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On the b-Functions of Hypergeometric Systems

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For any integer $d \times (n+1)$ matrix A and parameter $\beta \in \mathbb{C}^d$ let $M_A(\beta)$ be the associated A-hypergeometric (or GKZ) system in the variables $x_0, \ldots, x_n$. We describe bounds for the (roots of the) b-functions of both $M_A(\beta)$ and its Fourier transform along the hyperplanes $(x_j = 0)$. We also give an estimate for the b-function for restricting $M_A(\beta)$ to a generic point.

1 Introduction

Let D be the ring of algebraic $\mathbb{C}$-linear differential operators on $\mathbb{C}^{n+1}$ with coordinates $x_0, \ldots, x_n$.

Definition 1.1 (Compare [4, 5]). Let M be a left D-module and pick an element $m \in M$ with annihilator $I \subseteq D$. If $(V^i D)$ is the vector space spanned by the monomials $x^\alpha \partial^\beta$ with $\alpha_0 - \beta_0 \geq i$ then the *b-function* of $m \in M$ along the coordinate hyperplane $x_0 = 0$ is the minimal monic polynomial $b(s)$ that satisfies: $b(x_0\partial_0) \cdot m \in (V^1 D) \cdot m$ in M, which is to say $b(x_0\partial_0) \in I + (V^1 D)$ in D.

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If M is cyclic, that is, $M = D/I$, then we call b-function of M the b-function in the above sense of the element $1 + I \in M$. □

The b-function exists in greater generality along any hypersurface ($f = 0$), as long as the module M is holonomic, cf. [4]. The V-filtration of Kashiwara and Malgrange then takes the form $(V^i D) = \{P \in D \mid f^{i+k} \text{ divides } P \bullet f^k \text{ for } k \gg 0\}$. Both the V-filtration and the b-function are intimately connected to the restriction of the given D-module to the hypersurface. The purpose of this note is to give, for any A-hypergeometric system as well as its Fourier transform, an explicit arithmetic description of a bound for the root set of the b-function along any coordinate hyperplane that involves the parameter β in a very elementary way.

We have several applications in mind: first, it is a longstanding question to understand the monodromy of A-hypergeometric systems, and for this purpose the roots of the b-function can be of some use. On the other hand, the Fourier transform of an A-hypergeometric system often (see [15]) appears as a direct image module under a natural torus embedding given by the columns of the matrix A. This point of view turns out to be extremely useful for Hodge theoretic considerations of A-hypergeometric systems (see [9]). It is one of the fundamental insights of Morihiko Saito (see [11, Section 3.2]) that the boundary behavior of variations of Hodge structures (or, more generally, of mixed Hodge modules) is controlled by the Kashiwara–Malgrange filtration along such a boundary divisor. In the case of a cyclic D-module, such as A-hypergeometric systems or their Fourier transforms, one can often deduce a large part of this filtration from the values of the b-function. We refer to [10] for an immediate application of our results. In a third direction, one can also see our calculation of the b-function of the Fourier transform as a refinement of [1, 15] geared towards restriction of A-hypergeometric systems.

In the last part, we compute an upper bound for the b-function of restriction of the A-hypergeometric system to a generic point, again in elementary terms of A and β. Since the restriction of a D-module to a point is a dual object to the zeroth level solution functor, our estimate can be viewed as a step towards a sheafification in β of the solution space, a problem that remains unsolved.

2 Basic notions and results

Notation. Throughout, the base field is $\mathbb{C}$ and we consider a $\mathbb{C}$-vector space V of dimension $n + 1$. □

In this introductory section, we review basic facts on A-hypergeometric systems as well as the Euler–Koszul functor. Readers are advised to refer to [6] for more detailed explanations.

Notation 2.1. For any integer matrix A, let R_A (respectively O_A) be the polynomial ring over $\mathbb{C}$ generated by the variables ∂_j (respectively x_j) corresponding to the columns $\mathbf{a}_j$ of A. We identify O_A with the symmetric algebra on $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \cong \bigoplus \mathbb{C} \cdot x_j$. Further, let D_A be the ring of $\mathbb{C}$-linear differential operators on O_A, where we identify $\frac{\partial}{\partial x_j}$ with ∂_j and multiplication by x_j with x_j so that both R_A and O_A become subrings of D_A. □

2.1 A-hypergeometric systems

Let $A = (\mathbf{a}_0, \ldots, \mathbf{a}_n)$ be an integer $d \times (n+1)$ matrix, $d \leq n+1$. For convenience, we assume that $\mathbb{Z}A = \mathbb{Z}^d$. For $(v_1, \ldots, v_r) = \mathbf{v} \in \mathbb{Z}^r$, we denote by $\mathbf{v}_+, \mathbf{v}_-$ the vectors given by

$$(\mathbf{v}_+)_j = \max(0, v_j) \qquad \text{and} \qquad (\mathbf{v}_-)_j = \max(0, -v_j).$$

For the complex parameter vector $\beta \in \mathbb{C}^d$ consider the system of d *homogeneity equations*

$$E_i \bullet \phi = \beta_i \cdot \phi, \tag{2.1}$$

where $E_i = \sum_{j=0}^{n} a_{i,j} x_j \partial_j$ is the i-th *Euler operator*, together with the *toric* (partial differential) *equations*

$$\{(\underbrace{\partial^{\mathbf{v}_+} - \partial^{\mathbf{v}_-}}_{:=\Delta_{\mathbf{v}}}) \bullet \phi = 0 \mid A \cdot \mathbf{v} = 0\}. \tag{2.2}$$

In R_A, the toric operators $\{\Delta_{\mathbf{v}} | A \cdot \mathbf{v} = 0\}$ generate the *toric ideal* I_A. The quotient

$$S_A := R_A / I_A$$

is naturally isomorphic to the semigroup ring $\mathbb{C}[\mathbb{N}A]$. In D_A, the left ideal generated by all equations (2.1) and (2.2) is the *hypergeometric ideal* $H_A(\beta)$. We put

$$M_A(\beta) := D_A / H_A(\beta);$$

this is the A-hypergeometric system introduced and first investigated by Gelfand, Graev, Kapranov, and Zelevinsky, in [2] and a string of other articles. ◇

2.2 A-degrees

If the rowspan of A contains $\mathbf{1}_A$ we call A *homogeneous*. Homogeneity is equivalent to I_A defining a projective variety, and also to the system $H_A(\beta)$ having only regular singularities [3, 13]. A more general *A-degree function* on R_A and D_A is induced by:

$$-\deg_A(x_j) := \mathbf{a}_j =: \deg_A(\partial_j).$$

We denote $\deg_{A,i}(-)$ the A-degree function associated to the weight given by the i-th row of A, so $\deg_A = (\deg_{A,1}, \dots, \deg_{A,d})$.

An R_A- (respectively D_A-)module M is *A-graded* if it has a decomposition $M = \bigoplus_{\alpha\in\mathbb{Z}^d} M_\alpha$ such that the module structure respects the grading $\deg_A(-)$ on R_A (respectively D_A) and M. If N is an A-graded R_A-module, then we denote $\deg_A(N) \subseteq \mathbb{Z}^d$ the set of all degrees of all non-zero homogeneous elements of N. The *quasi-degrees* $\mathrm{qdeg}_A(N)$ of N are the points in the Zariski closure in $\mathbb{C}^d$ of $\deg_A(N)$.

As is common, if M is A-graded then $M(\mathbf{b})$ denotes for each $\mathbf{b} \in \mathbb{Z}A$ its shift with graded structure $(M(\mathbf{b}))_{\mathbf{b}'} = M_{\mathbf{b}+\mathbf{b}'}$.

2.3 Euler–Koszul complex

Since

$$x^{\mathbf{u}}E_i - E_i x^{\mathbf{u}} = -(A\cdot\mathbf{u})_i x^{\mathbf{u}},$$
$$\partial^{\mathbf{u}}E_i - E_i\partial^{\mathbf{u}} = (A\cdot\mathbf{u})_i \partial^{\mathbf{u}},$$

we have

$$E_iP = P(E_i - \deg_{A,i}(P)) \tag{2.3}$$

for any A-homogeneous $P \in D_A$ and all i.

On the A-graded D_A-module M one can thus define commuting D_A-linear endomorphisms E_i via

$$E_i \circ m := (E_i + \deg_{A,i}(m)) \cdot m$$

for A-homogeneous elements $m \in M$. In particular, if N is an A-graded R_A-module one obtains commuting sets of D_A-endomorphisms on the left D_A-module $D_A \otimes_{R_A} N$ by

$$E_i \circ (P \otimes Q) := (E_i + \deg_{A,i}(P) + \deg_{A,i}(Q))P \otimes Q.$$

The *Euler–Koszul complex* $\mathscr{K}_\bullet(N;\beta)$ of the A-graded R_A-module N is the homological Koszul complex induced by $E-\beta := \{(E_i-\beta_i)\circ\}_1^d$ on $D_A \otimes_{R_A} N$. In particular, the terminal module $D_A \otimes_{R_A} N$ sits in homological degree zero. We denote the homology groups of $\mathscr{K}_\bullet(N;\beta)$ by $\mathscr{H}_\bullet(N;\beta)$. Implicit in the notation is "A": different presentations of semigroup rings that act on N yield different Euler–Koszul complexes.

If $N(\mathbf{b})$ denotes the usual shift-of-degree functor on the category of graded R_A-modules, then $\mathscr{K}_\bullet(N;\beta)(\mathbf{b})$ and $\mathscr{K}_\bullet(N(\mathbf{b});\beta-\mathbf{b})$ are identical.

2.4 The toric category

There is a bijection between faces τ of the cone $\mathbb{R}_{\geq 0}A$ and A-graded prime ideals $I_A^\tau = I_A + R_A\{\partial_j \mid j \notin \tau\}$ of R_A containing I_A. If the origin is a face of $\mathbb{R}_{\geq 0}A$, it corresponds to the ideal $I_A^\emptyset = (\partial_0,\dots,\partial_n)$. In general, $R_A/I_A^\tau \cong \mathbb{C}[\mathbb{N}\tau]$.

An R_A-module N is *toric* if it is A-graded and has a (finite) A-graded composition chain

$$0 = N_0 \subsetneq N_1 \subsetneq N_2 \cdots \subsetneq N_k = N$$

such that each composition factor N_i/N_{i-1} is isomorphic as A-graded R_A-module to an A-graded shift $(R_A/I_A^\tau)(\mathbf{b})$ for some $\mathbf{b} \in \mathbb{Z}A$ and some face τ. The category of toric modules is closed under the formation of subquotients and extensions.

For toric input N, the modules $\mathscr{H}_\bullet(N;\beta)$ are holonomic. As D_A is R_A-free, any short exact sequence $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ of A-graded R_A-modules produces a long exact sequence of Euler–Koszul homology. If β is not a quasi-degree of N then the complex $\mathscr{K}_\bullet(N;\beta)$ is exact, and if N is a maximal Cohen–Macaulay module then $\mathscr{K}_\bullet(N;\beta)$ is a resolution of $\mathscr{H}_0(N;\beta)$.

2.5 The Euler space

Notation 2.2. The $\mathbb{C}$-linear span of the Euler operators $\{E_i\}_1^d$ is called the *Euler space*. Let E be in the Euler space. Then E is in a unique fashion (as $\operatorname{rk}(A)=d$) a linear combination $E=\sum c_iE_i$. With $\beta_E := \sum c_i\beta_i$, we have $E-\beta_E \in H_A(\beta)$. We further write $\deg_E(-)$ for the degree function $\sum c_i \deg_{A,i}(-)$. □

Denote $\theta_j = x_j\partial_j$ and $\theta = (\theta_0,\dots,\theta_n)$. A linear combination $\sum_j v_j\theta_j$ is in the Euler space if and only if the coefficient vector $\mathbf{v}=(v_0,\dots,v_n)$, interpreted as a linear functional on $\mathbb{C}^{n+1}$ via $\mathbf{v}((q_0,\dots,q_n)) := \sum v_iq_i$, is the pull-back via A of a linear functional on $\mathbb{C}^d$. In other words,

$$[\mathbf{v}\cdot\theta^T = \sum_j v_j\theta_j \text{ is in the Euler space}] \Leftrightarrow [\mathbf{v} = \mathbf{c}\cdot A \text{ for some } \mathbf{c}\in\mathbb{C}^d].$$

If $L\colon \mathbb{C}^d \longrightarrow \mathbb{C}$ is a linear functional then the Euler operator in $H_A(\beta)$ corresponding to its image under $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^d,\mathbb{C}) \xrightarrow{\cdot A} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{n+1},\mathbb{C})$ is denoted $E_L - \beta_L$.

Lemma 2.3. For any set F of columns of A contained in a hyperplane that passes through the origin of $\mathbb{C}^d$ but does not contain $\mathbf{a}_k$, there is an Euler operator $E_F - \beta_F$ in $H_A(\beta)$ such that the coefficient of θ_j in E_F is zero for all $j \in F$, and equal to 1 for $j = k$. If $\mathbb{R}_{\geq 0}F$ is a facet of $\mathbb{R}_{\geq 0}A$ then $E_F - \beta_F$ is unique. □

Proof. Choose for any such set F a linear functional $L\colon \mathbb{Q}^d \longrightarrow \mathbb{Q}$ that vanishes on F while $L(\mathbf{a}_k) = 1$. The corresponding Euler operator $E_L - \beta_L$ has the desired properties, and if we define numbers $a_{L,j}$ by

$$E_L =: \sum_j a_{L,j} x_j \partial_j$$

then $a_{L,j} = L(\mathbf{a}_j)$. The uniqueness in the facet case is obvious. ■

3 Restricting the Fourier transform

The Fourier transform $\mathscr{F}(-)$ is a functor from the category of D-modules on V to the category of D-modules on the dual space $V^* = \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$. In this section, we bound the b-function along a coordinate hyperplane of the Fourier transform $\mathscr{F}(M_A(\beta))$ of the hypergeometric system. This module is called $\check{M}_A^\beta$ in [10].

The square of the Fourier transform is the involution induced by $x \mapsto -x$, which has no effect on the analytic properties of the modules we study. In particular, b-functions along coordinate hyperplanes are unaffected by this involution and we therefore consider $\mathscr{F}^{-1}(M_A(\beta))$ without harm.

We start with introducing some notation.

Notation 3.1. Let $\{y_j\}$ be the coordinates on V^* such that $\mathscr{F}^{-1}(\partial_j) = y_j$ on the level of differential operators. We let $\tilde{D}_A$ be the ring of $\mathbb{C}$-linear differential operators on $\tilde{O}_A := \mathbb{C}[y_0,\dots,y_n]$, generated by $\{y_j,\delta_j\}_0^n$ where δ_j denotes $\frac{\partial}{\partial y_j}$. Then $\mathscr{F}^{-1}(x_j) = -\delta_j$. The subring $\mathbb{C}[\delta_1,\dots,\delta_n]$ of $\tilde{D}_A$ is denoted $\tilde{R}_A$. The isomorphism $(\tilde{-})\colon D_A \longrightarrow \tilde{D}_A$ induced by $\tilde{\partial}_j := y_j$ and $\tilde{x}_j = \delta_j$ sends O_A to $\tilde{R}_A$ and R_A to $\tilde{O}_A$.

Thus, $\tilde{I}_A := \mathscr{F}^{-1}(I_A)$ is an ideal of $\tilde{O}_A$; the advantage of considering $\mathscr{F}^{-1}$ rather than $\mathscr{F}$ is that $\tilde{I}_A$ retains the shape of the generators of I_A as differences of monomials. For each j set $\tilde{\theta}_j := \mathscr{F}^{-1}(\theta_j) = -\delta_j y_j$. The i-th level V-filtration on $\tilde{D}_A$ along y_t is spanned by $\delta^\alpha y^\beta$ with $\beta_t - \alpha_t \geq i$. $\square$

Before, we get into the technical part, let us show by example an outline of what is to happen.

Example 3.2. Let $A = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, a matrix whose associated semigroup ring is a normal complete intersection. We will estimate the b-function for restriction to the hyperplane $y_1 = 0$ (corresponding to the middle column) of $\mathscr{F}^{-1}(M_A(\beta))$.

The ideal $\tilde{H}_A(\beta) := \mathscr{F}^{-1}(H_A(\beta))$ is generated by

$$-\tilde{\theta}_0 + \tilde{\theta}_2 - \beta_1, \qquad \tilde{\theta}_0 + \tilde{\theta}_1 + \tilde{\theta}_2 - \beta_2, \qquad y_0 y_2 - y_1^2. \tag{3.1}$$

Since $y_1 \in (V^1\tilde{D}_A)$, y_0y_2 and hence also $\tilde{\theta}_0\tilde{\theta}_2$ are in $(V^1\tilde{D}_A) + \tilde{H}_A(\beta)$. The strategy of the example, and of the theorem in this section, is to multiply the element $1 \in \tilde{D}_A$ by suitable Euler operators so that the result is a sum of a polynomial $p(\tilde{\theta}_1)$ with an element of $\mathbb{C}[\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_2] \cdot \tilde{\theta}_0\tilde{\theta}_2$; this certifies $p(\tilde{\theta}_1)$ to be in $\tilde{H}_A(\beta) + (V^1\tilde{D}_A)$.

In the case at hand, the relevant Euler operators are $2\tilde{\theta}_0 + \tilde{\theta}_1 + \beta_1 - \beta_2$ and $\tilde{\theta}_1 + 2\tilde{\theta}_2 - \beta_1 - \beta_2$. Modulo $\tilde{H}_A(\beta)$ we can rewrite $(V^1\tilde{D}_A) \ni 4\delta_0\delta_2 y_1^2 \equiv 4\tilde{\theta}_0\tilde{\theta}_2 \equiv (-\tilde{\theta}_1 - \beta_1 + \beta_2)(-\tilde{\theta}_1 + \beta_1 + \beta_2)$. It follows that $(\tilde{s} + \beta_1 - \beta_2)(\tilde{s} - \beta_1 - \beta_2)$ is a multiple of the b-function, where $\tilde{s} = \tilde{\theta}_1 = -y_1\delta_1 - 1$. This Fourier twist in the argument of the b-function occurs naturally throughout and we will make our computations in this section in terms of $b(\tilde{s})$.

The expressions $\tilde{\theta}_1 + 2\tilde{\theta}_2$ and $2\tilde{\theta}_0 + \tilde{\theta}_1$ that appear in the Euler operators we used can be found systematically as follows. Let d_1, d_2 denote the coordinates on the degree group $\mathbb{Z}^2$ corresponding to E_1 and E_2; compare the discussion following Notation 2.2. An element of S_A has degree on the facet $\mathbb{R}_{\geq 0}\mathbf{a}_0$ if and only if the functional $L_1(d_1, d_2) = d_1 + d_2$ vanishes, and the Euler field that corresponds to this functional in the spirit of Lemma 2.3 is exactly $\theta_1 + 2\theta_2 - \beta_1 - \beta_2$. The elements of S_A with degree on the facet $\mathbb{R}_{\geq 0}\mathbf{a}_2$ are determined by the vanishing of $L_2(d_1, d_2) = d_2 - d_1$ and the Euler field corresponding to this functional is exactly $2\theta_0 + \theta_1 + \beta_1 - \beta_2$. It is no coincidence that the union of the kernels of these two functionals is exactly the set of quasi-degrees of $S_A/\partial_1 \cdot S_A$. The point is that modulo $\tilde{H}_A(\beta)$ all monomials in $\tilde{S}_A$ with degree in $\mathbb{R}_+ A$ are already in $(V^1\tilde{D}_A)$. The task is then to deal with those with degree on the boundary through multiplication with suitable expressions.

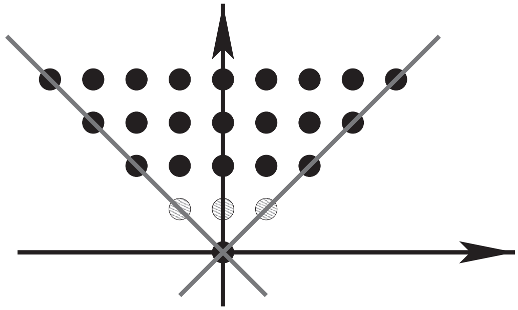

Fig. 1. Restriction of the Fourier transform to $y_1 = 0$.

Figure 1 shows as dots the elements of $\mathbb{N}$ A (where those inside A are shaded at height one); the quasi-degrees of $S_A/\partial_1 \cdot S_A$ are on the two indicated lines. Finally, $(\beta_2 - \beta_1)\mathbf{a}_1$ and $(\beta_1 + \beta_2)\mathbf{a}_1$ are the intersections of $\mathbb{R} \cdot \mathbf{a}_1$ with $\mathrm{qdeg}_A(S_A) + \beta$. □

We now generalize the computation of the example to the general case.

Convention 3.3. For the remainder of this section, we consider restriction to the hyperplane y_0 in order to save overhead (in terms of a further index variable). □

Consider the toric module $N = S_A/\partial_0 S_A$, and take a toric filtration

$$0 = N_0 \subsetneq N_1 \subsetneq \ldots \subsetneq N_k = N \tag{N}$$

with composition factors

$$\overline{N}_\alpha := N_\alpha / N_{\alpha-1},$$

each isomorphic to some shifted face ring $S_{F'_\alpha}(\mathbf{b}_\alpha)$, $F'_\alpha = \tau_\alpha \cap A$, attached to a face τ_α of $\mathbb{R}_{\geq 0}A$. (We will call such F'_α also a face.) Lifting the N_α to S_A yields an increasing sequence of A-graded ideals $J_\alpha \ni \partial_0$ of S_A with $N_\alpha = J_\alpha/\partial_0 \cdot S_A$.

Choose for each composition factor a facet F_α containing F'_α. None of the faces F'_α will contain $\mathbf{a}_0$ (as ∂_0 is zero on N but not nilpotent on any face ring of a face containing $\mathbf{a}_0$) and hence we can arrange that the corresponding facets do not contain $\mathbf{a}_0$ either.

Lemma 2.3 produces for each $\overline{N}_\alpha$ a facet F_α and corresponding functional L_{F_α} (which we abbreviate to L_α) that vanishes on the facet and evaluates to 1 on $\mathbf{a}_0$. The associated Euler operator in $H_A(\beta)$ is $E_{F_\alpha} - \beta_{F_\alpha}$. Since L_α is zero on all A-columns in F_α and since $\overline{N}_\alpha$ is a shifted quotient of S_{F_α}, there is a unique value for L_α on the A-degrees of all non-zero A-homogeneous elements of $\overline{N}_\alpha$. We denote this value by $L_\alpha(\overline{N}_\alpha)$. However, that $L_\alpha(\overline{N}_\alpha)$ does very much depend on the choice of the facet F_α even though the notation does not remember this.

Now let T_α be the image in $\mathscr{F}^{-1}(M_A(\beta))$ of $\mathscr{F}^{-1}(J_\alpha)$ under the map induced by $\tilde{O}_A \longrightarrow \tilde{D}_A \longrightarrow \mathscr{F}^{-1}(M_A(\beta))$. The image of $T_0 = y_0\tilde{O}_A$ in $\mathscr{F}^{-1}(M_A(\beta))$ is in $(V^1\tilde{D}_A)\cdot\overline{1}$, the bar denoting cosets in $\mathscr{F}^{-1}(M_A(\beta))$.

Lemma 3.4. In the context of the three preceding parapgraphs, let κ_α be the constant $L_\alpha(\overline{N}_\alpha)$. Then in $\mathscr{F}^{-1}(M_A(\beta))$, modulo the image of $(V^1\tilde{D}_A)$,

$$(\tilde{\theta}_0+\kappa_\alpha-\beta_\alpha)\cdot(V^0\tilde{D}_A)\cdot T_\alpha = (V^0\tilde{D}_A)\cdot(\tilde{\theta}_0+\kappa_\alpha-\beta_\alpha)\cdot T_\alpha \subseteq (V^0\tilde{D}_A)\cdot T_{\alpha-1}. \qquad \square$$

Proof. Since the commutators $[\tilde{\theta}_0,(V^0\tilde{D}_A)]$ are in $(V^1\tilde{D}_A)$, it suffices to show that $(\tilde{\theta}_0+\kappa_\alpha-\beta_\alpha)\cdot T_\alpha \subseteq (V^0\tilde{D}_A)\cdot T_{\alpha-1}$ modulo $\mathscr{F}^{-1}(H_A(\beta))$.

By definition, $\tilde{E}_\alpha-\beta_\alpha := \mathscr{F}^{-1}(E_\alpha-\beta_\alpha)$ is zero in $\mathscr{F}^{-1}(M_A(\beta))$. Take a monomial $\tilde{m}\in\tilde{O}_A$ whose coset lies in $T_\alpha\setminus T_{\alpha-1}$. By Equation (2.3), $\tilde{E}_\alpha\cdot\tilde{m} = \tilde{m}(\tilde{E}_\alpha-\kappa_\alpha)$ since $\mathscr{F}^{-1}(-)$ is a homomorphism. Now write $E_\alpha = \sum a_{\alpha,j}\theta_j$; as before, we have $a_{\alpha,j} = L_\alpha(\mathbf{a}_j)$.

Since the coefficient of θ_0 in E_α is 1, it follows that in $\mathscr{F}^{-1}(M_A(\beta))$:

$$\begin{aligned}\tilde{\theta}_0\tilde{m} &= (-\tilde{E}_\alpha+\tilde{\theta}_0)\tilde{m}+\tilde{E}_\alpha\tilde{m}\\ &= \sum_{\substack{j\neq 0\\ L_\alpha(\mathbf{a}_j)\neq 0}} a_{\alpha,j}\delta_j y_j\tilde{m}+\tilde{m}(\tilde{E}_\alpha-\kappa_\alpha)\\ &= \sum_{\substack{j\neq 0\\ \mathbf{a}_j\notin F_\alpha}} a_{\alpha,j}\delta_j y_j\tilde{m}+\tilde{m}(\beta_\alpha-\kappa_\alpha).\end{aligned}$$

Recall that F_α contains F'_α and that $\overline{N}_\alpha$ is a $\mathbb{Z}A$-shift of $S_{F'_\alpha} = R_A/I_A^\tau$, whence each y_j with $\mathbf{a}_j\notin F'$ annihilates $\mathscr{F}^{-1}(\overline{N}_\alpha)$. Therefore, each term $a_{\alpha,j}\delta_j(y_j m)$ in the last sum of the display is in $(V^0D_A)T_{\alpha-1}$. It follows that in $\mathscr{F}^{-1}(M_A(\beta))$, we have $(\tilde{\theta}_0+\kappa_\alpha-\beta_\alpha)T_\alpha\subseteq(V^0\tilde{D}_A)T_{\alpha-1}$ as claimed. ■

Theorem 3.5. For $t=0,\dots,n$, the number $\varepsilon\in\mathbb{C}$ is a root of the b-function $b(\tilde{s})$ (with $\tilde{s}=\tilde{\theta}_t=-\delta_t y_t$) of $\mathscr{F}^{-1}(M_A(\beta))$ along $y_t=0$, only if $\varepsilon\cdot\mathbf{a}_t$ is a point of intersection of the line $\mathbb{C}\cdot\mathbf{a}_t$ with the set $\beta-\operatorname{qdeg}_A(N)$, the quasi-degrees of the toric module $N=S_A/\partial_t S_A$ multiplied by -1 and shifted by β. $\square$

Proof. Without loss of generality we shall suppose that $t = 0$ by way of re-indexing.

We will show that a divisor of $\prod_\alpha(\tilde{\theta}_0+\kappa_\alpha-\beta_\alpha)$ is inside $H_A(\beta)+(V^1\tilde{D}_A)$, in notation from the previous lemma.

Indeed, it follows from Lemma 3.4 that $\prod_\alpha(\tilde{\theta}_0+\kappa_\alpha-\beta_\alpha)$ multiplies $\overline{1} \in \mathscr{F}^{-1}(M_A(\beta))$ into $(V^0\tilde{D}_A)\cdot y_0\cdot\overline{1} \subseteq (V^1\tilde{D}_A)\cdot\overline{1}$. Hence the root set of the b-function $b(\tilde{\theta}_0)$ in question is a subset of $\{\beta_\alpha - \kappa_\alpha\}$, α running through the indices of the chosen composition series of N. This set is determined by the composition series (N) and the choices of the facets F_α for each N_α. Varying over all choices of facets $\{F_\alpha\}$ for a given chain (N), the root set of $b(\tilde{\theta}_0)$ is in the intersection ρ_N of all possible sets $\{\beta_\alpha - \kappa_\alpha\}_{\alpha\in(N)}$.

Since $L_\alpha(\mathbf{a}_0) = 1$, the point $(\beta_\alpha - \kappa_\alpha)\cdot\mathbf{a}_0$ is the intersection of the hyperplane $L_\alpha = \beta_\alpha - \kappa_\alpha$ with the line $\mathbb{C}\cdot\mathbf{a}_0$. Thus, ρ_N is inside the intersection of $\mathbb{C}\cdot\mathbf{a}_0$ with all arrangements $\operatorname{Var}\prod_\alpha(L_\alpha-\beta_\alpha+\kappa_\alpha)$. The intersection of the arrangements $\operatorname{Var}\prod_\alpha(L_\alpha-\beta_\alpha+\kappa_\alpha)$ is the union of the quasi-degrees of all $\overline{N}_\alpha$ of the composition chain (N), multiplied by -1 and shifted by $-\beta_\alpha$. As N is finitely generated, $\operatorname{qdeg}_A(N) = \bigcup_\alpha \operatorname{qdeg}_A(\overline{N}_\alpha)$. Hence the root set of $b(\tilde{\theta}_0)$ is contained in the intersection $-\operatorname{qdeg}_A(S_A/\partial_0 S_A)+\beta$ with $\mathbb{C}\cdot\mathbf{a}_0$. ■

Remark 3.6. The quantity $\tilde{\theta}_t$ is the more natural argument for the b-function here. The roots of $b(y_t\delta_t)$ are those of $b(\tilde{\theta}_t)$ shifted up by 1 and then multiplied by -1. □

Example 3.7. Let $A = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2) = \begin{pmatrix} -1 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}$ and $\beta = \binom{\beta_1}{\beta_2}$. The ring S_A is a complete intersection but not normal.

Consider restriction to $y_1 = 0$ (the middle column). Then $N = S_A/\partial_1\cdot S_A$ has a toric filtration involving four steps, given by the ideals $0 \subsetneq \partial_0^3\cdot N \subsetneq \partial_0^2\cdot N \subsetneq \partial_0\cdot N \subsetneq N$. The corresponding A-graded composition factors are $S_A(-3\cdot\mathbf{a}_0)/(\partial_1,\partial_2)S_A$ and $\{S_A(-\alpha\cdot\mathbf{a}_0)/(\partial_0,\partial_1)S_A\}_{\alpha=0}^2$. The b-function $b(\tilde{\theta}_1)$ for the inverse Fourier transform is $(\tilde{\theta}_1-\beta_1-\beta_2)\prod_{\alpha=0}^2(\tilde{\theta}_1 - \frac{3\beta_2-\beta_1-4\alpha}{3})$.

Explicitly, $y_1^4 - y_0^3y_2 \in \tilde{H}_A(\beta)$ gives $(V^1\tilde{D}_A) \ni \delta_0^3\delta_2y_0^3y_2 = \tilde{\theta}_2\tilde{\theta}_0(\tilde{\theta}_0-1)(\tilde{\theta}_0-2)$ which modulo $\tilde{H}_A(\beta)$ equals $(-1)^4(\tilde{\theta}_1-\beta_1-\beta_2)\prod_{\alpha=0}^2(\tilde{\theta}_1-\frac{3\beta_2-\beta_1-4\alpha}{3})$. The relevant Euler operators are $\theta_1+4\theta_2-\beta_1-\beta_2$ and $3\theta_1+4\theta_0-3\beta_2+\beta_1$.

Figure 2 shows in the elements of $\mathbb{N}$ A (where those inside A are shaded at height one); the quasi-degrees of $N = S_A/\partial_1\cdot S_A$ are on the indicated four lines. The roots of $b(\delta_1y_1)$ (which are opposite to the roots of $b(\tilde{\theta}_1)$) are the intersections of the line $\mathbb{C}\cdot\binom{0}{1}$ with the shift of the indicated lines by $-\beta$.

In this example, each composition factor corresponds to a facet and to a component of the quasi-degrees of N. One checks that each composition chain must have these

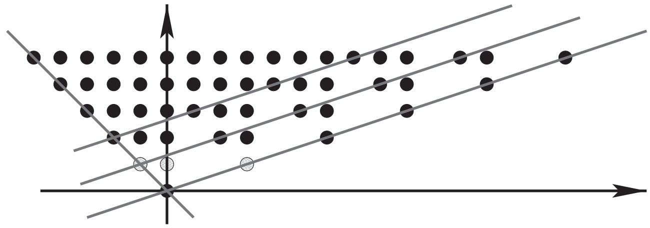

Fig. 2. Restriction of the Fourier transform to $y_1 = 0$.

four lines as quasi-degrees. However, that composition chains are far from unique and in general such correspondence will not exist. $\square$

Remark 3.8. The b-function for $\mathscr{F}^{-1}(M_A(\beta))$ along a coordinate hyperplane is generally not reduced, and its degree may be lower than the length of the shortest toric filtration for $N = S_A/\partial_t \cdot S_A$ would suggest. (Not every component of $\beta - \operatorname{qdeg}_A(N)$ needs to meet the line $\mathbb{C} \cdot \mathbf{a}_t$). $\square$

Corollary 3.9. The roots of the b-function $b(\delta_t y_t)$ of $\mathscr{F}^{-1}(M_A(\beta))$ along $y_t = 0$ are in the field $\mathbb{Q}(\beta)$.

Consider $\mathscr{F}^{-1}(M_A(0))$; then:

1. the roots of the b-function $b(\tilde{\theta}_t)$ are non-negative rationals;
2. if S_A is normal, all roots are in the interval $[0, 1)$;
3. if the interior ideal of S_A is contained in $\partial_t \cdot S_A$ then zero is the only root. $\square$

Proof. The first claim is a consequence of the intersection property in Theorem 3.5: the defining equations for the quasi-degrees are rational.

Let $N = S_A/\partial_t S_A$. For items 1.-3., we need to study the intersection of $\operatorname{qdeg}_A(N)$ with $\mathbb{C} \cdot \mathbf{a}_t$, since $\beta = 0$ and $\delta_t y_t = -\tilde{\theta}_t$. The quasi-degrees of N are covered by hyperplanes of the sort $L_\alpha = \varepsilon$ where L_α is a rational supporting functional of the facet F_α. In particular, we can arrange L_α to be zero on F_α, positive on the rest of A, and $L_\alpha(\mathbf{a}_t) = 1$. As $\deg_A(N) \subseteq \deg_A(S_A)$, $\varepsilon \geq 0$. Hence $\operatorname{Var}(L_\alpha - \varepsilon)$ meets $\mathbb{C} \cdot \mathbf{a}_t$ in the non-negative rational multiple $\varepsilon \mathbf{a}_t$ of $\mathbf{a}_t$. If S_A is normal, $\deg_A(S_A/\partial_A S_A)$ is covered by hyperplanes $\operatorname{Var}(L_\alpha - \varepsilon)$ that do not meet the cone $\mathbf{a}_t + \mathbb{R}_{\geq 0} A$. These are precisely the ones for which $\varepsilon < 1$.

If $\partial_t \cdot S_A$ contains the interior ideal then $\deg_A(N)$, and hence $\operatorname{qdeg}_A(N)$, is inside the supporting hyperplanes of the cone, which meet $\mathbb{C} \cdot \mathbf{a}_t$ at the origin. $\blacksquare$

Remark 3.10. One special case in which case 3 of Corollary 3.9 applies is when S_A is Gorenstein and where further ∂_t generates the canonical module. The matrix $A = (\mathbf{a}_0, \dots, \mathbf{a}_3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, with the interior ideal being generated by $\partial_1\partial_3$, provides an example that case (3) can occur in a Gorenstein situation without the boundary of $\mathbb{N}A$ being saturated. See [14] for a discussion on Cohen–Maculayness of face rings of Cohen–Macaulay semigroup rings. □

4 *b*-functions for the hypergeometric system

4.1 Restriction along a hyperplane

We are here interested in the b-function for the hypergeometric module $M_A(\beta)$ along the hyperplane $x_t = 0$. As in the previous section, apart from examples, we actually carry out all computations for $t = 0$, in order to have as few variables around as possible. On the other hand, the natural argument for expressing the b-function will be $s = x_0\partial_0$.

Notation 4.1. With $A = (\mathbf{a}_0, \dots, \mathbf{a}_n)$ and distinguished index 0, we denote $A' := (\mathbf{a}_1, \dots, \mathbf{a}_n)$. Via $\mathbb{N}A' \subseteq \mathbb{N}A$ we consider $S_{A'}$ as a subring of S_A.

For $k \in \mathbb{N}$ let $\overline{J}_{A,0;k} \subseteq S_{A'}$ be the vector space spanned by the monomials $\partial^{\mathbf{u}}$ with $u_0 = 0$ (so that $\partial^{\mathbf{u}} \in S_{A'}$) that satisfy $\partial_0^k \cdot \partial^{\mathbf{u}} \in S_{A'}$. We denote $J_{A,0;k} \subseteq R_{A'}$ the preimage of $\overline{J}_{A,0;k}$ under the natural surjection $R_{A'} \twoheadrightarrow S_{A'}$. Put $J_{A,0} = \sum_{k\geq 1} J_{A,0;k}$ and $\overline{J}_{A,0} = J_{A,0}/I_{A'} \subseteq S_{A'}$. □

Each $\overline{J}_{A,0;k}$ is a monomial ideal of $S_{A'}$ since $\partial_0^k(\partial^{\mathbf{v}}\partial^{\mathbf{u}}) = \partial^{\mathbf{v}}(\partial_0^k\partial^{\mathbf{u}})$. Note, however, that $\overline{J}_{A,0;k}$ need not be contained in $\overline{J}_{A,0;k+1}$. If $\mathbf{a}_0 \in \mathbb{R}_{\geq 0}A'$ then some power of ∂_0 is in $S_{A'}$ and so $\overline{J}_{A,0} = S_{A'}$.

Definition 4.2. For $\mathbf{a}_0 \in \mathbb{R}^d$ outside $\mathbb{R}_{\geq 0}A'$, a point $\mathbf{a} \in \mathbb{R}_{\geq 0}A'$ is $\mathbf{a}_0$-*visible* if $\mathbf{a} + \lambda \cdot \mathbf{a}_0$, $0 < \lambda \ll 1$ is outside $\mathbb{R}_{\geq 0}A'$. (The idea behind the choice of language is that the observer stands at the point of projective space given by the line $\mathbb{R}\mathbf{a}_0$.)

By abuse of notation, we say that $\partial^{\mathbf{a}}$ is $\mathbf{a}_0$-visible if $\mathbf{a}$ is. □

Lemma 4.3. Assume that $\mathbf{a}_0$ is not in the cone $\mathbb{R}_{\geq 0}A'$. Then the radical of $J_{A,0}$ is generated by the $\mathbf{a}_0$-invisible elements of $S_{A'}$, and in consequence the quasi-degrees of $S_{A'}/J_{A,0}$ are a union of shifted face spans where each face is in its entirety visible from $\mathbf{a}_0$. □

Proof. If $\mathbb{Z}A/\mathbb{Z}A'$ has positive rank then all points of $\mathbb{N}A$ are $\mathbf{a}_0$-visible while $J_{A,0}$ is clearly zero, so that in this case there is nothing to prove. We therefore assume that $\mathbb{Z}A/\mathbb{Z}A'$ is finite.

It is immediate that $\mathbf{a}$ is $\mathbf{a}_0$-visible if and only if any positive integer multiple of it is. This implies that no power of an $\mathbf{a}_0$-visible element $\partial^{\mathbf{a}}$ of $S_{A'}$ can be in the radical of $J_{A,0}$ since $\partial^{m\cdot\mathbf{a}+k\mathbf{a}_0}$ can't have its degree in the cone of A'.

For the converse, suppose $\mathbf{a}$ is not $\mathbf{a}_0$-visible, so that there are positive integers $p < q$ with $\mathbf{a} + (p/q)\cdot\mathbf{a}_0 \in \mathbb{R}_{\geq 0}A'$. Then a high power of $\partial^{q\cdot\mathbf{a}+p\cdot\mathbf{a}_0}$ is in $\mathbb{C}[\mathbb{Z}A \cap \mathbb{R}_{\geq 0}A']$ and a suitable power $\partial^{\mathbf{b}}$ of that will be in $\mathbb{C}[\mathbb{Z}A' \cap \mathbb{R}_{\geq 0}A']$ because of the finiteness of $\mathbb{Z}A/\mathbb{Z}A'$. Now let τ be the smallest face of $\mathbb{R}_{\geq 0}A'$ that contains $\mathbf{b}$; this makes $\mathbf{b}$ an interior point of τ. Since $\mathbb{C}[\tau \cap \mathbb{Z}A']$ is a finitely generated $\mathbb{C}[\tau \cap \mathbb{N}A']$-module, some power of $\partial^{\mathbf{b}}$ is in $\mathbb{C}[\tau \cap \mathbb{N}A'] \subseteq S_{A'}$. This shows that some power of $\partial^{q\cdot\mathbf{a}}$ times some power of $\partial^{p\cdot\mathbf{a}_0}$ is in $S_{A'}$, establishing the first claim of the lemma.

In every composition chain for $S_{A'}/J_{A,0}$, each composition factor is an $S_{A'}/\sqrt{J_{A,0}}$-module. Thus the quasi-degrees of $S_{A'}/J_{A,0}$ are inside a union of shifted quasi-degrees of $S_{A'}/\sqrt{J_{A,0}}$ and hence all $\mathbf{a}_0$-visible, which implies the second claim. ■

Our main theorem in this section is:

Theorem 4.4. The root locus of the b-function $b(x_0\partial_0)$ for restriction of $M_A(\beta)$ along $x_0 = 0$ is, up to inclusion of non-negative integers, contained in the locus of intersection $(-\operatorname{qdeg}_{A'}(S_{A'}/\overline{J}_{A,0}) + \beta) \cap \mathbb{C}\cdot\mathbf{a}_0$. The set of integers needed can be taken to be the integers $0, \ldots, k-1$ such that $J_{A,0} = \sum_{1\leq i\leq k} J_{A,0;i}$.

In two extreme cases, one can be explicit:

1. if $\dim S_A - 1 = \dim S_{A'}$ then the b-function is linear with root given by the intersection of $(-\operatorname{qdeg}_A(S_{A'}) + \beta) \cap \mathbb{C}\cdot\mathbf{a}_0$;
2. if $\mathbf{a}_0 \in \mathbb{R}_{\geq 0}A'$ then the b-function has integer roots in $\{0, 1, \ldots, k-1\}$, where $k = \min\{t \in \mathbb{N} \mid 0 \neq t\cdot\mathbf{a}_0 \in \mathbb{N}A'\}$. □

Proof. We first dispose of the extreme cases. If $\dim S_A - 1 = \dim S_{A'}$, then S_A is the polynomial ring $S_{A'}[\partial_0]$ and A' is a facet of A. By Lemma 2.3 there is $\mathbf{v} = (v_1, \ldots, v_d)$ such that the Euler operator

$$E - \beta_E = \sum v_i(E_i - \beta_i)$$

is in $H_A(\beta)$ and equals $\theta_0 - \beta_E$. In particular, the b-function is $s - \beta_E$. On the other hand: $\overline{J}_{A,0}$ is zero in this case, $\mathbf{v} = (v_1, \dots, v_d)$ is in the kernel of A'^T, and $\mathbf{a}_0^T \mathbf{v} = 1$. Therefore, the quasi-degrees of $S_{A'}/\overline{J}_{A,0}$ form the hyperplane given as the kernel of $\mathbf{v}$ and $(\mathbf{v}^T\beta)\mathbf{a}_0 = \beta_E \mathbf{a}_0$ is the intersection of $-\operatorname{qdeg}_A(S_{A'}) + \beta$ with $\mathbb{C}\mathbf{a}_0$.

If $\mathbf{a}_0 \in \mathbb{R}_{\geq 0}A'$ then $\mathbb{N}\mathbf{a}_0$ meets $\mathbb{N}A'$ and so $\partial_0^k = \partial^{\mathbf{u}}$ with $\mathbf{u} = (0, u_1, \dots, u_n) \in \mathbb{N}A'$. In particular, $J_{A,0} = S_{A'}$ in this case. Moreover, $(x_0\partial_0)(x_0\partial_0 - 1)\cdots(x_0\partial_0 - k + 1) = x_0^k\partial_0^k = x_0^k(\partial_0^k - \partial^{\mathbf{u}}) + x_0^k\partial^{\mathbf{u}} \in H_A(\beta) + V^1(D_A)$ shows the claim made in this case.

Now suppose that A and A' have equal rank but $\mathbf{a}_0 \notin \mathbb{R}_{\geq 0}A'$. In that case, $\overline{J}_{A,0}$ is a non-trivial ideal of $S_{A'}$. We shall use a toric filtration

$$(N) \quad : \quad 0 = N_0 \subsetneq N_1 \subsetneq \ldots \subsetneq N_t = S_{A'}/\overline{J}_{A,0}$$

and let $J_\alpha \supseteq J_{A,0}$ be the $R_{A'}$-ideal such that $N_\alpha = J_\alpha/J_{A,0}$. We will view J_α as subset of $D_{A'}$ or even D_A. In analogy to the previous case, for any $\partial^{\mathbf{u}}$ in $J_{A,0;k}$ the b-function along x_0 of the coset of $\partial^{\mathbf{u}}$ in $M_A(\beta)$ divides $s(s-1)\cdots(s-k+1)$. Indeed, $\partial^{\mathbf{u}} \in J_{A,0;k}$ implies that $\partial_0^k\partial^{\mathbf{u}} - \partial^{\mathbf{v}} \in I_A$ for some $\mathbf{v}$ with $v_0 = 0$, and so $x_0^k\partial_0^k\partial^{\mathbf{u}} \in H_A(\beta) + V^1(D_A)$. In particular, the root set of the b-function of the coset of $\partial^{\mathbf{u}}$ in $M_{A'}(\beta)$ is inside the set of integers described in the statement of the theorem.

For each composition factor $\overline{N}_\alpha = N_\alpha/N_{\alpha-1}$ choose now a facet τ_α of A' and an element $\partial^{\mathbf{u}_\alpha}$ of $S_{A'}$ $\mathbf{u}_\alpha \in \{0\} \times \mathbb{N}^n$ such that N_α is a quotient of $S_{A'} \cdot \partial^{\mathbf{u}_\alpha}$ and such that the annihilator of $\partial^{\mathbf{u}_\alpha}$ in $\overline{N}_\alpha$ contains the toric ideal $I_{A'}^{\tau_\alpha}$. Then $\operatorname{qdeg}_{A'}(\overline{N}_\alpha)$ is contained in $A' \cdot \mathbf{u}_\alpha + \operatorname{qdeg}_{A'}(S_{\tau_\alpha})$.

Since $\mathbf{a}_0$ is not in $\mathbb{R}_{\geq 0}A'$, Lemma 4.3 shows that the facet τ_α can be chosen such that $\mathbf{a}_0 \notin \mathbb{Q} \cdot \tau_\alpha$. Indeed, if an entire face of $\mathbb{R}_{\geq 0}A'$ is visible from $\mathbf{a}_0$ then it sits in at least one facet whose span does not contain $\mathbf{a}_0$. By Lemma 2.3 there is an element E_α of the Euler space of A that does not involve any element of τ_α, but which has coefficient 1 for θ_0. Notation 2.2 then associates a degree function $\deg_{E_\alpha}(-)$ to α.

As $\partial_j \cdot \partial^{\mathbf{u}_\alpha} \in N_{\alpha-1}$ for $j \notin \tau_\alpha$ it follows that the difference of $(E_\alpha - \beta_\alpha) \cdot \partial^{\mathbf{u}_\alpha}$ and $(\theta_0 - \beta_\alpha) \cdot \partial^{\mathbf{u}_\alpha}$ is inside $(V^0D_A)N_{\alpha-1}$. Since $E_\alpha - \beta_\alpha$ is in $H_A(\beta)$, so is $\partial^{\mathbf{u}_\alpha}(E_\alpha - \beta_\alpha) = (E_\alpha - \beta_\alpha + \deg_{E_\alpha}(\partial^{\mathbf{u}_\alpha}))\partial^{\mathbf{u}_\alpha}$. Therefore, $(\theta_0 - \beta_\alpha + \deg_{E_\alpha}(\partial^{\mathbf{u}_\alpha}))\partial^{\mathbf{u}_\alpha}$ is in $H_A(\beta) + (V^0D_A)N_{\alpha-1}$. Then, in parallel to how Lemma 3.4 was used in the proof of Theorem 3.5, the product

$$\prod_\alpha(\theta_0 - \beta_\alpha + \deg_{E_\alpha}(\partial^{\mathbf{u}_\alpha}))$$

multiplies $1 \in D_A$ into $H_A(\beta) + (V^0D_A)J_{A,0} + (V^1D_A)$. Multiplying by $x_0^k\partial_0^k$ for suitable k one obtains the desired bound for the b-function as in the second paragraph of the proof.

It follows as in Theorem 3.5 (with the modification that we have here θ_0 rather than $\mathscr{F}^{-1}(\theta_0)$, which affects signs) that the intersection of the roots of all such bounds is the intersection of $(-\operatorname{qdeg}_{A'}(S_{A'}/\overline{J}_{A,0})+\beta)$ with the line $\mathbb{C}\cdot\mathbf{a}_0$. ■

Example 4.5. With $A=(\mathbf{a}_0,\mathbf{a}_1,\mathbf{a}_2)=\begin{pmatrix}-1 & 0 & 3\\ 1 & 1 & 1\end{pmatrix}$, consider the b-function along x_1 of the A-hypergeometric system. The ideal $J_{A,1}$ is generated by $1\in S_{A'}=\mathbb{C}[\mathbb{N}(\mathbf{a}_0,\mathbf{a}_2)]$ since ∂_1^4 is in $S_{A'}$. The set of necessary integer roots is then $\{0,1,2,3\}$. No other roots are needed since $S_A/J_{A,1}$ is zero, irrespective of β. Figure 3 shows the situation in this case.

Restriction to $(x_2=0)$ behaves differently. As $S_{A'}=\mathbb{C}[\mathbb{N}(\mathbf{a}_0,\mathbf{a}_1)]$ now, $J_{A,2}=J_{A,2;1}$ is generated by ∂_0^3, and the quasi-degrees of $S_{A'}/J_{A,2}$ are the lines $\mathbb{C}\cdot(0,1)+(i,0)$ with $i=0,-1,-2$. The intersection of the negative of these three lines, shifted by β, with the line $\mathbb{C}\cdot\mathbf{a}_2$ is $\mathbf{a}_2\cdot\{(i+\beta_1)/3\}_{i=0,1,2}$. So the b-function has (at worst) roots $\{0,\beta_1,\beta_1+1,\beta_1+2\}/3$. Figure 4 shows the quasi-degrees of $S_A/J_{A,2}$. □

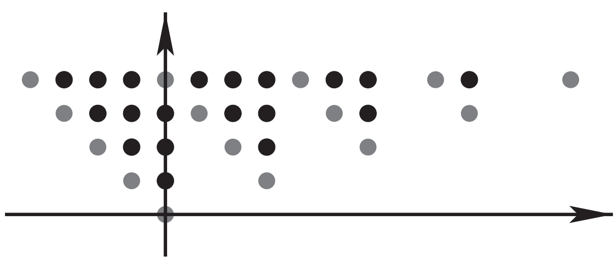

Fig. 3. The elements of $S_A\smallsetminus S_{A'}$ (black) and $S_{A'}$ (shaded) for restriction to x_1.

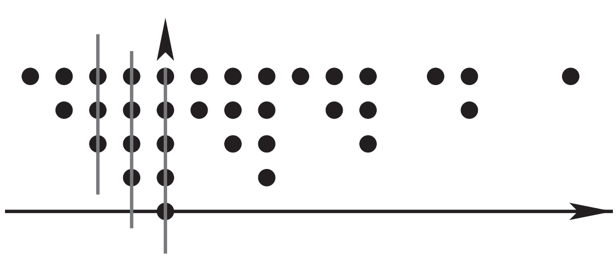

Fig. 4. The quasi-degrees of $S_A/J_{A,2}$ form three parallel lines.

Remark 4.6. We believe that both bounds in Theorems 3.5 (as is) and 4.4 (up to integers) are sharp. □

4.2 Restriction to a generic point

We suppose here that A is homogeneous; in other words, the Euler space contains a homothety. Let $p = (p_0, \ldots, p_n)$ be a point of $\mathbb{C}^{n+1}$. We wish to estimate here the b-function for restriction of $M_A(\beta)$ to the point $-p$ if p is generic. As a holonomic module is a connection near any generic point, this restriction yields a vector space isomorphic to the space of solutions to $H_A(\beta)$ near $-p$, see [12, Section 5.2].

Definition 4.7. Let $\theta_p = (x_0 + p_0)\partial_0 + \ldots + (x_n + p_n)\partial_n$ and write θ for θ_p if $p = 0$. The b-function for restriction of a principal D-module $M = D/I$ to the point $x + p = 0$ is the minimal polynomial $b_p(s)$ such that $b_p(\theta_p) \in I + (V_p^1 D)$ where $V_p^k D$ is the Kashiwara–Malgrange V-filtration along $\mathrm{Var}(x + p)$:

$$V_p^k D = \mathbb{C} \cdot \{(x + p)^{\mathbf{u}} \partial^{\mathbf{v}} \quad | \quad |\mathbf{u}| - |\mathbf{v}| \geq k\}.$$ □

Remark 4.8.

1. For any pair of manifolds $Y \subseteq X$ and given a D-module M on X one can define a b-function of restriction for the section $m \in M$ along Y by a formula generalizing both Definitions 1.1 and 4.7. Kashiwara proved their existence for holonomic M.
2. The roots of this b-function here relate to restriction of solution sheaves as follows. Near a generic point $x + p = 0$, a D-module M is a connection whose solution space has a basis consisting of a certain number of holomorphic functions. The germs of these functions form a vector space that can be identified with the dual of the zeroth homology group of $(D/(x + p)D) \otimes_D^L M$. Filtering this complex by $V_p^\bullet D$, $b_p(k)$ annihilates the k-th graded part of its homology, compare [7, 8, 16]. In particular, $b_p(s)$ carries information on the starting terms of the solution sheaf of M near $x + p = 0$. □

The purpose of this section is to bound $b_p(s)$ for $I = H_A(\beta)$ and generic p with the following strategy. We first show that a polynomial $b(s)$ is a multiple of $b_p(s)$ if $b(\theta)$

is in $D_A(I_A, A \cdot \mathscr{E} \cdot \partial)$ where

$$\mathscr{E} = \begin{pmatrix} p_0 & 0 & \cdots & 0 \\ 0 & p_1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & p_n \end{pmatrix},$$

provided that p is component-wise non-zero. The generators of $D_A(I_A, A \cdot \mathscr{E} \cdot \partial)$ are independent of x and we next observe that the radical of $R_A(I_A, A \cdot \mathscr{E} \cdot \partial)$ is $R_A \cdot \partial$, provided that p is generic. Thus, $b_p(s)$ will be a factor of any polynomial that annihilates the finite length module $R_A/(I_A, A \cdot \mathscr{E} \cdot \partial)$ as long as p is generic. We exhibit a particular such polynomial with all roots integral. In the case of a normal semigroup ring, we show that the (necessarily integral) roots of $b_p(s)$ are in the interval $[0, d-1]$.

We begin with pointing out that $b(\theta_p) \in I + (V_p^1 D)$ is equivalent to $b(\theta) \in I_p + (V_0^1 D)$, where I_p is the image of I under the morphism induced by $x \mapsto x - p$, $\partial \mapsto \partial$ and $(V_0^k D)$ is the Kashiwara–Malgrange filtration along the origin. Among the generators of $I = H_A(\beta)$, only the Euler operators depend on x while $(I_A)_p = I_A$ for any p; one has $(E_i - \beta_i)_p = \sum a_{i,j}(x_j - p_j)\partial_j - \beta_i = E_i - \beta_i - \sum a_{i,j} p_j \partial_j$. We hence seek a relation $b(\theta) \in D_A \cdot (I_A, E - \beta - A \cdot \mathscr{E} \cdot \partial) + (V_0^1 D_A)$ with $\mathscr{E}$ as in the display above.

Generally, a statement $b(\theta) \in I + (V_0^1 D_A)$ is equivalent to $b(\theta)$ being in the degree zero part $\mathrm{gr}^0_{V_0}(I)$ of the associated graded object. Note that $\mathrm{gr}_{V_0}(D_A)$ is a Weyl algebra again (although of course the symbol map $D_A \longrightarrow \mathrm{gr}_{V_0}(D_A)$ is not an isomorphism). Abusing notation, we denote x and ∂ also the symbols in $\mathrm{gr}_{V_0}(D_A)$ of the respective elements of D_A. By the previous paragraph then, the graded ideal $\mathrm{gr}_{V_0}(H_A(\beta)_p)$ contains the elements that generate I_A (since I_A is homogeneous!), as well as the elements $A \cdot \mathscr{E} \cdot \partial$ which arise as the V_0-symbols of $E_p - \beta$.

We need the following folklore result) for which we know no explicit reference.

Claim. The R_A-ideal generated by I_A and $A \cdot \mathscr{E} \cdot \partial$ has, for generic $\mathscr{E}$, radical $R_A \cdot \partial$. □

A sequence of d generic linear forms is of course a system of parameters on S_A; the issue is to show that linear forms of the type $A \cdot \mathscr{E} \cdot \partial$ are sufficiently generic.

Proof. As I_A and $A \cdot \mathscr{E} \cdot \partial$ are standard graded, $\mathrm{Var}(I_A, A \cdot \mathscr{E} \cdot \partial)$ is a conical variety. It thus suffices to show that the ideal $\mathrm{Var}(I_A, A \cdot \mathscr{E} \cdot \partial)$ is of height $n + 1$.

The ideal $R_A[x](I_A, A \cdot \theta)$ in the polynomial ring $R_A[x]$ defines in the cotangent bundle $\mathrm{Spec}(R_A[x])$ of $\mathbb{C}^{n+1}$ the union of the conormals to each torus orbit since the

Euler fields are tangent to the torus and span a space of the correct dimension in each orbit point. Suppose the claim is false, so that there is a non-zero point $y \in \mathrm{Var}(I_A)$ such that (the generically chosen vector) p is a conormal vector to the orbit of y. If y is in a torus orbit O_τ associated to a proper face τ of A then its coordinates corresponding to $A \setminus \tau$ are zero and we can reduce the question to the case where $A = \tau$. It is hence enough to show that there is $p \in \mathbb{C}^{n+1}$ such that p is not a conormal vector to any smooth point of $\mathrm{Var}(I_A)$.

Let $X \subseteq \mathbb{C}^{n+1}$ be any reduced affine variety and denote X_0 its smooth locus. We define a set $C(X)$ inside $\mathbb{C}^{n+1}$ by setting

$$[\eta \in C(X)] \iff [\exists y \in X_0, \quad \eta \in (T^*_{X_0}(\mathbb{C}^{n+1}))_y]$$

where $(T^*_{X_0}(\mathbb{C}^{n+1}))_y$ is the fiber of the conormal bundle at y of the pair $X_0 \subseteq \mathbb{C}^{n+1}$. This is a constructible, analytically parameterized union of a $\dim(X)$-dimensional family of vector spaces of dimension $n+1-\dim(X)$, which hence might fill $\mathbb{C}^{n+1}$.

Now suppose that X is a conical variety; then the conormals of y and λy agree for all $\lambda \in \mathbb{C}^*$. In particular,

$$C(X) = \bigcup_{\overline{y} \in \mathrm{Proj}(X)} (T^*_{X_0}(\mathbb{C}^{n+1}))_y,$$

where $\mathrm{Proj}(X)$ is the associated projective variety. But this is now an analytically parameterized union of a $(\dim(X)-1)$-dimensional family of vector spaces of dimension $n+1-\dim(X)$. It follows that most elements of $\mathbb{C}^{n+1}$ are outside $C(X)$ in this case, and the claim follows. ■

It follows from the Claim that $\mathrm{gr}_{V_0}(H_A(\beta)_p)$ contains all monomials in ∂ of a certain degree k that depends on A. Let $E = \theta_0 + \cdots + \theta_n$; by hypothesis $E - \beta_E \in H_A(\beta)$.

Lemma 4.9. Denote ∂_A^k the set of all monomials of degree k in $\partial_0, \ldots, \partial_n$, and $D_A \cdot \partial_A^k$ the left D_A-ideal generated by ∂_A^k. Then in $D_A / D_A \cdot \partial_A^k$, the identity $E(E-1)\ldots(E-k+1) \cong 0$ holds. □

Proof. This is clear if $k = 1$. In general, by induction,

$$E(E-1)\ldots(E-k+1) \in D_A \cdot \partial_A^{k-1} \cdot (E-k+1) = D_A \cdot E \cdot \partial_A^{k-1} \subseteq D_A \cdot \partial_A^k. \qquad ■$$

Remark 4.10. The homogeneity of X is necessary in the Claim, since otherwise $C(X)$ does not need to be contained in a hypersurface. Consider, for example, $A = (2, 1)$ in which case the union of all tangent lines (nearly) fills the plane, and where the zero locus of I_A and $A \cdot \mathscr{E} \cdot \partial$ contains always at least two points. $\square$

The lemma implies that $\mathrm{gr}^0_{V_0}(H_A(\beta)_p)$ contains $E(E-1)\dots(E-k+1)$ if p is generic. In other words, the b-function for restriction of $M_A(\beta)$ to a generic point divides $s(s-1)\dots(s-k+1)$.

In some cases, one can be more explicit about $k-1$, the top degree in which $R_A/R_A(I_A, A\cdot\mathscr{E}\cdot\partial)$ is non-zero. Suppose S_A is a Cohen–Macaulay ring, then systems of parameters are regular sequences. In particular, the Hilbert series of $Q_A := R_A/R_A(I_A, A\cdot\mathscr{E}\cdot\partial)$ is that of S_A multiplied by $(1-t)^d$. Suppose in addition, that S_A is normal. Since, we already assume that S_A is standard graded, let P be the polytope that forms the convex hull of the columns of A. The Hilbert series of S_A is then of the form $\sum_{m=0}^{\infty} p_m \cdot t^m$, where p_m is the number of lattice points in the dilated polytope $m\cdot P$. This number of lattice points is counted by the Erhart polynomial $E_P(m)$ of P, a polynomial of degree $d-1 = \dim(P)$. If one writes the Hilbert series of S_A in standard form $Q(t)/(1-t)^d$ then the Hilbert series of Q_A is just the polynomial $Q(t)$. In particular, the highest degree of a non-vanishing element of Q_A is the degree of $Q(t)$.

In order to determine $\deg(Q(t))$ let $E_P(m) = e_{d-1}m^{d-1} + \cdots + e_0$. Now in

$$\sum_{m=0}^{\infty} E_P(m)t^m = \sum_{i=0}^{d-1}\left(e_i \cdot \sum_{m=0}^{\infty} m^i \cdot t^m\right),$$

each term $\sum_{m=0}^{\infty} m^i \cdot t^m$, for $m > 0$, is a polylogarithm $\mathrm{Li}_{-i}(t)$ given by $(t\frac{\mathrm{d}}{\mathrm{d}t})^n(\frac{t}{1-t})$. A simple calculation shows that $\mathrm{Li}_{-i}(t)$ is the quotient of a polynomial of degree $i-1$ by $(1-t)^i$. Hence the sum in the display is the quotient of a polynomial of degree at most $d-1$ by $(1-t)^d$. The degree is truly $d-1$ as one can check from the differential expression for $\mathrm{Li}_{-i}(t)$.

Therefore, the Hilbert series $Q(t)$ of Q_A is a polynomial of degree $d-1$. We have proved

Theorem 4.11. Let S_A be standard graded. The b-function for restriction of $M_A(\beta)$ to a generic point $x+p=0$ divides $s(s-1)\dots(s-k+1)$ where k denotes the highest degree in which the quotient $S_A/S_A\cdot(A\cdot\mathscr{E}\cdot\partial)$ is non-zero. If, in addition, S_A is normal then one may take $k=d$.

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