For we have

$$\begin{aligned} I_1(f_1 + f_2) &= I_1(\varphi_1 + \varphi_2) - I_1(\psi_1 + \psi_2) \\ &= I_1(\varphi_1) + I_1(\varphi_2) - I_1(\psi_1) - I_1(\psi_2) \qquad \text{[By 3(3)} \\ &= I_1(f_1) + I_1(f_2). \end{aligned}$$

We have already proved that $I_1(f)$ also satisfies conditions (L) (P). Hence $I_1(f)$ is an I-integral.

4. We define further

$$I_2(f) = I_1(f) - S(f),$$

the negative integral associated with S. Also we define

$$I(f) = I_1(f) + I_2(f) = 2I_1(f) - S(f),$$

the modular integral associated with S. Evidently $I_2(f)$, $I(f)$ satisfy all the conditions for I-integrals. Another definition of the modular integral $I(f)$ when f is non-negative is the upper bound of $S(\varphi)$ for all functions φ of class T_0 such that $-f \leqq \varphi \leqq f$. For then

$$0 \leqq \varphi + f \leqq 2f, \quad S(\varphi) = S(\varphi + f) - S(f).$$

Varying φ, keeping f fixed, we see that the upper bound of $S(\varphi)$ is equal to

$$I_1(2f) - S(f) = 2I_1(f) - S(f) = I(f).$$

4(1). $|S(f)| \leqq I(|f|)$, if I is the modular integral associated with S.

For

$$\begin{aligned} |S(f)| = |I_1(f) - I_2(f)| &\leqq |I_1(f)| + |I_2(f)| \\ &\leqq I_1(|f|) + I_2(|f|) \text{[} \quad \text{By 2(2)} \\ &\text{or } I(|f|). \end{aligned}$$

It can be seen that if we extend the definitions of $I_1(f)$, $I_2(f)$ to functions of a wider class T so as still to satisfy $(C)(A)$ (L) (P), and if we define

$$S(f) = I_1(f) - I_2(f), \quad I(f) = I_1(f) + I_2(f),$$

$S(f)$ will satisfy (C) (A) (L) (M) and $I(f)$ will satisfy (C) (A) (L) (P).

5. **Extension to class T_1 for any I-integral.** If $f_1 \leqq f_2 \leqq \cdots$ is a non-decreasing sequence of functions of class T_0, $\lim f_n$ exists (if we allow $+\infty$ as a value) and we say that $\lim f_n = f$ *is of class* T_1.

Then

$$I(f_1) \leqq I(f_2) \leqq \cdots,$$

and $\lim I(f_n)$ exists (if we allow $+\infty$ as a value).

5(1). If $f_1 \leqq f_2 \leqq \cdots$ are of class T_0, and if $\lim f_n \geqq$ some function h of class T_0, $\lim I(f_n) \geqq I(h)$. Let $g_n = f_n \wedge h$. Then g_n is of class T_0 and since $\lim f_n \geqq h$, $\lim g_n = h$. Also $f_{n-1} \leqq f_n$, therefore $g_{n-1} \leqq g_n$.

$$\therefore \quad h - g_1 \geqq h - g_2 \geqq \cdots \text{ and } \lim (h - g_n) = 0.$$

$$\therefore \quad \lim I(h - g_n) = 0 \qquad \text{[By } (L).$$

$$\therefore \quad \lim I(g_n) = I(h).$$

But $f_n \geqq g_n$ or $I(f_n) \geqq I(g_n)$ [By 2(1).

$$\therefore \quad \lim I(f_n) \geqq \lim I(g_n) \text{ or } I(h).$$

5(2). If $f_1 \leqq f_2 \leqq \cdots$, $g_1 \leqq g_2 \leqq \cdots$ are of class T_0, and if

$$\lim f_n \geqq \lim g_n, \text{ then } \lim I(f_n) \geqq \lim I(g_n).$$

For $\lim f_n$ is of class T_1 and $\geqq \lim g_n$, and therefore $\geqq g_n$.

$$\therefore \quad \lim I(f_n) \geqq I(g_n) \quad (n = 1, 2, \cdots) \qquad \text{[By 5(1).}$$

$$\therefore \quad \lim I(f_n) \geqq \lim I(g_n).$$

5(3). If $f_1 \leqq f_2 \leqq \cdots$, $g_1 \leqq g_2 \leqq \cdots$ are of class T_0 and if

$$\lim f_n = \lim g_n, \text{ then } \lim I(f_n) = \lim I(g_n).$$

Apply 5(2) twice.

We define $I(f) = \lim I(f_n)$, *if f is of class T_1 and defined by the nondecreasing sequence (f_n) of functions of class T_0*. By 5(3) this definition will be self-consistent. By 5(1), if $f \geqq 0$, $I(f) \geqq 0$. Hence condition (P) is satisfied. Evidently conditions (A) (C) will be satisfied so long as in (C) the constant c is positive.

Note. We have allowed $\lim I(f_n)$ to be $+\infty$ and this necessitates a reconsideration of the above theorems. In 5(1), h is of class T_0 and therefore $I(h)$ is finite, or the statement will hold even if $\lim I(f_n) = +\infty$. In 5(2) the theorem must be taken to mean, in the case where either $\lim I(f_n)$ or $\lim I(g_n)$ is $+\infty$, that at least $\lim I(f_n)$ is $+\infty$. In 5(3) if either limit is $+\infty$ so is the other.

If $I(f) = \lim I(f_n)$ is finite and f is of class T_1 we say that *f is summable*.

5(4). If $f_1 \leqq f_2 \leqq \cdots$ is a nondecreasing sequence of functions of class T_1, then $\lim f_n = f$ is also of class T_1 and $I(f) = \lim I(f_n)$. For any integer r, f_r is of class T_1 and is the limit of a nondecreasing sequence of functions of class T_0,

$$f_{r,1} \leqq f_{r,2} \leqq \cdots \leqq f_{r,s} \leqq \cdots.$$

Let g_n be the logical sum of all functions $f_{r,s}$ for which $r \leqq n$, $s \leqq n$. Then g_n is of class T_0 and $g_n \leqq g_{n+1}$.

$$\therefore \quad \lim g_n \text{ is of class } T_1.$$

If $r \leqq n$, $f_{r,s} \leqq f_r \leqq f_n$.

$$\therefore \quad g_n \leqq f_n \quad \text{and} \quad I(g_n) \leqq I(f_n).$$

If $r \leqq n$, $g_n \geqq f_{r,n}$

$$\therefore \quad \lim g_n \geqq \lim f_{r,n} \text{ or } f_r.$$

$$\therefore \quad \lim g_n \geqq f_r \quad (r = 1, 2, \cdots)$$

and

$$I(\lim g_n) \geqq I(f_r).$$

$$g_n \leqq f_n, \qquad \lim g_n \leqq \lim f_n.$$

$$\lim g_n \geqq f_r.$$

$$\therefore \quad \lim g_n \geqq \lim f_n.$$

$$\therefore \quad \lim g_n = \lim f_n.$$

But $\lim g_n$ is of class T_1, therefore $\lim f_n$ is of class T_1.

$$I(f) = I(\lim g_n) \geqq I(f_r) \text{ for all } r.$$

$$\therefore \quad I(f) \geqq \lim I(f_n).$$

$$I(f) = I(\lim g_n) \leqq I(g_n).$$

But

$$I(g_n) \leqq I(f_n).$$

$$\therefore \quad I(f) \leqq \lim I(f_n).$$

$$\therefore \quad I(f) = \lim I(f_n).$$

6. **Semi-integrals.** For any function f we define $\dot{I}(f)$, *the upper semi-integral of f, as the lower bound of $I(\varphi)$ for all functions φ of class T_1, such that $\varphi \geqq f$.*

6(1). If c is a positive constant, $\dot{I}(cf) = c\dot{I}(f)$. For if $\varphi \geqq f$, $c\varphi \geqq cf$ and vice versa, and $I(c\varphi) = cI(\varphi)$.

6(2) $$\dot{I}(f_1 + f_2) \leqq \dot{I}(f_1) + \dot{I}(f_2).$$

For if φ_1, φ_2 are any functions of class T_1 such that $\varphi_1 \geqq f_1$, $\varphi_2 \geqq f_2$,

$$\varphi_1 + \varphi_2 \geqq f_1 + f_2.$$

$$\therefore \quad \dot{I}(f_1 + f_2) \leqq I(\varphi_1 + \varphi_2), \text{ or } I(\varphi_1) + I(\varphi_2).$$

Varying φ_1, φ_2 independently, we obtain the theorem.

6(3). If $f \leqq g$ for all p,

$$\dot{I}(f) \leqq \dot{I}(g).$$

For if φ is any function of class T_1 such that $\varphi \geqq g$, $\varphi \geqq f$.

$$\dot{I}(f) \leqq I(\varphi), \text{ etc.}$$

6(4). We define $\underset{\cdot}{I}(f) = -\dot{I}(-f)$; then $\underset{\cdot}{I}(f)$ is called *the lower semi-integral of* f, and $\underset{\cdot}{I}(f) \leqq \dot{I}(f)$. For by 6(2)

$$0 = \dot{I}(0) = \dot{I}(f - f) \leqq \dot{I}(f) + \dot{I}(-f) \text{ or } \dot{I}(f) - \underset{\cdot}{I}(f).$$

6(5) $$\dot{I}(f \vee g) + \dot{I}(f \wedge g) \leqq \dot{I}(f) + \dot{I}(g).$$

For if φ_1, φ_2 are any functions of class T_1 such that $\varphi_1 \geqq f$, $\varphi_2 \geqq g$.

$$\varphi_1 \vee \varphi_2 \geqq f \vee g. \qquad \varphi_1 \wedge \varphi_2 \geqq f \wedge g.$$

$$\therefore \quad \dot{I}(f \vee g) + \dot{I}(f \wedge g) \leqq I(\varphi_1 \vee \varphi_2) + I(\varphi_1 \wedge \varphi_2) \quad \text{or} \quad I(\varphi_1) + I(\varphi_2).$$

For

$$\varphi_1 \vee \varphi_2 + \varphi_1 \wedge \varphi_2 = \varphi_1 + \varphi_2.$$

Varying φ_1, φ_2 independently, we obtain the theorem.

Corollary.

$$\dot{I}(|f|) - \underset{\cdot}{I}(|f|) \leqq \dot{I}(f) - \underset{\cdot}{I}(f).$$

For

$$|f| = f \vee (-f), \qquad -|f| = f \wedge (-f).$$

$$\therefore \quad \dot{I}(|f|) + \dot{I}(-|f|) \leqq \dot{I}(f) + \dot{I}(-f).$$

7. **Summability.** If $\dot{I}(f) = \underset{\cdot}{I}(f) =$ finite, f *is said to be summable,* and we define

$$I(f) = \dot{I}(f) = \underset{\cdot}{I}(f).$$

7(1). If f is summable and $f \geqq 0$ for all p, $I(f) \geqq 0$. For by 6(3), $\dot{I}(f) \geqq \dot{I}(0)$ or 0.

7(2). If c is any constant and f is summable, cf is summable and $I(cf) = cI(f)$.

If c is positive,

$$\dot{I}(cf) = c\dot{I}(f) = cI(f) \qquad \text{[By 6(1).}$$

$$-\underset{\cdot}{I}(cf) = \dot{I}(-cf) = cI(-f) = -cI(f).$$

$$\therefore \quad \dot{I}(cf) = \underset{\cdot}{I}(cf) = cI(f).$$

If c is negative,

$$\dot{I}(cf) = -c\dot{I}(-f) \qquad \text{[By 6(1)}$$

$$= cI(f)$$

$$-\underset{\cdot}{I}(cf) = \dot{I}(-cf) = -c\dot{I}(f). \quad \therefore \text{ etc.}$$

7(3). If f_1, f_2 are summable, so is $f_1 + f_2$ and $I(f_1 + f_2) = I(f_1) + I(f_2)$. For

$$\dot{I}(f_1 + f_2) \leqq \dot{I}(f_1) + \dot{I}(f_2) \qquad \text{[By 6(2),}$$

$$\text{or } I(f_1) + I(f_2).$$

$$-\underset{\cdot}{I}(f_1+f_2) = \dot{I}(-f_1-f_2) \leqq \dot{I}(-f_1)+\dot{I}(-f_2).$$

$$\text{or } -I(f_1)-I(f_2).$$

$$\therefore\quad \underset{\cdot}{I}(f_1+f_2) \geqq I(f_1)+I(f_2)$$

But

$$\dot{I}(f_1+f_2) \geqq \underset{\cdot}{I}(f_1+f_2). \quad \therefore \quad \text{etc.}$$

7(4). If f is summable, so is $|f|$ and $|I(f)| \leqq I(|f|)$. By 6(5) Cor.

$$\dot{I}(|f|) - \underset{\cdot}{I}(|f|) \leqq \dot{I}(f) - \underset{\cdot}{I}(f) = 0.$$

But $\dot{I}(|f|) \geqq \underset{\cdot}{I}(|f|)$. Hence the first part follows. Moreover,

$$-|f| \leqq f \leqq |f|.$$

$$\begin{aligned} \therefore\quad -I(|f|) &= \dot{I}(-|f|) \\ &\leqq \dot{I}(f) \text{ or } I(f) \\ &\leqq \dot{I}(|f|) \text{ or } I(|f|). \end{aligned}$$

The second part follows immediately.

7(5). If f_1, f_2 are summable, so are $f_1 \vee f_2, f_1 \wedge f_2$. For by 6(5)

$$\dot{I}(f_1 \vee f_2) + \dot{I}(f_1 \wedge f_2) \leqq \dot{I}(f_1) + \dot{I}(f_2), \text{ or } I(f_1)+I(f_2).$$

If we replace f_1 by $-f_1$, f_2 by $-f_2$, we shall replace $f_1 \vee f_2$ by $-(f_1 \wedge f_2)$ and $f_1 \wedge f_2$ by $-(f_1 \vee f_2)$.

$$\therefore\quad \dot{I}[-(f_1 \wedge f_2)] + \dot{I}[-(f_1 \vee f_2)] \leqq \dot{I}(-f_1) + \dot{I}(-f_2),$$

$$\therefore\quad \underset{\cdot}{I}(f_1 \wedge f_2) + \underset{\cdot}{I}(f_1 \vee f_2) \geqq \underset{\cdot}{I}(f_1) + \underset{\cdot}{I}(f_2), \text{ or } I(f_1)+I(f_2).$$

$$\therefore\quad \dot{I}(f_1 \vee f_2) - \underset{\cdot}{I}(f_1 \vee f_2) + \dot{I}(f_1 \wedge f_2) - \underset{\cdot}{I}(f_1 \wedge f_2) \leqq 0.$$

But each of these differences is non-negative, therefore they are both zero.

7(6). If $f_1 \leqq f_2 \leqq \cdots$ is a nondecreasing sequence of summable functions, and if $\lim I(f_n)$ is finite, $\lim f_n = f$ is summable, and $I(f) = \lim I(f_n)$; while if $\lim I(f_n) = +\infty$, $\underset{\cdot}{I}(f) = +\infty$. For $-f \leqq -f_n$.

$$\therefore\quad \dot{I}(-f) \leqq I(-f_n) \qquad \text{[By 6(3).}$$

$$\underset{\cdot}{I}(f) \geqq I(f_n) \quad (n = 1, 2, \cdots).$$

$$\therefore\quad \underset{\cdot}{I}(f) \geqq \lim I(f_n).$$

This proves the last part of the theorem. Given any positive e, we can choose $\varphi_1, \varphi_2, \cdots$ of class T_1 such that

$$\varphi_1 \geqq f_1, \qquad \varphi_2 \geqq f_2 - f_1, \qquad \varphi_3 \geqq f_3 - f_2, \quad \cdots,$$

and so that

$$I(\varphi_1) < I(f_1) + e/2,$$
$$I(\varphi_2) < I(f_2 - f_1) + e/4,$$
$$\cdots\cdots\cdots$$

If $n \geqq 2$, $\varphi_n \geqq f_n - f_{n-1} \geqq 0$. We define $\psi_n = \varphi_1 + \varphi_2 + \cdots + \varphi_n$. Then ψ_n is of class T_1 and $\psi_1 \leqq \psi_2 \leqq \cdots$. By 5(4), $\lim \psi_n$ is of class T_1 and $I(\lim \psi_n) = \lim I(\psi_n)$. $\psi_n \geqq f_n$, or $\lim \psi_n \geqq \lim f_n$ or f.

$$\therefore \quad \dot{I}(f) \leqq I(\lim \psi_n) \qquad \text{[By 6(3),}$$
$$\text{or } \lim I(\psi_n).$$

$$\begin{aligned} I(\psi_n) &= I(\varphi_1) + I(\varphi_2) + \cdots + I(\varphi_n) \\ &< I(f_1) + I(f_2 - f_1) + \cdots + I(f_n - f_{n-1}) \\ &\quad + e(\tfrac{1}{2} + \tfrac{1}{4} + \cdots + 2^{-n}) \\ &< I(f_n) + e. \end{aligned}$$

$$\therefore \quad \lim I(\psi_n) \leqq \lim I(f_n) + e.$$
$$\therefore \quad \dot{I}(f) \leqq \lim I(f_n) + e.$$

But e is any positive quantity.

$$\therefore \quad \dot{I}(f) \leqq \lim I(f_n).$$

We have already shown that $\underset{\cdot}{I}(f) \geqq \lim I(f_n)$.

$$\therefore \quad \dot{I}(f) = \underset{\cdot}{I}(f) = \lim I(f_n).$$

$\therefore$ if $\lim I(f_n)$ exists, f is summable and $I(f) = \lim I(f_n)$.

7(7). If $f_1, f_2, \cdots$ is a sequence of summable functions with limit f, and if a summable function φ exists such that $|f_n| \leqq \varphi$ for all n, f is summable, $\lim I(f_n)$ exists and $= I(f)$. We must recall the method whereby the limit of a sequence is obtained. Let $g_{r,s}$ be the logical sum of $f_r, f_{r+1}, \cdots, f_{r+1}$; then $g_{r,s} \leqq g_{r,s+1} \leqq \cdots$ with limit g_r. Then $g_r \geqq g_{r+1} \geqq \cdots$, and $\lim g_r = f$, if $\lim f_n = f$. Similarly we let $h_{r,s}$ be the logical product of $f_r, f_{r+1}, \cdots f_{r+s}$, and then $h_{r,s} \geqq h_{r,s+1} \geqq \cdots$ with limit h_r. Then $h_r \leqq h_{r+1} \leqq \cdots$, and $\lim h_r = f$ if $\lim f_n = f$. f_n is summable for all n, therefore by 7(5), $g_{r,s}$ is summable. Since $f_n \leqq \varphi$, $g_{r,s} \leqq \varphi_1$. $\therefore$ $I(g_{r,s}) \leqq I(\varphi)$. Therefore g_r is summable by 7(6). Again $g_{r,s} \geqq -\varphi$ or $-g_{r,s} \leqq \varphi$. $\therefore$ $-g_r \leqq \varphi$. $\therefore$ $I(-g_r) \leqq I(\varphi)$. But

$$-g_r \leqq -g_{r+1} \leqq \cdots$$

with limit $-f$. Therefore $-f$ is summable by 7(6) and

$$I(-f) = \lim I(-g_r).$$

Therefore by 7(2), f is summable and $I(f) = \lim I(g_r)$. Given any positive e we can find r_1, so that

$$I(g_r) < I(f) + e \quad (r \geqq r_1).$$

Now

$$f_r \leqq g_{r,s} \leqq g_r.$$

$$\therefore \quad I(f_r) \leqq I(g_r)$$

$$< I(f) + e \quad (r \geqq r_1).$$

Similarly we can prove that h_r is summable and $I(f) = \lim I(h_r)$. We can find r_2 so that

$$I(h_r) > I(f) - e \quad (r \geqq r_2).$$

$$f_r \geqq h_{r,s} \geqq h_r.$$

$$\therefore \quad I(f_r) \geqq I(h_r)$$

$$> I(f) - e \quad (r \geqq r_2).$$

Therefore if r_0 is the greater of r_1 and r_2,

$$|I(f_r) - I(f)| < e \quad (r \geqq r_0).$$

Hence lim $I(f_n)$ exists and equals $I(f)$. From these theorems it follows that $I(f)$ satisfies the conditions (C) (A) (L) (P), where the functions now belong to the class of summable functions.

8. **S-integrals.** Associated with any S-integral $S(f)$ for functions of class T_0 we have three I-integrals, namely $I_1(f)$, $I_2(f)$, $I(f)$, such that

$$S(f) = I_1(f) - I_2(f), \quad I(f) = I_1(f) + I_2(f).$$

If we extend our definitions of the I-integrals to functions of class T_1, we shall still have

$$I(f) = I_1(f) + I_2(f).$$

8(1). If f is any function,

$$\dot{I}_1(f) + \dot{I}_2(f) = \dot{I}(f).$$

For if φ is any function of class T_1 such that $\varphi \geqq f$,

$$\dot{I}_1(f) + \dot{I}_2(f) \leqq I_1(\varphi) + I_2(\varphi), \text{ or } I(\varphi).$$

Varying φ, we obtain

$$\dot{I}_1(f) + \dot{I}_2(f) \leqq \dot{I}(f).$$

Again given any positive e we can choose φ_1, φ_2 of class T_1 so that $\varphi_1 \geqq f$, $\varphi_2 \geqq f$ and $\dot{I}_1(f) > I_1(\varphi_1) - \frac{1}{2}e$, $\dot{I}_2(f) > I_2(\varphi_2) - \frac{1}{2}e$. Let $\psi = \varphi_1 \wedge \varphi_2$, then $\psi \leqq \varphi_1$ and $\psi \leqq \varphi_2$.

$$\therefore \quad I_1(\varphi_1) + I_2(\varphi_2) \geqq I_1(\psi) + I_2(\psi), \text{ or } I(\psi).$$

$$\therefore \quad \dot{I}_1(f) + \dot{I}_2(f) > I(\psi) - e.$$

But $\psi \geqq f$. $\therefore \quad I(\psi) \geqq \dot{I}(f)$.

$$\therefore \quad \dot{I}_1(f) + \dot{I}_2(f) > \dot{I}(f) - e.$$

This is true for any positive e.

$$\therefore \quad \dot{I}_1(f) + \dot{I}_2(f) \geqq \dot{I}(f).$$

But we have already proved that

$$\dot{I}_1(f) + \dot{I}_2(f) \leqq \dot{I}(f). \quad \therefore \text{ etc.}$$

8(2). If f is summable (I), it is summable (I_1) and (I_2), and

$$I(f) = I_1(f) + I_2(f).$$

For

$$\dot{I}_1(f) + \dot{I}_2(f) = \dot{I}(f) = I(f).$$

$$\dot{I}_1(-f) + \dot{I}_2(-f) = \dot{I}(-f) = -I(f).$$

$$\therefore \quad \underset{\cdot}{I}_1(f) + \underset{\cdot}{I}_2(f) = I(f).$$

$$\therefore \quad \dot{I}_1(f) - \underset{\cdot}{I}_1(f) + \dot{I}_2(f) - \underset{\cdot}{I}_2(f) = 0.$$

But each of these differences is non-negative, therefore each must be zero separately. Then f is summable (I_1) and (I_2) and

$$I(f) = \dot{I}_1(f) + \dot{I}_2(f) = I_1(f) + I_2(f).$$

f is said to be summable (S) if, and only if, it is summable (I), where I is the modular integral associated with S. Hence if f is summable (S), it is summable (I_1) and (I_2) by 8(2).

We define

$$S(f) = I_1(f) - I_2(f).$$

Then $S(f)$ satisfies all the conditions (C) (A) (L) (M) for functions summable (S).

Many of the theorems already obtained for the I-integral can be immediately stated also for the S-integral.

Thus 7(2, 3, 5, 7) are true if we replace I everywhere by S. 7(4) becomes:

If f is summable (S), so is $|f|$, and

$$|S(f)| \leqq I(|f|),$$

where I is the modular integral associated with S. 7(6) becomes:

If $f_1 \leqq f_2 \leqq \cdots$ are summable (S) and if $\lim I(f_n)$ exists, where I is the associate modular integral, then $\lim f_n = f$ is summable (S) and

$$S(f) = \lim S(f_n).$$

8(3). The necessary and sufficient condition that f be summable (S) is that given any positive e there exists a function f_e of class T_0 such that

$$\dot{I}(|f - f_e|) < e,$$

where I is the modular integral associated with S.

Also in this case

$$S(f) = \lim_{e \doteq 0} S(f_e).$$

The condition is sufficient for

$$f = f_e + f - f_e \leqq f_e + |f - f_e|.$$

$$\therefore \quad \dot{I}(f) \leqq I(f_e) + \dot{I}(|f - f_e|) < I(f_e) + e.$$

Similarly

$$\dot{I}(-f) < I(-f_e) + e,$$

$$\underset{\cdot}{I}(f) > I(f_e) - e.$$

$$\therefore \quad \dot{I}(f) - \underset{\cdot}{I}(f) < 2e.$$

This is true for any positive e, therefore $\dot{I}(f) = \underset{\cdot}{I}(f)$. They are also finite for $I(f_e)$ and e are finite. Also

$$|S(f) - S(f_e)| = |S(f - f_e)| \leqq I(|f - f_e|) < e.$$

$$\therefore \quad S(f) = \lim S(f_e).$$

The condition is necessary, for given any positive e there exists a summable function φ of class T_1 such that $\varphi \geqq f$ and

$$\dot{I}(f) \leqq I(\varphi) < \dot{I}(f) + \tfrac{1}{2}e.$$

$$\dot{I}(|\varphi - f|) = \dot{I}(\varphi - f)$$

(since $\varphi \geqq f$)

$$\leqq I(\varphi) + \dot{I}(-f), \quad \text{or} \quad I(\varphi) - \underset{\cdot}{I}(f), \quad \text{or} \quad \dot{I}(\varphi) - \dot{I}(f),$$

for f is summable.

$$\therefore \quad \dot{I}(|\varphi - f|) < \tfrac{1}{2}e.$$

φ is summable and of class T_1, therefore there exists a function f_e of class T_0 such that $f_e \leqq \varphi$ and

$$I(f_e) > I(\varphi) - \tfrac{1}{2}e.$$

Then

$$I(|\varphi - f_e|) < \tfrac{1}{2}e.$$

$$\therefore \quad I(|f - f_e|) \leqq \dot{I}(|\varphi - f| + |\varphi - f_e|)$$

$$\leqq \dot{I}(|\varphi - f|) + \dot{I}(|\varphi - f_e|) < e.$$

Evidently the condition for summability, or the class of summable functions depends both on the operation S and on the class T_0.

9. **Measure.** It is usual, though not necessary, to define the integral in terms of the measure of certain fundamental sets. Let us suppose that the measures of a certain class of elementary, or initial sets, or collections E, of the p are given. In connection with a collection E we can define a function $= 1$ when p belongs to E, $= 0$ otherwise. We can agree to call the measure of E, the integral of the corresponding function. The class T_0 is then taken as the class of all functions which are linear combinations of these elementary set-functions. It will then be closed with respect to the operations (C) (A). For any set E whatever we can say that it is measurable if the corresponding function is summable, and we can identify its measure with the integral of that function. This question requires however a separate and careful consideration. The author wishes to point out, without proof, a simple manner in which the Stieltjes integral can be generalized.

In the ordinary Stieltjes integral $\int_a^b f(x)d\alpha(x)$, $f(x)$ is a continuous function $(a \leqq x \leqq b)$ and $\alpha(x)$ of limited variation, that is such that

$$\sum_{i=1}^{n} | \alpha(x_i) - \alpha(x_{i-1}) | \leqq M,$$

for all subdivisions $a = x_0 < x_1 < \cdots < x_n = b$. Suppose that $\gamma(x)$ is not a function of limited variation but that

$$\sum_{i=1}^{n} x_i | \gamma(x_i) - \gamma(x_{i-1}) | \leqq M,$$

for all subdivisions $0 = x_0 < x_1 < \cdots < x_n = 1$. Then if $f(x)/x$ is continuous $(0 \leqq x \leqq 1)$, we can define

$$\int_0^1 f(x)d\gamma(x) = \int_0^1 \frac{f(x)}{x} xd\gamma(x).$$

Of course this is only a transformation of the integral, $\int_0^1 \frac{f(x)}{x} d\alpha(x)$, where $\alpha(x) = \int_0^x xd\gamma(x)$ is of limited variation. It shows nevertheless that so long as suitable restrictions are placed on the integrand, integrals similar to that of Stieltjes can be defined with respect to functions which are not of limited variation.

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