Hypergeometric systems from groups with torsion

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Abstract

We consider A-hypergeometric (or GKZ-)systems in the case where the grading (character) group is an arbitrary finitely generated Abelian group. Emulating the approach taken for classical GKZ-systems in [MMW05] that allows for a coefficient module, we show that these D-modules are holonomic systems. For this purpose we formulate an Euler–Koszul complex in this context, built on an extension of the category of A-toric modules. We derive that these new systems are regular holonomic under circumstances that are similar to those that lead to regular holonomic classical GKZ-systems.

For the appropriate coefficient module, our D-modules specialize to the "better behaved GKZ-systems" introduced by Borisov and Horja. We certify the corresponding D-modules as regular holonomic, and establish a holonomic duality on the level of D-modules that was suggested on the level of solutions by Borisov and Horja and later shown by Borisov and Han in a special situation, [BH15, BH23].

Contents

1 Introduction

In a series of papers ([BH06, BPH13, BH15, BHW21, BH23]), Borisov together with various collaborators has introduced a system of linear partial differential equations called the *better*

behaved GKZ-system. It has considerable similarities with the members of a class of D-modules that generalize the widely studied GKZ-systems, which themselves are far-reaching generalizations of classical hypergeometric differential equations. We refer to our survey [RSSW21] for a comprehensive overview on GKZ-systems.

A GKZ-system arises from a finite set of elements in a finite-dimensional lattice (the character group), together with a complex parameter vector. In a more general setup, an additional component enters in the form of a coefficient module as introduced in [MMW05]. The systems studied by Borisov *et al.* are special cases in a more general setup. The generalization arises from replacing the lattice by an arbitrary finitely generated Abelian group. This introduction of possible torsion creates interesting new phenomena: for example, the associated toric ideal is now replaced by a smaller ideal that is no longer prime. Within this more general situation, Borisov and his collaborators consider two specific systems. One arises from taking the coefficient module to be the normalization of the associated toric ring, while the other comes from the canonical module of this normalization. In the classical (torsion-free) situation, these choices for the coefficient module assure that the rank of these systems is independent of the parameter vector (which otherwise is not always true, even in the classical case without coefficient module). This absence of rank jumps is why Borisov called these system "better behaved".

In the present paper, we study these systems in general: we allow torsion and arbitrary coefficent modules. We begin with a study of the ideals that replace of toric ideals, and in particular prove that they are still "reasonably close" to the toric ideal. This is necessary to bring in our main tool, the *Euler–Koszul complex* from [MMW05]. We show that these new systems are the terminal homology group of an appropriate Euler–Koszul complex, but in order to allow for torsion in the extended character group, we must extend the category of toric modules (which serve as coefficient modules) from [MMW05] to the category of *twisted toric module*, or *ttoric modules* for short, see Subsection 2.3.

For this extended category we show that the homology groups of the Euler–Koszul complex are holonomic, and we establish a criterion in terms of the coefficient module and the parameter vector to determine (non-)vanishing. As a consequence, we infer that all GKZ-systems with finitely generated Abelian character group and with arbitrary coefficient module are holonomic, and we characterize when those attached to the natural coefficient module are regular holonomic. These statements apply in particular to Borisov's systems.

Following the trail of ideas of [MMW05], we then investigate the behavior of these systems under holonomic duality. Euler–Koszul functors are self-dual up to a shift in the parameter vector, and thus holonomic duality reduces to standard duality over the underlying ttoric ring. This allows us to give a compact and conceptional proof of the duality statement in [BH23] regarding the systems considered in [BPH13], but for an arbitrary character group (allowing torsion) and without the assumption of normality of the underlying toric ring.

2   The $\mathcal{A}$-graded Category, Toric and Ttoric Modules

2.1   General Setup

For any set S, and any ring $\mathbb{K}$ we write $\mathbb{K}^S$ for the free $\mathbb{K}$-module $\bigoplus_{s \in S} \mathbb{K}$.

Let
$$N = F \oplus \mathbb{Z}^d$$

be a finitely generated Abelian group where F is the torsion part of N. The lattice
$$N^\vee := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$$

embeds naturally into $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}) \simeq N^{\vee} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}^d$. As in [BH06], let

$$\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subseteq N$$

be a finite multi-set of elements of N (*i.e.*, possibly with repetitions). We set

$$\ell := |F|$$

to be the torsion index, and choose

$$\beta \in N \otimes_{\mathbb{Z}} \mathbb{C}.$$

There is a morphism

$$\begin{aligned} \pi : N &\rightarrow N/F =: \bar{N} = \mathbb{Z}^d, \\ \mathbf{u} &\mapsto \bar{\mathbf{u}}, \end{aligned}$$

and whenever it suits us, we identify $\pi(N)$ with its image under the inclusion

$$\pi(N) \hookrightarrow \pi(N) \otimes_{\mathbb{Z}} \mathbb{R} =: N_{\mathbb{R}}.$$

We can identify non-canonically

$$\bar{N} = \bigoplus_1^d \mathbb{Z} \cdot \varepsilon_i$$

where $\varepsilon_1, \ldots, \varepsilon_d$ is a chosen basis for N. Denote by

$$K_{\mathbb{R}} := \mathbb{R}_{\geq 0} \cdot \pi(\mathcal{A}) \subseteq N_{\mathbb{R}}$$

the rational polyhedral cone generated by the images $\pi(\mathbf{a}_j) \in N_{\mathbb{R}}$.

Hypothesis 2.1. We shall work throughout under the following assumptions:

1. The set $A := \{\pi(\mathbf{a}_j) \mid 1 \leq j \leq n\}$ spans the lattice $\bar{N} = \pi(N)$ as a $\mathbb{Z}$-module; this implies in particular that the index

$$\delta := [N : \mathbb{Z}\mathcal{A}]$$

   is a (possibly strict) divisor of $\ell = |F|$.

2. The cone $K_{\mathbb{R}}$ is *pointed*: the origin is the only unit of the additive semigroup inherited from $\mathbb{R}^d$ on $K_{\mathbb{R}}$.

$\Diamond$

Among the set of all vector space morphisms $\{\tau \colon N_{\mathbb{R}} \to \mathbb{R}\}$ choose a minimal set $\mathcal{F}_K$ defined such that

$$K_{\mathbb{R}} = \bigcap_{\tau \in \mathcal{F}_K} (\tau^{-1}([0, \infty))$$

(minimality means that the intersection is irredundant) and set

$$K := \pi^{-1}(K_{\mathbb{R}}) \subseteq N, \tag{2.2}$$

the preimage under π. Then $\mathcal{F}_K$ is finite, and $\{K_{\mathbb{R}} \cap \tau^{-1}(0)\}_{\tau \in \mathcal{F}_K}$ are the *facets of $\mathcal{A}$*. As is traditional, we identify a facet τ with the intersection $\tau \cap \pi(A)$ as well as with the semigroup spanned by this intersection, or with $\tau \cap K$.

Notation 2.3. We will further denote by

$$A := \pi(\mathcal{A}),$$

the finite multi-set consisting of the images in $\bar{N} = \mathbb{Z}^d$ of the elements of $\mathcal{A}$ under the morphism π. Fixing a lattice basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for

$$L := \bigoplus \mathbb{Z} \cdot \mathbf{e}_i \simeq \mathbb{Z}^{\mathcal{A}}$$

we will always allow ourselves to view A as an integer $d \times n$ matrix, with columns $\{\bar{\mathbf{a}}_j\}$ and entries $(a_{i,j})$; as A is a multi-set, this matrix might have repeated columns.

We define N- resp. $\bar{N}$-graded polynomial rings

$$R_{\mathcal{A}} := \mathbb{C}[\{\partial_j \mid \mathbf{a}_j \in \mathcal{A}\}], \qquad R_A = \mathbb{C}[\{\partial_j \mid \bar{\mathbf{a}}_j \in A\}],$$

by grading them according to

$$\deg_{\mathcal{A}}(\partial_j) = \mathbf{a}_j, \qquad \deg_A(\partial_j) = \bar{\mathbf{a}}_j.$$

(If $\mathcal{A}$ is truly a multi-set, so $\mathbf{a}_j = \mathbf{a}_{j'}$ for some j, j' with $j \neq j'$ then $R_{\mathcal{A}}$ and R_A have one generator each for both copies).

We further introduce the non-commutative *Weyl algebras*

$$D_{\mathcal{A}} := R_{\mathcal{A}}[\{x_j \mid \mathbf{a}_j \in \mathcal{A}\}], \qquad D_A := R_A[\{x_j \mid \bar{\mathbf{a}}_j \in A\}],$$

that we grade by by setting $\deg_{(-)}(x_j) = -\deg_{(-)}(\partial_j)$.

In R_A there is the *toric* (prime) *ideal*

$$I_A := R_A \cdot \{\partial^{\mathbf{u}} - \partial^{\mathbf{v}} \mid A \cdot (\mathbf{u} - \mathbf{v}) = 0\}.$$

In analogy (by replacing A with $\mathcal{A}$), we define the $R_{\mathcal{A}}$-ideal

$$I_{\mathcal{A}} := R_{\mathcal{A}} \cdot \{\partial^{\mathbf{u}} - \partial^{\mathbf{v}} \mid \mathcal{A} \cdot (\mathbf{u} - \mathbf{v}) = 0\}, \qquad \textbf{(2.4)}$$

which we also call *toric*; the dot product here expresses the process of forming $\mathbb{Z}$-linear combinations (inside N) of elements of $\mathcal{A}$.

The quotient rings

$$S_A := R_A/I_A \quad \text{and} \quad S_{\mathcal{A}} := R_{\mathcal{A}}/I_{\mathcal{A}}$$

are the semigroup rings $S_A = \mathbb{K}[\mathbb{N}A]$ and $S_{\mathcal{A}} = \mathbb{K}[\mathbb{N}\mathcal{A}]$ respectively. $\qquad\qquad\qquad \diamondsuit$

There are graded ring morphisms

$$\mathrm{id}_R(\pi) \colon R_{\mathcal{A}} \to R_A, \qquad \mathrm{id}_D(\pi) \colon D_{\mathcal{A}} \to D_A,$$

induced by the identity map on the (ungraded) underlying rings, and by the morphism of grading groups $N \to \bar{N}$. They induce functors (both denoted ι_π) from the categories of graded $R_{\mathcal{A}}$- and $D_{\mathcal{A}}$-modules to those of graded R_A- and D_A-modules that are the identity map on the underlying modules. They are, however, not equivalences of categories, unless F is trivial.

Remark 2.5. 1. Note that syzygies between elements of $\mathcal{A}$ are also syzygies between the corresponding elements of A, and so $\iota_\pi(I_{\mathcal{A}}) \subseteq I_A$.

2. The ideal $I_{\mathcal{A}}$ is, in contrast to I_A, usually not prime. For example, if $\mathcal{A} = \{(1 \mod 4\mathbb{Z}, 1), (2 \mod 4\mathbb{Z}, 2)\} \subseteq N := (\mathbb{Z}/4\mathbb{Z}) \oplus \mathbb{Z}$, then $R_{\mathcal{A}}$ and R_A are $\mathbb{C}[\partial_1, \partial_2]$, $I_{\mathcal{A}} = R_{\mathcal{A}} \cdot (\partial_1^4 - \partial_2^2)$, $I_A = R_A \cdot (\partial_1^2 - \partial_2)$, and $I_{\mathcal{A}}^{[\ell]} \subseteq R_{\mathcal{A}}$ (defined in the proof of Lemma 2.9 below) is $R_{\mathcal{A}} \cdot (\partial_1^8 - \partial_2^4)$.

$\diamondsuit$

2.2 Binomial Ideals

We review some facts from [ES96]. Let

$$\rho \colon L_\rho \to \mathbb{C}^*$$

be a character on a sublattice L_ρ of L. We say that $L_\rho \hookrightarrow L$ is *saturated* (and then call ρ a *saturated partial character*) if the group L/L_ρ is torsion-free—or, equivalently, if L_ρ is a direct summand of L.

Remark 2.6. Suppose ρ' is a saturated partial character with associated sublattice L'. Let L'' be a complement, $L = L' \oplus L''$. Expand ρ' to a character ρ on all of L by setting $\rho(L'') = 1$. Then there is a monomial isomorphism ρ^* of R_A given by $\rho^*(\partial^{\mathbf{u}'+\mathbf{u}''}) = \rho(\mathbf{u}') \cdot \partial^{\mathbf{u}'+\mathbf{u}''}$ for any $\mathbf{u}' \in L', \mathbf{u}'' \in L''$. $\diamond$

From any partial character ρ we define a binomial ideal

$$I_+(\rho) := R_A \cdot \{\partial^{\mathbf{u}} - \rho(\mathbf{u} - \mathbf{v})\partial^{\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathbb{N}^n, \mathbf{u} - \mathbf{v} \in L_\rho\}.$$

This ideal is prime if and only if ρ is saturated.

Let now $\mathfrak{p}$ be an arbitrary binomial prime ideal in R_A. There is an induced partition $\{\partial_1, \dots, \partial_n\} = \{y_1, \dots, y_s\} \cup \{z_1, \dots, z_{n-s}\}$ where $\{y_1, \dots, y_s\} = \mathfrak{p} \cap \{\partial_1, \dots, \partial_n\}$ are the variables inside $\mathfrak{p}$. (Note that this perhaps not the entire linear part of $\mathfrak{p}$; there might be linear binomials in $\mathfrak{p}$). There is a corresponding splitting of lattices $\mathbb{Z}^n = \mathbb{Z}^s \times \mathbb{Z}^{n-s}$. By [ES96, Cor. 2.4], $\mathfrak{p}$ has the form

$$\mathfrak{p} = R_A \cdot \{y_1, \dots, y_s\} + I_+(\rho)$$

for some partial saturated character ρ in the lattice $\mathbb{Z}^{n-s}$.

A special case is when $L_\rho = \ker(A)$ and ρ is the trivial character; this results in $I_+(\rho) = I_A$. More generally, if $L_\rho = \ker(A)$ and ρ is an arbitrary character, we denote the prime ideal $I_+(\rho)$ by $I_{A,\rho}$. Note that $I_{A,\rho}$ relates to I_A via the R_A-isomorphism from Remark 2.6.

For a face τ of the cone $K_\mathbb{R} = \mathbb{R}_{\geq 0}A$, denote by I_A^τ the prime ideal defining the toric ring of τ as a quotient of R_A,

$$R_A/I_A^\tau \simeq \mathbb{C}[\tau \cap \mathbb{N}A].$$

Then I_A^τ is the ideal sum of I_A with the ideal generated by all ∂_j with $\bar{\mathbf{a}}_j \notin \tau$.

For a partial character ρ with $L_\rho = \ker(A)$ we define similarly

$$I_{A,\rho}^\tau := R_A \cdot (\{\partial_j \mid \bar{\mathbf{a}}_j \notin \tau\} \cup \underbrace{I_+(\rho)}_{=I_{A,\rho}}).$$

We note that $I_{A,\rho}^\tau$ is a prime ideal. Indeed, $R_A/I_{A,\rho}^\tau = R_B/I_{B,\rho'}$ where $B = A \cap \tau$ corresponds to the variables that are not generators of $I_{A,\rho}^\tau$, and where ρ' is the restriction of ρ to $\mathbb{Z}B$.

We are going to need a statement of the following form.

Lemma 2.7. *If $\mathfrak{p}$ is a prime ideal in R_A that is A-homogeneous, contains $I_{A,\rho}$ for some character ρ on $\mathbb{Z}A$, then it is of the form $I_{A,\rho}^\tau$.*

Proof. In the A-graded module $R_A/\mathfrak{p} = \bigoplus_{\mathbf{u} \in \mathbb{Z}A}(R_A/\mathfrak{p})_{\mathbf{u}}$, each graded component is either zero or a 1-dimensional vector space. Indeed, already the relations coming from $I_{A,\rho}$ identify (up to nonzero scalars) monomials of R_A if they have equal A-degree. Hence, any A-homogeneous element of $\mathfrak{p}/I_{A,\rho}$ is the coset of a monomial in R_A. But $\mathfrak{p}$ being prime implies that $\mathfrak{p}$ contains a variable appearing in such monomial. Let $V_\mathfrak{p}$ be the ideal generated by the variables that are in $\mathfrak{p}$ and let τ be the face of A defined by $[\partial_j \in V_\mathfrak{p}] \Leftrightarrow [\mathbf{a}_j \notin \tau]$; then $\mathfrak{p} = I_{A,\rho}^\tau$. $\square$

2.3 Toric Modules

We recall from [MMW05] the notion of a *toric A-module*, as an A-graded R_A-module that has a finite filtration by A-graded modules such that each composition factor is graded isomorphic to a shifted copy of some R_A/I_A^τ where I_A^τ is the toric ideal of one of the faces of the cone spanned by A (the τ depending on the composition factor). We modify this concept as follows.

Definition 2.8. An $\mathcal{A}$-*toric module* is an $\mathcal{A}$-graded $R_\mathcal{A}$-module that has a finite filtration by $\mathcal{A}$-graded modules such that each composition factor is an $\mathcal{A}$-graded quotient of $R_\mathcal{A}/I_\mathcal{A}$. Note that $\mathcal{A}$-toric modules are necessarily finitely generated. $\diamond$

Lemma 2.9. *For any $\mathcal{A}$-toric module M, the A-graded module $\iota_\pi(M)$ has a finite filtration by A-graded submodules such that each composition factor is graded isomorphic to a shifted copy of some $R_A/I_{A,\rho}^\tau$ for suitable ρ, τ depending on the composition factor.*

Proof. Recall that $\ell = |F|$ is the torsion index, and introduce the $R_\mathcal{A}$-ideal

$$I_\mathcal{A}^{[\ell]} := R_\mathcal{A} \cdot \{\partial^{\ell \mathbf{u}} - \partial^{\ell \mathbf{v}} \mid A \cdot \mathbf{u} = A \cdot \mathbf{v}\}.$$

Note that this is an $R_\mathcal{A}$-ideal but the conditions defining it come from A. Since $\ell \mathbf{a} \in 0 \oplus \mathbb{Z}^d \subseteq F \oplus \mathbb{Z}^d$ for any $\mathbf{a} \in N$, it follows that if $\mathbf{a}$ is a linear relation on A then $\ell \mathbf{a}$ is a linear relation on $\mathcal{A}$. In particular, $I_\mathcal{A}^{[\ell]} \subseteq I_\mathcal{A}$, and so $I_\mathcal{A}^{[\ell]}$ is $\mathcal{A}$-graded and there is an A-graded surjection $\iota_\pi(R_\mathcal{A}/I_\mathcal{A}^{[\ell]}) \twoheadrightarrow \iota_\pi(R_\mathcal{A}/I_\mathcal{A})$.

By definition, $\iota_\pi(I_\mathcal{A}^{[\ell]})$ is an A-graded binomial ideal in a polynomial ring, and by [ES96] all its associated prime ideals $\{\mathfrak{p}_k\}$ are A-graded binomial primes.

We study now the binomial A-graded prime ideals $\mathfrak{p} \subseteq R_A$ containing $\iota_\pi(I_\mathcal{A}^{[\ell]})$. For any $\partial^\mathbf{u} - \partial^\mathbf{v} \in I_A$ we have $\partial^{\ell \mathbf{u}} - \partial^{\ell \mathbf{v}} = \prod(\partial^\mathbf{u} - \zeta_\ell^i \partial^\mathbf{v}) \in \iota_\pi(I_\mathcal{A}^{[\ell]})$, where ζ_ℓ is a primitive ℓ-th root of unity, and where i runs through the elements of $\mathbb{Z}/\ell\mathbb{Z}$. A prime ideal $\mathfrak{p}$ containing these products will contain at least one factor from each such product, and so for each binomial in I_A such prime ideal $\mathfrak{p}$ contains that same binomial twisted by a power of ζ_ℓ. On the other hand, by [ES96], $\mathfrak{p}$ is the sum of a binomial prime ideal $I_+(\rho)$ with an ideal $V_\mathfrak{p}$ of variables, and we can assume that $I_+(\rho)$ contains no monomial of degree one. Suppose one of the binomials $\partial^\mathbf{u} - \zeta_\ell^i \partial^\mathbf{v} \in \mathfrak{p}$ is not in $R_A \cdot V_\mathfrak{p}$. Then neither of its two monomials can be in $\mathfrak{p}$ since else they would have to be in $V_\mathfrak{p}$ as $\mathfrak{p}$ is prime. Thus, such binomial is in $I_+(\rho)$, which then forces $\mathbf{u} - \mathbf{v} \in L_\rho$. But then $\partial^\mathbf{u} - \zeta_\ell^i \partial^\mathbf{v}$ and $\partial^\mathbf{u} - \rho(\mathbf{u} - \mathbf{v})\partial^\mathbf{v}$ need to agree up to a constant factor, since both are in $\mathfrak{p}$ (and hence so are their linear combinations). It follows that for each $\mathbf{u} - \mathbf{v} \in \ker(A)$, either both $\partial^\mathbf{u}, \partial^\mathbf{v}$ are in $V_\mathfrak{p}$, or $I_+(\rho)$ contains a $\partial^\mathbf{u} - \zeta_\ell^i \partial^\mathbf{v}$ with i depending on $\mathbf{u}, \mathbf{v}$.

Pick an extension ρ' of ρ to $\mathbb{Z}^n$, which exists since $\mathfrak{p}$ is prime and hence ρ is saturated. Then the monomial automorphism of R_A from Remark 2.6 that sends $\partial^\mathbf{u}$ to $\rho'(\mathbf{u}) \cdot \partial^\mathbf{u}$ sends $V_\mathfrak{p}$ to itself and $V_\mathfrak{p} + I_+(\rho)$ to an A-homogeneous prime ideal generated by variables and binomials containing I_A. The only such ideals are, by Lemma 2.7 the ideals I_A^τ, and so $\mathfrak{p} = I_{A,\rho''}^\tau$ for some character ρ'' on L, and some face τ of $\mathbb{N}A$. In particular, the associated primes of $\iota_\pi(R_\mathcal{A}/I_\mathcal{A}^{[\ell]})$ are all of the form $I_{A,\rho''}^\tau$ for suitable τ, ρ''.

Suppose now that M is an arbitrary $\mathcal{A}$-toric module, and choose one of its composition factors $\tilde{M}_i := M_i/M_{i-1}$, an $\mathcal{A}$-graded quotient of $R_\mathcal{A}/I_\mathcal{A}$. Then $\iota_\pi(\tilde{M}_i)$ is an A-graded quotient of $\iota_\pi(R_\mathcal{A}/I_\mathcal{A}^{[\ell]})$.

It is well-known that any finitely generated ($\bar{N}$-graded) module $\tilde{M}$ over any ($\bar{N}$-graded) Noetherian ring R permits a finite filtration by ($\bar{N}$-graded) modules such that each composition factor is, up to a shift in the grading, of the form $R/\mathfrak{p}$ where $\mathfrak{p}$ is an associated ($\bar{N}$-graded) prime ideal of $\tilde{M}$ over R. We apply this to $R = \iota_\pi(R_\mathcal{A}/I_\mathcal{A}^{[\ell]})$ and $\tilde{M} = \iota_\pi(\tilde{M}_i)$. This means that

$\iota_\pi(\tilde{M}_i)$ has an A-graded filtration such that its composition factors $\tilde{M}_{i,j}$ are, up to a shift in the grading, of the form $R_A/\mathfrak{p}_{i,j}$ where each $\mathfrak{p}_{i,j}$ is an A-graded prime containing $\iota_\pi(I_\mathcal{A}^{[q]})$. By what we have proved earlier, each $\mathfrak{p}_{i,j}$ is thus of the form $I_{A,\rho}^\tau$ for some face of $\mathbb{N}A$ and some character on L, and hence each $\bar{M}_{i,j}$ has the claimed property.

It is now standard to assemble a composition chain for M from these composition chains of its composition factors. $\qquad\square$

Inspired be this lemma, we make the following definition.

Definition 2.10. An A-graded R_A-module M is A-*twisted-toric*, or short just A-*ttoric*, if it has a finite filtration such that each composition factor is A-graded isomorphic to a shifted copy of $R_A/I_{A,\rho}^\tau$ for some face τ of A and some character ρ on the (saturated) lattice $\mathbb{R}\tau \cap N$. $\qquad\diamondsuit$

The image under ι_π of any $\mathcal{A}$-toric module M is A-ttoric by Lemma 2.9.

3 Euler–Koszul Technology

We review from [MMW05] the following concepts and constructions. If $m \in M$ is an A-homogeneous element of A-degree $\mathbf{u} = \sum u_i \varepsilon_i$ inside an A-graded module M, set $\deg_{A,i}(m) := u_i$. For $1 \le i \le d$, let

$$E_i = \sum_{j=1}^n a_{i,j} x_j \partial_j \in D_A$$

be the i-th *Euler operator*. Then $\deg_A(E_i) = 0$ and we have the commutator relation

$$[E_i, P] = -\deg_{A,i}(P) \cdot P$$

for all A-homogeneous $P \in D_A$.

For M any A-graded R_A-module, and for all $\beta \in L \otimes_\mathbb{Z} \mathbb{C} = \bigoplus_1^d \mathbb{C} \cdot \varepsilon_i$, one can define an endomorphism (denoted $(E_i - \beta)\bullet$) of left D_A-modules on $D_A \otimes_{R_A} M$ by setting

$$(E_i - \beta) \bullet (P \otimes m) := (E_i + \deg_{A,i}(P) + \deg_{A,i}(m) - \beta_i))(P \otimes m).$$

If $M = R_A$ with A-grading that places $1 \in R_A$ into degree zero, this endomorphism on $D_A \otimes_{R_A} R_A = D_A$ of left D_A-modules is right-multiplication by $E_i - \beta_i$ on D_A as one checks from the commutator relations.

The endomorphisms $(E_i - \beta_i)\bullet$ and $(E_{i'} - \beta_{i'})\bullet$ commute for all i, i' and on all A-graded modules $D_A \otimes_{R_A} M$, and thus one can define the Koszul complex on the endomorphisms

$$\left(0 \to D_A \otimes_{R_A} M \xrightarrow{(E_i-\beta_i)\bullet} D_A \otimes_{R_A} M \to 0 \right).$$

The resulting complex $K_{A,\bullet}(M; \beta)$ (well-defined up to ordering the factors in the tensor product) is the *Euler–Koszul complex* to M and β and we view it as positioned in such a way that the terminal module $D_A \otimes M$ of this complex is situated in homological degree zero. We denote the i-th homology module by $H_{A,i}(M; \beta)$.

One of the main results in [MMW05] is that every homology group of $K_{A,\bullet}(M, \beta)$ is a holonomic left D_A-module, provided that M is an A-toric R_A-module. We prove next a generalization of this fact to A-ttoric modules.

Theorem 3.1. *If M is A-ttoric, then for every $\beta \in N_\mathbb{C}$, the Euler–Koszul complex $K_{A,\bullet}(M, \beta)$ is holonomic (in the sense that each of its homology groups is a holonomic D_A-module). Moreover, $K_{A,\bullet}(M; \beta)$ is exact (has zero homology) if and only if $\beta \notin \mathrm{qdeg}_A(M)$.*

Proof. We review the cornerstones of the proof in the A-toric case from [MMW05] and indicate the necessary changes for the A-ttoric case.

If $0 \to M' \to M \to M'' \to 0$ is an A-graded exact sequence of A-toric modules, the Euler–Koszul functor induces a long exact sequence of Euler–Koszul homology modules. Basic properties of holonomic modules imply that $H_{A,\bullet}(M;\beta)$ is holonomic if and only if both $H_{A,\bullet}(M',\beta)$ and $H_{A,\bullet}(M'';\beta)$ are holonomic. The definition of A-toric modules implies that holonomicity of $K_{A,\bullet}(M;\beta)$ follows from holonomicity of all $K_{A,\bullet}(R_A/I_A^\tau;\beta)$.

The *true A-degrees* $\mathrm{tdeg}_A(M)$ of an A-graded module $M = \bigoplus_{\mathbf{u}\in\bar N} M_{\mathbf{u}}$ are the collection $\{\mathbf{u} \in \bar N \mid M_{\mathbf{u}} \neq 0\}$. As a subset of $\bar N_{\mathbb{C}} = \bigoplus \mathbb{C}\cdot\boldsymbol{\varepsilon}_i$, the Zariski closure of $\mathrm{tdeg}_A(M)$ is the set of *A-quasi-degrees* $\mathrm{qdeg}_A(M)$. For a toric A-module, the quasi-degrees are a finite union of subspaces of the form $\mathbb{C}\cdot(A\cap\tau) + \mathbf{u}$ for faces τ of A and suitable shift vectors $\mathbf{u} \in \bar N$. The quasi-degrees of R_A/I_A^τ are exactly $\mathbb{C}\cdot(A\cap\tau)$. One shows next that $K_{A,\bullet}(R_A/I_A^\tau;\beta)$ is an exact complex if and only if β is not a quasi-degree of R_A/I_A^τ. Since the construction of an Euler–Koszul complex is designed to work well with degree shifts, it follows that $K_{A,\bullet}(M;\beta)$ is exact if and only if $\beta \in \mathrm{qdeg}_A(M)$.

Let now $\beta \in \mathrm{qdeg}_A(R_A/I_A^\tau) = \mathbb{C}\cdot(A\cap\tau)$. The main (*i.e.*, terminal) homology group $H_{A,0}(R_A/I_A^\tau;\beta)$ is holonomic since it can be shown to be isomorphic to the D-module inverse image under a projection, from $\bar N_{\mathbb{C}}$ to a quotient space of $\bar N_{\mathbb{C}}$, of a standard GKZ-system on that quotient space. (More precisely, $H_{A,0}(R_A/I_A^\tau;\beta)$ is of the form $H_{B,0}(R_B/I_B;\gamma)\otimes_{\mathbb{C}}\mathbb{C}[y_1,\ldots,y_k]$ where B is a full rank matrix with the same span as $\tau\cap A$ and k fewer columns than A, compare the proof of [MMW05, Lemma 4.9]). This then implies that all $H_{A,0}(M;\beta)$ are holonomic for toric M. The higher Euler–Koszul homologies of R_A/I_A^τ are the cosets of A-homogeneous elements m in the kernel of the Euler–Koszul complex. Their classes are annihilated by each $(E_i - \beta - \deg(m))\bullet$, and also by a power of I_A^τ (compare the proof of [MMW05, Prop. 5.1]). Using a filtration argument, they are a quotient of a finite direct sum of various $H_{A,0}(R_A/I_A^\tau;\gamma)$ and hence holonomic.

Inspecting this proof strategy shows that in order to adapt this proof to the A-ttoric case, one just needs to know that $H_{A,0}(R_A/I_{A,\rho}^\tau;\beta)$ is holonomic, and that furthermore it is nonzero if and only if β is an A-quasi-degree of $R_A/I_{A,\rho}^\tau$.

Let $M = R_A/I_{A,\rho}^\tau$. If β lies outside $\mathbb{C}(A\cap\tau) = \mathrm{qdeg}_A(M)$, then the $\mathbb{C}$-linear span of $E - \beta$ together with the products $\{x_j\partial_j \mid \mathbf{a}_j \notin \tau\}$ is a subset of $D_A\cdot I_{A,\rho}^\tau$ and contains a nonzero scalar (compare [MMW05, Lemma 4.9]); hence, $H_{A,0}(M;\beta) = D_A/D_A(I_{A,\rho}^\tau, \{E_i - \beta_i\})$ is zero. As in the untwisted case, this implies the vanishing of all higher homology groups. If, on the other hand, β is in the span of $A\cap\tau$ then choose $\dim(\tau)$ many rows of A such that its restriction to the columns in τ of these rows are linearly independent. Then write B for the corresponding $\mathrm{rank}(A\cap\tau)\times|A\cap\tau|$-submatrix, denote β^τ the restriction of β to the column span of B, and set $E_i^\tau := \sum_{j\in\tau} a_{i,j}x_j\partial_j$. If now ρ^B is the restriction of ρ to $\ker(B) \subseteq \ker(A)$, it follows that $H_{A,0}(M;\beta) = (D_B/D_B\cdot(I_{B,\rho^B}, \{E_i^\tau - \beta_i^\tau\}))\otimes_{\mathbb{C}}\mathbb{C}[\{x_j \mid \mathbf{a}_j \notin \tau\}]$. The partial saturated character ρ^B can be used to induce an automorphism of (R_B as in Remark 2.6, and hence also on) the Weyl algebra D_B that carries $D_B/D_B\cdot(I_{B,\rho^\rho}, \{E_i^\tau - \beta_i^\tau\})$ to the classical GKZ-system $H_{B,0}(R_B/I_B;\beta^\tau)$. The latter is, of course, nonzero. $\square$

The *rank* of a D_A-module $\mathscr{M}$ is the vector space dimension of its classical solution space near a generic point of $\mathbb{C}^n$. The Cauchy–Kovalevskaya–Kashiwara Theorem shows that this is also the dimension over $\mathbb{C}(\boldsymbol{x})$ of the vector space $\mathbb{C}(\boldsymbol{x})\otimes_{\mathbb{C}[\boldsymbol{x}]}\mathscr{M}$. For modules such as $H_{A,0}(M;\beta)$ that arise from toric modules M, it is shown in [MMW05] that the rank is positive precisely when β is a quasi-degree of M. This generalizes to the A-ttoric case.

Lemma 3.2. *The parameter β is a quasi-degree of the A-ttoric module M precisely if $H_{A,0}(M;\beta)$ has positive rank. More precisely, the rank function is upper-semicontinuous in β.*

Proof. Since rank is additive in short exact sequences, it suffices to prove the statement when $M = R_A/I_{A,\rho}^\tau$ for some face τ and some character ρ on $\ker(A)$. In that case, the automorphism on R_A arising from Remark 2.6, and the induced automorphism of D_A (that acts on x_j inversely to the action on ∂_j) reflects the issue from $\mathbb{C}(\boldsymbol{x}) \otimes_{R_A} \mathscr{M}$ back to $H_{A,0}(R_A/I_A^\tau; \beta)$, in which case the result is known from [MMW05].

The final claim follows the lines of [MMW05] since A-ttoric modules give holonomic families. $\square$

Corollary 3.3. *For any A-ttoric module M, there is a subspace arrangement $\mathcal{E}_M$ in $\mathbb{C}^d$ such that $\beta \notin \mathcal{E}_M$ implies that the rank of $H_{A,0}(M; \beta)$ is independent of β.*

Proof. Following the train of the proof of Lemma 3.2, one needs to inspect the statement of the lemma only when $M = R_A/I_{A,\rho}^\tau$. But because of the isomorphism induced via Remark 2.6, the rank of $H_{A,0}(R_A/I_{A,\rho}^\tau; \beta)$ is the same as that of $H_{A,0}(R_A/I_A^\tau; \beta)$, and so one may once again refer to [MMW05]. $\square$

4 D-modules from Modules over Semigroups

In the next section we will review the definition of the systems introduced by Borisov and Horja in [BH06, BPH13], show that they arise as terminal homology groups of an Euler–Koszul complex on a suitable ttoric A-module, and deduce finiteness conditions about it. Here, we lay some ground work regarding modules over semigroups.

4.1 Semigroup Modules

Convention 4.1. Throughout, we will assume that our semigroups are Abelian, have an identity, and inject into their Grothendieck group. $\diamondsuit$

If S is any semigroup, then a set T is an *S-module* if there is an action

$$\bullet \colon S \times T \to T$$

that is additive in S. We call T *finitely generated (over S)* if $T = \bigcup S \bullet t$ with t running through some finite subset of T. Let S_+ be the non-units in S and remark that there are only finitely many units in S (since the torsion is finite and $\pi(S)$ is pointed).

We refer to
$$T_{\mathrm{prim}} := T \smallsetminus (S_+ \bullet T)$$

as the *(S-)primitive elements of T*; this is the natural candidate set for a "collection of minimal generators of T over S".

Lemma 4.2. *Suppose S is a finitely generated semigroup and write S_+ for the non-invertible elements of S. Suppose $T = \bigcup_1^k S \bullet t_i$ is a finitely generated S-module, such that $s \bullet t = t$ only occurs when s is a unit. Then the set $T_{\mathrm{prim}} := T \smallsetminus S_+ \bullet T$ is finite and generates T over S.*

Proof. Let U be the units in S; then one may form the factor semigroup $\bar{S} := S/U$ which acts on T as well. The identification space $\bar{T}$ obtained from T by identifying for all $t \in T$ the elements $\{u \bullet t \mid u \in U\}$ is an $\bar{S}$-module and $\bar{s} \bullet \bar{t} = \bar{t}$ if and only if $\bar{s}$ is the coset of the identity of S. Then $S_+/U = (\bar{S})_+$ and cosets of primitive elements in T consist exclusively of primitive elements of T, forming primitive elements of $\bar{T}$. Finiteness of U now implies that for the purpose of the proof we may assume that U is trivial, $S = \bar{S}$ and $T = \bar{T}$. Note that this reduction eliminates all torsion from S.

In general, let $t \in T_{\mathrm{prim}}$; according to the setup it is in $\bigcup_1^k S \bullet t_i$ and so there is $i \in \{1, \ldots, k\}$ and $s_i \in S$ with $t = s_i \bullet t_i$. But for a primitive element this can only happen if s_i is a unit (in the reduced setting, the identity,) of S. Since T is assumed to be finitely generated over S, we have shown that $|T_{\mathrm{prim}}|$ is at most k times the number of units of S.

We introduce a descending *level* filtration on T by $\mathrm{level}(t) \geq h$ if $t \in \underbrace{(S_+ + S_+ + \ldots + S_+)}_{h \text{ copies}} \bullet T$.

We say that t is of *height* h if $\mathrm{level}(t) \geq h$ but $\mathrm{level}(t) \not\geq h + 1$. It is convenient to say that every element of T is in level zero, and thus we attach to $t \in T \setminus S_+ \bullet T$ the height zero.

Let now $g_1, \ldots, g_q$ be nonzero semigroup generators for S and let $t_1, \ldots, t_k$ be generators for T over S. The reduction $S = \bar{S}$ forces $g_i \in S_+$, and every element of S_+ can be written as a sum of the generators (perhaps in many ways).

We claim that every $t \in T$ has well-defined, finite, height. Indeed, suppose t is in every level. So there are equations $t = (\sum_{i=1}^{h'} g_{i,h}) \bullet t_{i_h}$ for each $h \in \mathbb{N}$, with $h' \geq h$ and $g_{i,h}$ one of our chosen generators.

Since there are only finitely many generators t_{i_h} for T, but infinitely many such expressions for t, we can choose a subsequence for which always the same t_{i_h} is used (and discard the other expressions for t).

Let $\mu_{h,j}$ be the number of times that the generator g_j appears in the chosen h-th sum $t = (\sum_{i=1}^{h'} g_{i,h}) \bullet t_{i_h}$. As S is finitely generated, there is at least one generator g_1 of S such that the set $\{\mu_{h,1}\}$ is finite. Choose a subsequence of the infinitely many expressions for t in which the number of times g_1 appears is strictly increasing, and discard all other expressions for t.

Now it can happen that for all other generators $g_i \in S_+$ of S the sets $\{\mu_{i,h}\}$ are finite. If so, move to the next paragraph. Otherwise, take a generator g_2 that appears with unbounded multiplicity, and choose a subsequence of expressions for t in which now both $\mu_{h,1}$ and $\mu_{h,2}$ are strictly increasing. Repeat if necessary.

In this manner, we find a sequence of expressions for t in which the multiplicities for $g_1, \ldots, g_r \in S_+$ are strictly increasing, and the multiplicities for $g_{r+1}, \ldots, g_q$ are uniformly bounded. The pigeon hole principle now dictates that there are two expressions for t in which for all g_i the multiplicity of g_i in the second expression strictly dominates the multiplicity in the first. So we have

$$(a_1 g_1 + \ldots + a_q g_q) \bullet t_i = t = (b_1 g_1 + \ldots b_q g_q) \bullet t_i,$$

where $a_j < b_j$ for $j = 1, \ldots, q$, and so $\sum_{j=1}^q (b_j - a_j) g_j \in S_+$ acts on t_i as identity. By hypothesis, this implies that this sum is a unit (that is, the identity in the reduced setting). By contradiction, t must have a finite level, as claimed.

We are now ready finish the proof of the lemma. By definition, T_{prim} are the elements of height zero. Any generator t_i of T that is not height zero is of the form $s \bullet t$ for some $s \in S_+$ and some $t \in T$. By definition of level and height, at least one such rewriting $t_i = s \bullet t$ exists where the height of t is smaller than the height of t_i. Then $\{t_1, \ldots, t_k, t\} \setminus \{t_i\}$ is a generating set for T over S. By iteration, we can replace any element of positive height among $\{t_1, \ldots, t_k\}$ by one of height zero, while preserving the fact that they are a generating set. It follows that T_{prim} generates T over S. $\qquad\square$

Notation 4.3. On the semigroups $\mathbb{N}\mathcal{A}$ and $\mathbb{N}A$ there is a tautological degree function, $\deg_A(-)$ with values in $\mathbb{Z}^d$ that sends an element to its natural image in $\mathbb{N}A$.

Suppose T is an S-module with $S = \mathbb{N}A$ or $\mathbb{N}\mathcal{A}$. We say T *has an A-grading* if there is a map $(-)_A \colon T \to \mathbb{Z}A$ such that for all $s \in S, t \in T$ we have $(s \bullet t)_A = \deg_A(s) + (t)_A$. All S-modules inside the Grothendieck group $\mathbb{Z}S$ have a natural A-grading. $\qquad\diamond$

Define for an $\mathbb{N}\mathcal{A}$-module T the $S_{\mathcal{A}}$-module

$$M_T := \bigoplus_{t \in T} \mathbb{K} \cdot t$$

via the rule $(c_s \cdot s) \bullet (c_t \cdot t) := (c_s c_t) \cdot (s \bullet t)$, for $c_s, c_t \in \mathbb{K}, s \in \mathbb{N}\mathcal{A}, t \in T$. If T has an A-grading $(-)_A$, then M_T becomes an A-graded $R_{\mathcal{A}}$-module via $\deg_A(c_t \cdot t) := (t)_A$. If T is finitely generated over $\mathbb{N}\mathcal{A}$ then M_T is a finitely generated A-graded $S_{\mathcal{A}} = \mathbb{K}[\mathbb{N}\mathcal{A}]$-module, and hence $\mathcal{A}$-toric.

Theorem 4.4. *For every finitely generated A-graded $\mathbb{N}\mathcal{A}$-module T and all $\beta \in N_{\mathbb{C}}$, the D_A-module $H_{A,0}(\iota_\pi(M_T); \beta)$ is holonomic. In fact, $H_{A,0}(\iota_\pi(M_T); \beta)$ is the terminal homology group of an Euler–Koszul complex in which every homology group is a holonomic D_A-module. This Euler–Koszul complex is exact if and only if β is not in the Zariski closure of $\deg_A(T)$.*

Proof. In view of Theorem 3.1, all we need to show is that $\iota_\pi(M_T)$ is A-ttoric. But that follows from Lemma 2.9. $\square$

Corollary 4.5. *If there is a linear functional $h \colon N \to \mathbb{R}$ such that $\mathcal{A} \subseteq h^{-1}(1)$ then for any finitely generated $\mathbb{N}\mathcal{A}$-module T and for all $\beta \in \mathrm{Hom}(N, \mathbb{C})$, the Euler–Koszul complex on M_T has regular holonomic homology.*

Proof. By [SW08], every GKZ-system $H_A(\beta)$ is regular holonomic if a functional h as in the corollary exists. In the proof of Theorem 3.1 it is shown that every homology group of the Euler–Koszul complex on M_T is a quotient of a finite sum of modules that are isomorphic to A-hypergeometric systems in the sense of [MMW05]. The corollary follows. $\square$

4.2 Rank and Duality

The transposition $\tau(x^{\mathbf{a}} \partial^{\mathbf{b}}) = (-\partial)^{\mathbf{b}} x^{\mathbf{a}}$ on D_A provides an equivalence of the categories of left and right D_A-modules, that corresponds on the level of sheaves to the tensor product with the canonical sheaf $\omega_{\mathbb{C}^n}$ in one direction, and $\mathscr{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\omega_{\mathbb{C}^n}, -)$ in the other.

The automorphism $x \mapsto -x$ on $\mathbb{C}^n$ induces, via $\partial \mapsto -\partial$, an auto-equivalence $(-) \mapsto (-)^-$ on the categories of $\mathbb{Z}A$-graded R_A-modules and D_A-modules (but not on the category of $\mathbb{Z}A$-graded S_A-modules, since I_A is not preserved under $(-)^-$ unless it is projective). The formation of Euler–Koszul complexes is equivariant under this sign change since $E_i - \beta_i = (E_i - \beta_i)^-$. Moreover, for $\mathbb{Z}A$-graded R_A-modules N we have $D_A \otimes_{R_A} N \simeq \tau(N \otimes_{R_A} D_A)^-$ as left D_A-modules, where the tensor products exploit the two different R_A-structures on D_A.

Let M be a finitely generated $\mathbb{Z}A$-graded R_A-module and choose a minimal free $\mathbb{Z}A$-graded resolution $F_\bullet$. Denoting

$$\varepsilon_A := \sum_{\mathbf{a} \in \mathcal{A}} \pi(\mathbf{a}) \tag{4.6}$$

the sum of all elements of the multi-set A, it is explained in [MMW05, Section 6], that there is a natural identification of the complexes

$$\tau \mathrm{Hom}_{D_A}(K_{A,\bullet}(F_\bullet; E - \beta), D_A)^- \simeq K_{,\bullet}(\mathrm{Hom}_{D_A}(F_\bullet, D_A); -E - \beta - \varepsilon_A).$$

The same construction works if $F_\bullet$ is any free finite $\mathbb{Z}A$-graded complex, and in this setting $F_\bullet$ having toric homology assures the resulting homology groups of the Euler–Koszul complex to be holonomic. In consequence, for a finite $\mathbb{Z}A$-graded complex $F_\bullet$ with toric homology, the holonomic dual of the higher Euler–Koszul homology $H_{A,i}(F_\bullet; \beta)$ is, up to a degree shift by ε_A, the Euler–Koszul homology $H_{A,n-i}(\mathrm{Hom}_{D_A}(F_\bullet, D_A); -\beta - \varepsilon_A)^-$. In particular, if $F_\bullet$

has only one homology module M, say in cohomological position zero, then $\mathbb{D}H_{A,i}(M,\beta) = H_{A,n-i}(\mathrm{Hom}_{D_A}(F_\bullet, D_A); -\beta - \varepsilon_A)^-$, and if M is a Cohen–Macaulay R_A-module of dimension d then (there is no higher Euler–Koszul homology of M, and)

$$\mathbb{D}H_{A,0}(M,\beta) = H_{A,0}(\mathrm{Ext}_{R_A}^{n-d}(M,R_A); -\beta - \varepsilon_A)^-. \tag{4.7}$$

We want this isomorphism for finite free $\mathbb{Z}A$-graded D_A-complexes with A-ttoric homology. Inspecting the proof of Thorem 6.3 in [MMW05] reveals that the only property of "toric F" that is used is that A-toric modules are A-graded and produce holonomic Euler–Koszul homology modules. That, however is also the case for A-ttoric modules.

Corollary 4.8. *The duality statement* (**4.7**) *applies in particular to $\mathcal{A}$-toric modules M.*

5   Applications

Definition 5.1 (Differential systems on $\mathcal{A}$-toric modules)**.** Choose $r \in \mathbb{N}$, $\beta \in N \otimes_{\mathbb{Z}} \mathbb{C}$, and let $T \subseteq N^r$ be a module over the semigroup $\mathbb{N}\mathcal{A} \subseteq N$. Let

$$\Phi_T = (\phi_{\mathbf{u}})_{\mathbf{u} \in T}$$

be an assignment of a (sufficiently differentiable) function in x_A to each element of T. The natural morphism $N^\vee = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{C}}(N \otimes \mathbb{C}, \mathbb{C})$ allows to define $\mu(\beta)$ for any $\mu \in N^\vee$.

Consider the following system of partial differential equations:

$$\{\partial_j \phi_{\mathbf{u}} = \phi_{\mathbf{u}+\mathbf{a}_j}\}_{\mathbf{u} \in T, \mathbf{a}_j \in \mathcal{A}} \qquad \cup \qquad \{\sum_{\mathbf{a}_j \in \mathcal{A}} \mu(\mathbf{a}_j) x_j \partial_j \phi_{\mathbf{u}} = \mu(\beta - \mathbf{u}) \phi_{\mathbf{u}}\}_{\mathbf{u} \in T, \mu \in N^\vee}.$$

Let us write

$$E_\mu := \sum_{\mathbf{a}_j \in \mathcal{A}} \mu(\mathbf{a}_j) x_j \partial_j \quad \in D_{\mathcal{A}}.$$

$$\diamondsuit$$

A system of type Φ_T induces a $D_{\mathcal{A}}$-module

$$\Phi_{\mathcal{A},0}(T;\beta) := D_{\mathcal{A}}^T / H_{\mathcal{A}}(T;\beta),$$

where $H_{\mathcal{A}}(T;\beta) \subseteq D_{\mathcal{A}}$ is the left $D_{\mathcal{A}}$-module

$$H_{\mathcal{A}}(T;\beta) := D_{\mathcal{A}} \cdot \left(\{\partial_j 1_{\mathbf{u}} - 1_{\mathbf{u}+\mathbf{a}_j}\}_{\mathbf{u} \in T, \mathbf{a}_j \in \mathcal{A}} \quad \cup \quad \{(E_\mu - \mu(\beta - \mathbf{u})) \cdot 1_{\mathbf{u}}\}_{\mathbf{u} \in T, \mu \in N^\vee}\right) \tag{5.2}$$

and $1_{\mathbf{u}}$ is the element of $\mathcal{D}^T$ that is 1 in coordinate $\mathbf{u}$ and zero everywhere else.

The system Φ_T is the space of classical solutions to $\Phi_{\mathcal{A}}(T;\beta)$. In particular, it can be identified with the localization of $\Phi_{\mathcal{A}}(T;\beta)$ to a generic point.

Since T_{prim} generates T as $\mathbb{N}\mathcal{A}$-module by Lemma 4.2, the defining relations spelled out in (**5.2**) imply that $\Phi_{\mathcal{A}}(T;\beta)$ is as $D_{\mathcal{A}}$-module generated by the cosets of all $1_{\mathbf{u}}$ with $\mathbf{u} \in T_{\mathrm{prim}}$; it is hence a finitely generated $D_{\mathcal{A}}$-module and we can replace in the definition of $\Phi_{\mathcal{A},0}(T;\beta)$ the set T by T_{prim}:

$$\Phi_{\mathcal{A},0}(T;\beta) \simeq_{D_{\mathcal{A}}} D_{\mathcal{A}}^{T_{\mathrm{prim}}} / H_{\mathcal{A},\mathrm{prim}}(T;\beta)$$

where

$$H_{\mathcal{A},\mathrm{prim}}(T;\beta) := H_{\mathcal{A}}(T;\beta) \cap D_{\mathcal{A}}^{T_{\mathrm{prim}}}.$$

Note that if we set

$$I_{\mathcal{A},\mathrm{prim}}(T) := D_{\mathcal{A}} \cdot \{\partial_{\mathcal{A}}^{\mathbf{u}} \cdot 1_{\mathbf{u}'} - \partial_{\mathcal{A}}^{\mathbf{v}} \cdot 1_{\mathbf{v}'} \mid \mathbf{u} + \mathbf{u}' = \mathbf{v} + \mathbf{v}'\}_{\mathbf{u}', \mathbf{v}' \in T_{\mathrm{prim}}}$$

then

$$H_{\mathcal{A},\mathrm{prim}}(T;\beta) = D_{\mathcal{A}} \cdot (I_{\mathcal{A},\mathrm{prim}}(T), \{(E - \mu(\beta - \mathbf{u})) \cdot 1_{\mathbf{u}}\}_{\mu \in N^\vee, \mathbf{u} \in T_{\mathrm{prim}}}). \tag{5.3}$$

Example 5.4. One example of a finitely generated $\mathbb{N}\mathcal{A}$-module is given by K, as defined in Formula (**2.2**). Indeed, the image $\pi(K)$ is the saturation semigroup of $\mathbb{N}A$, the set of all lattice points $\mathbf{u} \in \mathbb{Z}^d$ such that $\mathbb{N}\mathbf{u}$ meets $\mathbb{N}A$. That $\pi(K)$ is a finitely generated module over $\mathbb{N}A$ is Gordan's Lemma (see, e.g., [CLS11, Proposition 1.2.17]). That K is finite over $\mathbb{N}\mathcal{A}$ then follows from the fact that the fibers of π are finite.

A second important finitely generated $\mathbb{N}\mathcal{A}$-module is the set K°, sitting over the interior points of K,

$$K^\circ = \{\mathbf{u} \in K \mid \pi(\mathbf{u}) \notin \tau \; \forall \tau \in \mathcal{F}_K\}. \tag{5.5}$$

The shortest argument that K° is finitely generated is slightly roundabout. Let $\mathbb{K}$ be any field, $S \subseteq \mathbb{Z}^d$ a semigroup without non-trivial units, $T \subseteq \mathbb{Z}^d$ an S-module, and consider the semigroup ring $\mathbb{K}[S] = \bigoplus_S \mathbb{K} \cdot s$ and the vector space $\bigoplus_T \mathbb{K} \cdot t$, which in natural fashion has a $\mathbb{K}[S]$-module structure. If S is finitely generated by τ elements, then S is a quotient of $\mathbb{N}^\tau$ and so $\mathbb{K}[S]$ is the quotient of a polynomial ring in τ variables, and in particular Noetherian. If then T were an infinitely generated module over S, one would have an infinite ascending chain of $\mathbb{K}[S]$-submodules of $\bigoplus_T \mathbb{K} \cdot t$, which could then not be Noetherian. Now consider the $\mathbb{N}\mathcal{A}$-module K° and the corresponding $\mathbb{K}[\mathbb{N}A]$-module $\mathbb{K}[\pi(K^\circ)]$. Since $\pi(K)$ is a finite $\mathbb{N}A$-module, $\mathbb{K}[\pi(K)]$ is module-finite over $\mathbb{K}[\mathbb{N}A]$. But $\mathbb{K}[\pi(K^\circ)]$ is the canonical module of the normal Cohen–Macaulay ring $\mathbb{K}[\pi(K)]$ (by work of Hochster, Danilov and Stanley), and thus certainly module-finite over $\mathbb{K}[\pi(K)]$. Finiteness of fibers of π then dictates that K° is finitely generated over $\mathbb{N}\mathcal{A}$. $\diamond$

Remark 5.6. It follows from the description (**5.3**) that for every $\mathcal{A}$-toric module T,

$$\iota_\pi(\Phi_{\mathcal{A},0}(T;\beta)) = H_{A,0}(M_T;\beta)$$

is the distinguished (terminal) homology group of the Euler–Koszul complex to β on M_T.

The case $T = K$ inspired Borisov and Horja to write down the differential system Φ_K and the corresponding D_A-module $\Phi_A(K;\beta)$ which they termed the *better behaved GKZ-system.*

In particular, the better behaved GKZ-system to β in the torsion-free case $F = 0$ is precisely the A-hypergeometric module that corresponds to the toric S_A-module $M_K = \tilde{S}_A$. $\diamond$

Example 5.7. Let $N = (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$, and assume $\mathcal{A}$ is the singleton $\mathbf{a}_1 = (\bar{1}, 1)$. Then $\mathbb{N}\mathcal{A} = \mathbb{N} \cdot (\bar{1}, 1) \subseteq N$ is isomorphic to $\mathbb{N} = \pi(\mathbb{N}\mathcal{A})$ and so $R_\mathcal{A}$ and R_A are isomorphic to the polynomial ring $\mathbb{K}[\partial_1]$, with corresponding $\mathcal{A}$- or A-grading. The semigroup K is $(\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{N}$ and its semigroup ring is isomorphic to $(\mathbb{K}[y_0]/(y_0^2 - 1)) \otimes (\mathbb{K}[y_1])$.

However, under this isomorphism, ∂_1 acts as the element $y_0 y_1$, and in particular does not act trivially on the y_0-factor. Thus, (here and in general) there is an obvious abstract vector space isomorphism between $\mathbb{K}[K]$ and a degree-$|F|$ ring extension of $\tilde{S}_A$, but it is less clear (in general) that there is such an isomorphism that is an isomorphism of R_A-algebras. $\diamond$

Corollary 5.8. *We consider the D_A-modules $H_{A,0}(M_K;\beta)$ and $H_{A,0}(M_{K^\circ};\beta)$. Both have rank equal to $|F| \cdot \mathrm{vol}(A \cup \{0\})$.*

Proof. By definition, K is the preimage under π of the lattice points in the cone spanned by $A = \pi(\mathcal{A})$. In particular, as a semigroup, $K = F \oplus \pi(K)$. As a semigroup ring, $\mathbb{K}[\pi(\mathcal{A})]$ is the saturated semigroup ring to $\mathbb{K}[\mathbb{N}A]$, and

$$\mathbb{K}[K] \simeq \tilde{S}_A \otimes_\mathbb{K} \mathbb{K}[y_1, \ldots, y_k]/(\{y_i^{\ell_i} - 1\}_1^k),$$

where $F = \bigoplus_1^k (\mathbb{Z}/\ell_i\mathbb{Z})$ and $\ell = \prod \ell_i$. While the R_A-structure on $\mathbb{K}[K]$ in the above description is not compatible with the "natural" subring $\tilde{S}_A = \tilde{S}_A \otimes 1 \hookrightarrow \mathbb{K}[K]$ (as seen in the above

Example 5.7), we show below that there *is* such a direct sum decomposition that is compatible with this natural structure; we are interested in it because it certifies $\mathbb{K}[K]$ as a Cohen–Macaulay R_A-module.

We start with considering R_A-algebra morphisms from $\mathbb{K}[K]$ to $\tilde{S}_A$. For $1 \leq i \leq k$, let ζ_i be a primitive ℓ_i-th root of unity, and choose $\underline{r} := (r_i \in \mathbb{N} \mid 0 \leq r_i \leq \ell_i - 1)_{i=1}^k$. Then

$$\phi_{\underline{r}} \colon \mathbb{K}[K] \quad \to \quad \tilde{S}_A,$$
$$\{y_i \quad \mapsto \quad (\zeta_i)^{r_i}\}_1^k$$

is a surjective R_A-algebra morphism. Now set $\phi := \bigoplus_r \phi_{\underline{r}} \colon \mathbb{K}[K] \to \bigoplus_r \tilde{S}_A$.

The map ϕ is an injective R_A-module map. Indeed, if $f \in \mathbb{K}[K]$ is in the kernel of $\phi_{\underline{r}}$, then f is in the ideal generated by $\{y_i - \zeta_i^{r_i}\}_1^k$. Thus, if f is in the kernel of ϕ then f is in the ideal generated by $\{y_i^{\ell_i} - 1\}_1^k$ and hence zero in $\mathbb{K}[K]$. But ϕ is also surjective, which becomes obvious if one considers one variable at the time: for any $\mathbb{K}$-algebra R, the morphism that sends $(\mathbb{K}[y]/(y^q - 1)) \otimes_{\mathbb{K}} R$ to $\bigoplus_0^{q-1} R$ by evaluating y to ζ_q^i on the i-th summand is onto since

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \zeta_q^1 & \zeta_q^2 & \cdots & \zeta_q^{q-1} \\ \vdots & & & \vdots \\ \zeta_q^{q-1} & \zeta_q^{2q-2} & \cdots & \zeta_q^{(q-1)(q-1)} \end{pmatrix}$$

is a non-singular Vandermonde matrix.

It follows that $\mathbb{K}[K]$ is R_A-isomorphic to $\bigoplus_{|F|} \tilde{S}_A$ when $\tilde{S}_A$ is given its natural R_A-structure inherited from $\pi(K)$ being the saturation of the semigroup $\pi(\mathbb{N}A) = \mathbb{N}A$.

Restriction of the map ϕ to $\mathbb{K}[K^\circ]$ shows that $\mathbb{K}[K^\circ]$ is R_A-linearly isomorphic to the free sum of $|F|$ copies of $\mathbb{K}[\pi(K^\circ)]$. The decompositions show that $H_{A,0}(M_T; \beta)$ is isomorphic to $|F|$ copies of $H_{A,0}(M_{\pi(T)}; \beta)$, for $T \in \{K, K^\circ\}$.

It is well-known that $H_{A,0}(M_{\pi(K)}; \beta)$ has rank equal to the (simplicial) volume of the convex hull spanned by the origin and A, independently of β. Moreover, $\mathbb{K}[\pi(K^\circ)]$ is a maximal Cohen–Macaulay S_A-module, and so the rank of $H_{A,0}(M_{K^\circ}; \beta)$ is independent of β as well. It suffices then to consider a generic β. But the quasi-degrees of $\pi(K)/\pi(K^\circ)$ are contained in the facets of $\mathcal{A}$, and thus do not contain a generic β. By Theorem 3.1, applied to the long Euler–Koszul homology sequence derived from $0 \to \pi(K^\circ) \to \pi(K) \to \pi(K)/\pi(K^\circ) \to 0$, one concludes that the ranks of $H_{A,0}(M_{\pi K^\circ}; \beta)$ and $H_{A,0}(M_{\pi(K^\circ)}; \beta)$ are always equal. $\qquad\square$

Finally, we apply the results on duality of Euler–Koszul homology groups to the module $H_{A,0}(M_K; \beta)$.

Theorem 5.9. *There is a natural non-degenerate pairing*

$$\big(\mathbb{C}(\mathbf{x}) \otimes_{\mathbb{C}[\boldsymbol{x}]} H_{A,0}(M_K; \beta) \big) \otimes \big(\mathbb{C}(\mathbf{x}) \otimes_{\mathbb{C}[\boldsymbol{x}]} H_{A,0}(M_{K^\circ}; -\beta - \varepsilon_A)^- \big) \longrightarrow \mathbb{C}(\mathbf{x}).$$

In particular, if $\Gamma = H^0(\mathrm{Sol}(H_{A,0}(M_K; \beta)_\xi))$ resp. $\Gamma' = H^0(\mathrm{Sol}(H_{A,0}(M_{K^\circ}; -\beta - \varepsilon_A)_\xi)^-)$ is the space of (classical) solutions of $H_{A,0}(M_K; \beta)$ resp. $H_{A,0}(M_{K^\circ}; -\beta - \varepsilon_A)^-$ near a generic point ξ, then we have a non-degenerate pairing $\Gamma \times \Gamma' \to \mathbb{C}$.

Proof. It follows from Formula (**4.7**) that

$$\mathbb{D}H_{A,0}(M_K, \beta) \cong H_{A,0}(M_{K^\circ}; -\beta - \varepsilon_A)^-(\varepsilon_A),$$

since by what has been shown in the proof of Corollary 5.8, M_K is a Cohen–Macaulay R_A-module, and then its dualizing module is M_{K°. Forgetting the graded structure, localize at the

generic point. Then we are looking at holonomically dual, smooth D-modules. These are hence mutually dual bundles with mutually dual connections, as follows from the Spencer resolution. (See, *e.g.*, [DS03, Lemma A.11] for the argument in a slightly different but related situation). The theorem follows. $\square$

Remark 5.10. If $\bar{\mathbf{a}}_1, \ldots, \bar{\mathbf{a}}_n$ lie on an affine hyperplane in $\bar{N}$, the functor $(-)^-$ is an auto-equivalence of graded S_A-modules, since I_A is then projective. If in this case β is the zero vector, we obtain a duality result similar to [BH23, Theorem 2.4.]. Formula (**4.7**) is more general, since the D_A-modules $H_{A,0}(M_K; \beta)$ and $H_{A,0}(M_{K^\circ}; -\beta - \varepsilon_A)^-$ cannot be reconstructed from their (classical) solutions, nor from their restrictions to a generic point. On the other hand, holonomic duality of modules always yields a pairing on the level of solutions at a generic point.

It is an interesting project to check whether the concrete construction in *loc. cit.* agrees with our functorial pairing. $\diamond$

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