Frobenius manifolds and variation of twistor structures in singularity theory

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Christian Sevenheck

1 Introduction

In this text I give an overview on research I have done during the last 5 years. The unifying theme is the understanding of algebraic and geometric structures on deformation resp. moduli spaces in complex analytic geometry, and more specifically in singularity theory. Two (related) notions that occur in such situations are *Frobenius manifold* and *Variations of (integrable) twistor structures*. Both can be expressed through certain meromorphic connections on the product of the parameter space in question with the projective line, and hence the theory of systems of differential equations with singularities enters into play. Some of the classical questions of this theory, like e.g., finding normal forms for such systems of differential equations are directly related to the existence of the above structures on the parameter spaces. A striking difference between Frobenius manifolds and variations of twistor structures is that the former are purely holomorphic data, whereas the latter involve a rather subtle mixture between holomorphic and anti-holomorphic objects.
The following text is a short overview of the main results obtained. All definitions and notions are explained, and all results are stated in detail, but without proofs for which I refer to the original research papers listed below.
Acknowledgments: I would like to thank Claus Hertling for a very fruitful and intense cooperation on tt^*-geometry during the last years. I also learned much about Frobenius manifolds and classical singularity theory from his papers and from the many discussions we had.
It is a true pleasure to thank Claude Sabbah for his continuing support and interest in my work since my Ph.D. thesis. I also thank David Mond and Ignacio de Gregorio for the stimulating cooperation on linear free divisors and Frobenius manifolds.

1.1 Submitted papers

The following papers are submitted for the "Habilitation":

1. *Claus Hertling and Christian Sevenheck,* ***Nilpotent orbits of a generalization of Hodge structures****, J. Reine Angew. Math.* ***609*** *(2007), 23–80.*

2. *Claus Hertling and Christian Sevenheck,* ***Curvature of classifying spaces for Brieskorn lattices****, J. Geom. Phys.* ***58*** *(2008), no. 11, 1591–1606.*

3. *Claus Hertling and Christian Sevenheck,* ***Limits of families of Brieskorn lattices and compactified classifying spaces****, Preprint math.AG/0806.2011, 51 pages, 2008.*

4. *David Mond, Ignacio de Gregorio and Christian Sevenheck,* ***Linear free divisors and Frobenius manifolds****, Preprint math.AG/0802.4188, 46 pages, to appear in "Compositio Mathematica", 2008.*

5. *Christian Sevenheck,* ***Spectral numbers and Bernstein polynomials for linear free divisors****, Preprint math.AG/0905.0971, 14 pages, 2009.*

The first three papers deal with variation of integrable twistor structures, also known as "tt^*-geometry", whereas the last two are concerned with Frobenius structures and related objects for some particular classes of non-isolated singularities, the so-called free divisors. A unifying point of view in all of the

submitted articles is the theory of (variation of) Brieskorn lattices, these are algebraic or analytic objects naturally attached to holomorphic resp. algebraic functions on complex resp. algebraic manifolds.
The main results contained in these papers are described in some detail in the following sections.

1.2 Short description of other scientific works

Below is a list and short description of other scientific work.

- *Christian Sevenheck and Duco van Straten,* ***Deformation of singular Lagrangian subvarieties****, Math. Ann.* ***327*** *(2003), no. 1, 79–102.*

  In this paper, a deformation theory for singular lagrangian subvarieties in a symplectic manifold is developed. A variant of the de Rham complex for such a variety is introduced and it is shown that under a suitable hypothesis on the geometry of the variety, its cohomology form constructible sheaves of finite-dimensional vector spaces. As a consequence, such lagrangian varieties, although their singularities are often non-isolated, have a (formal) versal deformation space. The dimension of these deformation spaces are explicitly computed for some interesting examples.

- *Christian Sevenheck and Duco van Straten,* ***Rigid and complete intersection Lagrangian singularities****, Manuscripta Math.* ***114*** *(2004), no. 2, 197–209.*

  The general results on deformation of lagrangian varieties are applied to study the question of *rigidity* for these objects. It is shown that certain classes of lagrangian varieties, in particular, the so called "open swallowtails" have no lagrangian deformations. This follows from a cohomological argument on the lagrangian de Rham complex. Similar arguments show that this complex is *pervers* in the case of lagrangian complete intersections, for which we also give an estimate on the codimension of the singular locus.

- *Christian Sevenheck,* ***Unobstructed Lagrangian deformations****, C. R. Math. Acad. Sci. Paris* ***338*** *(2004), no. 8, 617–622.*

  In this paper, a criterion is given which ensure smoothness of the local moduli space for lagrangian singularities. This is done via the so-called T^1-lifting principle, and it involves studying a relative version of the lagrangian de Rham complex.

- *Claus Hertling and Christian Sevenheck,* ***Twistor structures, tt^*-geometry and singularity theory****, From Hodge theory to integrability and TQFT tt^*-geometry, Proc. Sympos. Pure Math., vol. 78, Amer. Math. Soc., Providence, RI, 2008, pp. 49–73.*

  This article gives an introduction to the papers [HS07], [HS08a] and [HS08b], with some details on the theory of Brieskorn lattices associated to local singularities and tame functions.

2 tt^*-geometry and variation of twistor structures

tt^*-geometry was introduced in [CV91], [CV93]. It can be considered as a generalization of variations of Hodge structures. A convenient framework is the notion of a variation of TERP-structures, which we recall first.

Definition 1 ([Her03], [HS07])**.** *Let M be a complex manifold and $w \in \mathbb{Z}$. A variation of TERP-structures on M of weight w is a tuple $(H, H'_{\mathbb{R}}, \nabla, P, w)$ consisting of:*

1. *a holomorphic vector bundle H on $\mathbb{C} \times M$, equipped with a meromorphic connection $\nabla : \mathcal{H} \to \mathcal{H} \otimes z^{-1}\Omega^1_{\mathbb{C}\times M}(\log\{0\}\times M)$, satisfying $\nabla^2 = 0$. Here and throughout this text, z is a fixed coordinate on $\mathbb{C}$. The restriction $H' := H_{|\mathbb{C}^*\times M}$ is then necessarily flat, corresponding to a local system given by a monodromy automorphism $M \in \mathrm{Aut}(H^\infty)$, where H^∞ is the space of flat multivalued sections. For simplicity, we will make the assumption that the eigenvalues of M are in S^1, which is virtually always the case in applications.*

2. *a flat real subbundle $H'_{\mathbb{R}}$ of maximal rank of the restriction H'. In particular, M is actually an element in $\mathrm{Aut}(H^\infty_{\mathbb{R}})$.*

3. *a $(-1)^w$-symmetric, non-degenerate and flat pairing*

$$P : \mathcal{H}' \otimes j^*\mathcal{H}' \to \mathcal{O}_{\mathbb{C}^*}$$

where $j(z,x) = (-z,x)$, and which takes values in $i^w\mathbb{R}$ on the real subbundle $H'_{\mathbb{R}}$. It extends to a pairing on $i_*\mathcal{H}'$, where $i : \mathbb{C}^* \times M \hookrightarrow \mathbb{C} \times M$, and has the following two properties on the subsheaf $\mathcal{H} \subset i_*\mathcal{H}'$:

   (a) $P(\mathcal{H},\mathcal{H}) \subset z^w\mathcal{O}_{\mathbb{C}\times M}$.

   (b) P induces a non-degenerate symmetric pairing

$$[z^{-w}P] : \mathcal{H}/z\mathcal{H} \otimes \mathcal{H}/z\mathcal{H} \to \mathcal{O}_M.$$

Remark: TERP-structures can be divided very roughly into two classes: By definition, the pole order of ∇ at zero is smaller or equal than two, but it may still define a regular singularity on the underlying meromorphic bundle $(\mathcal{H}(*0),\nabla)$. This will be referred to as the *regular singular case*, otherwise, we will says that $(H, H'_{\mathbb{R}}, \nabla, P, w)$ is irregular. As we will see later, both cases actually occur in geometric situations.
As TERP-structures are supposed to generalize Hodge structures, one has to define in a natural way pure resp. pure polarized TERP-structures. This is done as follows. Consider the involution $\gamma : \mathbb{P}^1 \to \mathbb{P}^1$, $z \mapsto \overline{z}^{-1}$. For any given flat real bundle $H'_{\mathbb{R}}$ on $\mathbb{C}^*$, with complexification $H' := H'_{\mathbb{R}} \otimes \mathbb{C}$, and a given weight $w \in \mathbb{Z}$, define

$$\begin{array}{rcl} \tau : H'_{|z} & \longrightarrow & H'_{|\gamma(z)} \\ s & \longmapsto & \nabla\text{-parallel transport of } \overline{z^{-w}\cdot s} \end{array}$$

The induced map on sections can be seen as an isomorphism of $\mathcal{O}_{\mathbb{C}^*}$-modules $\tau : \mathcal{H}' \to \overline{\gamma^*\mathcal{H}'}$ and as such, it extends to a morphism $\tau : \widehat{i}_*\mathcal{H}' \to \widehat{i}_*(\overline{\gamma^*\mathcal{H}'})$, where $\widehat{i} : \mathbb{C}^* \hookrightarrow \mathbb{P}^1$.
Suppose now that we are given a TERP-structure $(H, H'_{\mathbb{R}}, \nabla, P, w)$, then we can use τ to identify the $\mathcal{O}_{\mathbb{C}}$-bundle $\mathcal{H}$ and the $\mathcal{O}_{\mathbb{P}^1\backslash\{0\}}$-bundle $\overline{\gamma^*\mathcal{H}}$ on the common intersection $\mathbb{C}^*$, and hence define a holomorphic vector bundle over $\mathbb{P}^1$, denoted by $\widehat{\mathcal{H}}$.

Definition 2 ([Her03, HS07]). *1. $(H, H'_{\mathbb{R}}, \nabla, P, w)$ is called pure iff $\widehat{\mathcal{H}}$ is trivial, i.e. $\widehat{\mathcal{H}} \cong \mathcal{O}_{\mathbb{P}^1}^{\mathrm{rk}(H)}$.*

2. The morphism τ defines a $\mathbb{C}$-antilinear involution on $H^0(\mathbb{P}^1,\widehat{\mathcal{H}})$. If $(H, H'_{\mathbb{R}}, \nabla, P, w)$ is pure, put

$$\begin{array}{rcl} h : H^0(\mathbb{P}^1,\widehat{\mathcal{H}}) \times H^0(\mathbb{P}^1,\widehat{\mathcal{H}}) & \longrightarrow & \mathbb{C} \\ (a,b) & \longmapsto & z^{-w}P(a,\tau b), \end{array}$$

then h is a hermitian form on $H^0(\mathbb{P}^1,\widehat{\mathcal{H}})$ and we say that $(H, H'_{\mathbb{R}}, \nabla, P, w)$ is pure polarized iff h is positive definite.

An easy generalization of the above construction shows that for a given *variation of TERP-structures*, the extension to infinity $\widehat{\mathcal{H}}$ is no longer holomorphic, but only a (locally free) $\mathcal{O}_{\mathbb{P}^1}\mathcal{C}^{an}_M$-module, where $\mathcal{O}_{\mathbb{P}^1}\mathcal{C}^{an}_M$ is the sheaf of real-analytic functions on $\mathbb{P}^1 \times M$ which are annihilated by $\partial_{\overline{z}}$. In this case, the extension of the connection ∇ on $\mathcal{H}$ has a holomorphic and an anti-holomorphic part. More precisely, we have the following.

Definition 3 ([Sim97]). *Let M be a complex manifold and $\mathcal{F}$ a locally free $\mathcal{O}_{\mathbb{P}^1}\mathcal{C}^{an}_M$-module.*
Consider the map $\sigma : \mathbb{P}^1 \times M \to \mathbb{P}^1 \times M$, $\sigma(z,x) = (-\overline{z}^{-1},x)$ and the two sheaves of meromorphic functions $\mathcal{O}_{\mathbb{P}^1}(1,0) = l_*\mathcal{O}_{\mathbb{P}^1\backslash\{0\}} \cap \widetilde{i}_*z^{-1}\mathcal{O}_{\mathbb{C}}$ and $\mathcal{O}_{\mathbb{P}^1}(0,1) = l_*z\mathcal{O}_{\mathbb{P}^1\backslash\{0\}} \cap \widetilde{i}_*\mathcal{O}_{\mathbb{C}}$ where $l : \mathbb{P}^1\backslash\{0\} \hookrightarrow \mathbb{P}^1$ and $\widetilde{i} : \mathbb{C} \hookrightarrow \mathbb{P}^1$. Then we say that $\mathcal{F}$ underlies a variation of twistor structures if it comes equipped with an operator

$$\mathbf{D} : \mathcal{F} \longrightarrow \mathcal{F} \otimes (\mathcal{O}_{\mathbb{P}^1}(1,0) \otimes \mathcal{A}^{1,0}_M \oplus \mathcal{O}_{\mathbb{P}^1}(0,1) \otimes \mathcal{A}^{0,1}_M)$$

which is $\mathcal{O}_{\mathbb{P}^1}$-linear and such that for any $z \in \mathbb{C}^*$, its restriction $\mathbf{D}_{|\{z\}\times M}$ is a flat connection. $\mathcal{F}$ is pure (of weight zero) if it is fibrewise trivial and in that case a polarization of $\mathcal{F}$ is a symmetric $\mathbf{D}$-flat non-degenerate pairing

$$\widehat{S} : \mathcal{F} \otimes \sigma^*\mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^1}$$

which is a morphism of twistors. It induces a Hermitian pairing h on $E := \pi_*\mathcal{F}$, where $\pi : \mathbb{P}^1 \times M \to M$ is the projection. Then $\mathcal{F}$ is called polarized if h is positive definite.

One checks that the two definitions are compatible, i.e., a variation of TERP-structures gives rise to a variation of twistor structures via the above construction, and the former is pure resp. pure polarized iff the same is true for the latter.
One of the main features of this definition is that it is related to some other interesting differential-geometric object, namely, a harmonic bundle.

Definition 4 ([Sim88, Sim90, Sim97])**.** *Let M be a complex manifold, and $(\mathcal{E},\theta)$ be a Higgs bundle bundle on M, i.e., a holomorphic vector bundle with an $\mathcal{O}_M$-linear morphism $\theta:\mathcal{E}\to\mathcal{E}\otimes\Omega^1_M$ such that $\theta\wedge\theta=0$. Suppose moreover that we are given a hermitian metric $h:\mathcal{E}\otimes\overline{\mathcal{E}}\to\mathcal{C}^{an}_M$ on $\mathcal{E}$ and denote by $D'+D''$ its Chern connection. Then h is called harmonic, and the tuple $(\mathcal{E},\theta,h)$ is called a harmonic bundle, iff $(D'+D''+\theta+\overline{\theta})^2=0$, where $\overline{\theta}$ is the h-adjoint of θ.*

Given a variation of pure polarized twistor structures $\widehat{\mathcal{H}}$, e.g., one coming from a variation of pure polarized TERP-structures, the metric h defined on the bundle $E=\pi_*\widehat{\mathcal{H}}$ of fibrewise global sections (here $\pi:\mathbb{P}^1\times M\to M$ is the projection) is actually harmonic. Even more, we have the following basic correspondence.

Proposition 5 ([Sim97])**.** *The category of variations of pure polarized twistor structures on M is equivalent to the category of harmonic bundles on M.*

For a pure polarized variation of TERP-structures, the corresponding harmonic bundle is equipped with some additional structure, explained in the following result.

Lemma 6. *[Her03, theorem 2.19] For any variation of pure TERP-structures, the connection operator ∇ takes the following form on fibrewise global sections (i.e., sections from $E:=\pi_*\widehat{\mathcal{H}}$).*

$$\nabla = D'+D''+z^{-1}\theta+z\overline{\theta}+\frac{dz}{z}\left(\frac{1}{z}\mathcal{U}-\mathcal{Q}+\frac{w}{2}-z\tau\mathcal{U}\tau\right)$$

where $\mathcal{U},\mathcal{Q}\in\mathcal{E}nd_{\mathcal{C}^{an}_M}(E)$. These objects satisfy the following relations, summarized under the name CV⊕-structure in [Her03].

$$\begin{aligned}
h(\theta-,-)-h(-,\overline{\theta})=0, &\qquad (D'+D'')(h)=0 &\quad(1)\\
(D''+\theta)^2=0, &\qquad (D'+\overline{\theta})^2=0 &\quad(2)\\
D'(\theta)=0, &\qquad D''(\overline{\theta})=0 &\quad(3)\\
D'D''+D''D' &\;=\; -(\theta\overline{\theta}+\overline{\theta}\theta) &\quad(4)\\
[\theta,\mathcal{U}]=0, &\qquad D'(\mathcal{U})-[\theta,\mathcal{Q}]+\theta=0 &\quad(5)\\
D''(\mathcal{U})=0, &\qquad D'(\mathcal{Q})+[\theta,\tau\mathcal{U}\tau]=0 &\quad(6)\\
\tau\theta\tau=\overline{\theta} &\qquad (D'+D'')(\tau)=0 &\\
h(\mathcal{U}-,-)=h(-,\tau\mathcal{U}\tau-), &\qquad h(\mathcal{Q}-,-)=h(-,\mathcal{Q}-) &\\
\mathcal{Q} &\;=\; -\tau\mathcal{Q}\tau &
\end{aligned}$$

If H is pure polarized, then $\mathcal{Q}$ is a Hermitian endomorphism of the bundle (E,h) and its real-analytically varying real eigenvalues are distributed symmetrically around zero.

As already remarked, equations (1) to (4) say that the metric h on E is harmonic. The two identities (3) and (4) were called tt^*-equations in [CV91]. Variation of twistor structures corresponding to harmonic bundles with operators $\mathcal{U},\mathcal{Q},\tau\mathcal{U}\tau$ are studied under the name “integrable” in [Sab05, Chapter 7], and the identities (5) and (6) are called “integrability equations” in loc.cit.
Any variation of pure polarized Hodge structures gives rise to harmonic bundle, and even to a variation of pure polarized TERP-structures. These variations have the special property that the endomorphism $\mathcal{U}$ from above is zero (see [Her03, section 3.2]). The following result shows that in some cases, it may happen that even for non-vanishing $\mathcal{U}$, a variation of pure polarized TERP-structures actually comes from a variation of pure polarized Hodge structures.

Theorem 7. *[Sab05, corollary 7.2.8], [HS08b, theorem 6.2] Let M be either a compact Kähler manifold or the complement of a simple normal crossing divisor in a smooth projective variety. Let $(H,H'_{\mathbb{R}},\nabla,P,w)$ be a variation of pure polarized TERP-structures on M with associated harmonic bundle (E,D'',θ,h). In the second case, suppose moreover that E is* tame *along the boundary divisor (see, e.g., [HS08b, section 3]). Then E underlies a (sum of two) variation(s) of pure polarized Hodge structures.*

We end this part with a short description of how (variation of) TERP-structures actually arise in singularity theory. More details on this construction can be found in the first section of [HS08c]. Two cases are of interest: The classical situation of a holomorphic function germ $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ with an isolated critical point, and the more recently studied case of polynomial functions $f : U \to \mathbb{C}$ (where U is a smooth affine complex manifold of dimension m) satisfying a condition of topological triviality at infinity, e.g., M-tameness ([NS99]) or cohomological tameness ([Sab06]). In both situations, we denote by t a coordinate on the base space, centered at zero. In the first case, by choosing an appropriate representative $f : X \to S$ for the given function germ, we define

$$M_0^{an} := \frac{f_*\Omega_X^m}{df \wedge df_*\Omega_X^{m-2}}$$

which is a free $\mathcal{O}_S$-module of rank $\mu := \dim_{\mathbb{C}}(\mathcal{O}_X/J_f)$. It is naturally equipped with a connection operator $\nabla : M_0^{an} \to M_0^{an} \otimes \Omega_S^1(*0)$ which is regular singular at $0 \in S$ (see [Bri70]). Hence, there is a unique extension (which we denote by the same symbol) M_0^{an} to a free $\mathcal{O}_{\mathbb{C}}$-module with a connection of the same type. Consider the intersection $\widetilde{i}_* M_0^{an} \cap l_* V_{<\infty}$ (here $\widetilde{i} : \mathbb{C} \hookrightarrow \mathbb{P}^1$ and $l : \mathbb{P}^1 \backslash \{0\} \hookrightarrow \mathbb{P}^1$), where the latter module is the (unique) meromorphic regular singular extension of M_0 to infinity. The space $M_0^{alg} := H^0(\mathbb{P}^1, \widetilde{i}_* M_0^{an} \cap l_* V_{<\infty})$ is then a free $\mathbb{C}[t]$-module, i.e., corresponds to an algebraic vector bundle on $\mathbb{C}$.
In the second case mentioned above, one can directly define

$$M_0^{alg} := \frac{f_*\Omega_U^{m,alg}}{df \wedge df_*\Omega_U^{m-2,alg}}$$

using algebraic differential forms. By [Sab06], tameness implies that this module is $\mathbb{C}[t]$-free of finite rank, which may, however, differ from the sum of Milnor numbers of the critical points of f on U.
Now define $G_0^{alg} := M_0^{alg} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t^{-1}]$, and equip it with a $\mathbb{C}[z]$-action, where z acts by ∂_t^{-1}. Notice that in the first case (and also in the second one if U is contractible), any element in M_0^{alg} has a unique preimage under the operator ∇_t, so that actually $G_0^{alg} = M_0^{alg}$ (as $\mathbb{C}$-vector spaces) in these cases. The $\mathbb{C}[t]$-module structure on M_0^{alg} yields an action of $z^2\nabla_z$ on G_0^{alg}, and we write G_0^{an} for the associated analytic bundle, which is thus equipped with a connection with a pole of order at most two at 0. Notice finally that in the second case, G_0^{alg} is simply given as the top-cohomology group of a twisted de Rham complex, i.e.,

$$G_0^{alg} := \frac{f_*\Omega_U^{m,alg}[z]}{(zd - df\wedge) f_*\Omega_U^{m-1,alg}[z]}$$

and the connection operator $z^2\nabla_z$ is defined by $(z^2\nabla_z)(\omega) := f \cdot \omega$ for $\omega \in f_*\Omega_U^{m,alg}$, and extended by the Leibniz rule. We will use this direct description in section 3.
Summarizing the above construction, we obtain a holomorphic vector bundle on $\mathbb{C}$ with a connection with pole of order at most two at zero. A detailed discussion of the duality theory involved yields the definition of a pairing $P : G_0^{an} \otimes j^* G_0^{an} \to z^m \mathcal{O}_{\mathbb{C}}$ (see [Her03] for the local and again [Sab06] for the global case). Moreover, an essentially topological argument shows that the flat bundle $(G_0^{an})_{|\mathbb{C}^*}$ carries a natural real structure.

Theorem 8 ([Her03],[Sab06],[HS07])**.** *Consider one of the two cases discussed above.*

1. *Putting $H := G_0^{an}$, then $(H, H'_{\mathbb{R}}, \nabla, P, w)$ is a TERP-structure of weight $w := m$.*
2. *For a deformation $F : X \times M \to S$ resp. $F : U \times M \to \mathbb{C}$ of $f : X \to S$ resp. $f : U \to \mathbb{C}$, a variant of the above construction yields a holomorphic bundle $H := G_0^{an} \to \mathbb{C} \times M$ which underlies a variation of TERP-structures on M. (Notice that in the second case it is not supposed that $F_t : U \times \{t\} \to \mathbb{C}$ is tame for all $t \in M$).*

The above rigidity result (theorem 7) says heuristically that interesting variations of pure polarized TERP-structures occur on parameter spaces which are neither compact nor quasi-projective (this is the case for local singularities), or otherwise that the corresponding harmonic bundle is not tame along all components of the boundary divisor (this happens for variation of TERP-structures defined by families of tame polynomial functions).

2.1 Nilpotent orbits of TERP-structures

A classical theorem of Cattani, Kaplan and Schmid ([Sch73], [CK82], [CK89]) characterizes polarized mixed Hodge structures through a geometric object, the so called "nilpotent orbits". For TERP-structures, there is a natural generalization of this notion.

Definition 9 ([HS07])**.** *Let $(H, H'_{\mathbb{R}}, \nabla, P, w)$ be a given TERP-structure of weight w. Write, for any $r \in \mathbb{C}^*$, π_r for the the multiplication map $\mathbb{C} \times \mathbb{C}^* \to \mathbb{C}, (z, r) \mapsto zr$ and π'_r for the map $\mathbb{C} \times \mathbb{C}^* \to \mathbb{C}, (z, r) \mapsto zr^{-1}$. Then we say that $(H, H'_{\mathbb{R}}, \nabla, P, w)$ induces a nilpotent orbit of TERP-structures iff $\pi_r^*(H, H'_{\mathbb{R}}, \nabla, P, w)$ is pure polarized for any r such that $|r| \ll 1$, and a Sabbah orbit of TERP-structures iff the same holds for $(\pi')_r^*(H, H'_{\mathbb{R}}, \nabla, P, w)$.*

One of the main results of [HS07] establishes a correspondence between such nilpotent orbits and so called "mixed TERP-structures" which is the appropriate generalization of mixed Hodge structures. Its definition is rather involved, we concentrate on the regular singular case here. In this case, a TERP-structure is mixed if a certain associated filtration defines a polarized mixed Hodge structure. For the geometric situation of an isolated hypersurface singularity mentioned above, this filtration is actually rather classical, it corresponds to Steenbrink's Hodge filtration on the cohomology of the Milnor fibre of the singularity.

Definition 10 ([Her03],[HS07])**.** *Let $(H, H'_{\mathbb{R}}, \nabla, P, w)$ be a given TERP-structure of weight w.*

1. *Consider the space of multivalued flat sections (i.e., nearby cycles) $H^\infty := \psi_z(H_{|\mathbb{C}^*})$. Write $H^\infty = \oplus_{\lambda \in \mathbb{C}} H^\infty_\lambda$ for the generalized eigen-decomposition with respect to the monodromy operator $M = M_s \cdot M_u$, where M_s resp. M_u denotes the semi-simple resp. unipotent part. Put $N := \log(M_u)$ and denote by $W_\bullet$ the weight filtration defined by the nilpotent endomorphism N. Define an automorphism of $H^\infty_{e^{-2\pi i \alpha}}$ by*

$$G^{(\alpha)} := \sum_{k \geq 0} \frac{1}{k!} \Gamma^{(k)}(\alpha) \left(\frac{-N}{2\pi i} \right)^k =: \Gamma \left(\alpha \cdot \mathrm{id} - \frac{N}{2\pi i} \right).$$

   Here $\Gamma^{(k)}$ is the k-th derivative of the gamma function. In particular, G depends only on the flat bundle H' and induces the identity on $\mathrm{Gr}^W_\bullet$.

2. *We define a polarizing form on H^∞ by putting, for $A, B \in H^\infty$, $S(A, B) := (-1)(2\pi i)^w P(A, t(B))$ where*

$$t(B) = \begin{cases} (M - \mathrm{Id})^{-1}(B) & \forall B \in H^\infty_{\neq 1} \\ -(\sum\limits_{k \geq 1} \frac{1}{k!} N^{k-1})^{-1}(B) & \forall B \in H^\infty_1. \end{cases}$$

   Here $H^\infty_{\neq 1} := \oplus_{\lambda \neq 1} H^\infty_\lambda$ and P is seen as defined on the local system corresponding to the flat bundle H'.

3. *Suppose that (H, ∇) is regular singular at zero. Then put for $\alpha \in (0, 1]$*

$$\widetilde{F}^p H^\infty_{e^{-2\pi i \alpha}} := (G^{(\alpha)})^{-1} \left(z^{p+1-w-\alpha+\frac{N}{2\pi i}} Gr_V^{\alpha+w-1-p} \mathcal{H} \right), \tag{7}$$

   where $V^\bullet$ is the decreasing filtration by Deligne lattices of the meromorphic bundle $\mathcal{H}(*0)$ at $z = 0$.

4. *Let (H, ∇) be arbitrary and put $G_0 := H^0(\mathbb{P}^1, l_* V_{<\infty} \cap \widetilde{i}_* \mathcal{H})$. In the geometric situation from above, this G_0 is exactly the module G_0^{alg}. Define*

$$\widetilde{F}^p_{Sab} H^\infty_{e^{-2\pi i \alpha}} := (G^{(\alpha)})^{-1} \left(z^{p+1-w-\alpha+\frac{N}{2\pi i}} Gr^V_{\alpha+w-1-p} G_0 \right), \tag{8}$$

   here $V_\bullet$ is the increasing filtration by Deligne lattices of G_0 at $z = \infty$.

With these notions at hand, we can state the following result.

Theorem 11. *[HS07, theorem 6.6, theorem 7.3] Let $(H, H'_{\mathbb{R}}, \nabla, P, w)$ be a regular singular TERP-structure. Then the following two conditions are equivalent.*

1. It induces a nilpotent orbit.

2. $(H^\infty, H^\infty_{\mathbb{R}}, -N, S, \widetilde{F}^\bullet)$ *defines a polarized mixed Hodge structure (PMHS) of weight* $w-1$ *on* $H^\infty_{\neq 1}$ *and a PMHS of weight* w *on* H^∞_1*.*

Similarly, let $(H, H'_{\mathbb{R}}, \nabla, P, w)$ *be an arbitrary TERP-structure. Then the following two conditions are equivalent.*

1. It induces a Sabbah orbit.

2. $(H^\infty, H^\infty_{\mathbb{R}}, N, S, \widetilde{F}^\bullet_{Sab})$ *defines a PMHS of weight* $w-1$ *on* $H^\infty_{\neq 1}$ *and a PMHS of weight* w *on* H^∞_1*.*

The main part of the proof of these theorems relies on the asymptotic analysis of harmonic bundles from [Moc07], in particular, it uses the construction of a limit mixed twistor structure.
We outline the following two consequences of these results which apply to TERP-structures in the geometric situations discussed above.

Corollary 12. *[HS07, corollary 11.4]*

1. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ *be an isolated hypersurface singularity, and denote by* $TERP(f)$ *the TERP-structure defined by the construction in theorem 8, 1. Then there is a sufficiently big real number* r *such that* $TERP(r \cdot f)$ *is pure polarized.*

2. Let $f : U \to \mathbb{C}$ *be a tame function, and consider the TERP structure* $TERP(f)$ *from theorem 8, 2. Then the filtration* $\widetilde{F}^\bullet_{Sab} H^\infty$ *gives rise to a (sum of two) PMHS, which is a twist by the automorphism* G *of the MHS considered in [Sab06].*

In applications, the case of irregular TERP-structures is as important as the regular singular case. A similar result holds in this case, one direction was proved in [HS07]. We refrain here from explaining it in detail, let us only mention that the notion of a *mixed TERP-structure* (see [HS07, definition 9.1]), which in the regular singular case is defined by condition 2. in theorem 11 is more complicated in general. Basically it says that the formal decomposition of the irregular connection $(\mathcal{H}, \nabla)$ can be done without ramification, that the associated Stokes structure is compatible with the real structures and that the regular singular parts in the decomposition satisfy condition 2. in theorem 11. Under these hypotheses, it is shown in [HS07, theorem 9.3, 2.] that the family $\pi_r^*(H, H'_{\mathbb{R}}, \nabla, P, w)$ is a nilpotent orbit. As a consequence, the first part of corollary 12 from above also holds for any family $TERP(r \cdot F_t)$, where $F : (\mathbb{C}^m \times M, 0) \to (\mathbb{C}, 0)$ is a semi-universal unfolding of the given germ f and $t \in$M.
The converse of [HS07, theorem 9.3, 2.] was conjectured in [HS07]. It was proved very recently in [Moc08a], using deep structure results on wild harmonic bundles from [Moc08b].

2.2 Classifying spaces of integrable twistor structures

The philosophy of generalizing Hodge structures to TERP-structures naturally lead to the question of studying period mappings associated to variations of TERP-structures, in particular, for those variations defined by families of holomorphic resp. algebraic functions, as explained in the beginning of this section. An important step in this program is the study of appropriate classifying spaces for TERP-structures. We first recall some facts about classifying spaces of Hodge resp. mixed Hodge structures which are needed later.
Fix a real vector space $H^\infty_{\mathbb{R}}$, an automorphism $M \in \mathrm{Aut}(H^\infty_{\mathbb{R}})$ with eigenvalues in S^1 (and again we write $M = M_s \cdot M_u$ and $N = \log(M_u)$), an integer $w \in \mathbb{Z}$ and a $(-1)^w$-symmetric bilinear pairing $S : H^\infty_{\mathbb{R}} \times H^\infty_{\mathbb{R}} \to \mathbb{R}$ such that $S(N-,-) + S(-,N-) = 0$. Write $H^\infty := H^\infty_{\mathbb{R}} \otimes \mathbb{C}$. Fix moreover a reference filtration $F_0^\bullet$ on H^∞ such that $(H^\infty, H^\infty_{\mathbb{R}}, N, S, F_0^\bullet)$ is a PMHS of weight w (see, e.g., [HS07, section 2] for a definition) and such that M_s is a semi-simple automorphism of this PMHS. Denote by $P_l \subset \mathrm{Gr}^W_l$ the primitive subspaces of the weight filtrations $W_\bullet$ of N, centered at w. Define (see [Her99, HS08a])

$$\begin{aligned} \check{D}_{PMHS} \; := \; & \big\{ F^\bullet H^\infty \,|\, F^\bullet \text{ is } M_s\text{-invariant; } \dim F^p P_l = \dim F_0^p P_l;\; S(F^p, F^{w+1-p}) = 0; \\ & N(F^p) \subset F^{p-1}, \text{ and all powers of } N \text{ are strict with respect to } F^\bullet \big\} \end{aligned}$$

There is a projection map $\check{\beta}:\check{D}_{PMHS}\to\check{D}_{PHS}$, sending $F^\bullet$ to $(F^\bullet P_l)_{l\in\mathbb{Z}}$, here $\check{D}_{PHS}$ is a product of classifying spaces of Hodge-like filtrations on the primitive subspaces P_l. $\check{D}_{PHS}$ is a complex homogeneous space, it contains the open submanifold D_{PHS} which is a product of classifying spaces of polarized Hodge structures, and which has the structure of a real homogeneous space. $\check{\beta}$ is a locally trivial fibration with affine spaces as fibres. Define D_{PMHS} to be the restriction of this fibration to D_{PHS}. Then also $\check{D}_{PMHS}$ (resp. D_{PMHS}) is a complex (resp. real) homogeneous space.
In order to construct classifying spaces of TERP-structures, we will need to consider a classical invariant of a regular singular TERP-structure, namely the spectral numbers resp. spectral pairs. We recall the definition.

Definition 13. *[HS07] Let $(H,H'_\mathbb{R},\nabla,P,w)$ be a regular singular TERP-structure of weight w.*

1. *The spectrum of (H,∇) at zero is defined as $\mathrm{Sp}(H,\nabla)=\sum_{\alpha\in\mathbb{Q}}d(\alpha)\cdot\alpha\in\mathbb{Z}[\mathbb{R}]$ where*

$$d(\alpha):=\dim_\mathbb{C}\left(\frac{Gr_V^\alpha\mathcal{H}}{Gr_V^\alpha z\mathcal{H}}\right)=\dim_\mathbb{C}\mathrm{Gr}_{\widetilde{F}}^{\lfloor w-\alpha\rfloor}H^\infty_{e^{-2\pi i\alpha}}.$$

   We also write $\mathrm{Sp}(H,\nabla)$ as a tuple $\alpha_1,\ldots,\alpha_\mu$ of μ numbers (with $\mu=\mathrm{rk}(H)$), ordered by $\alpha_1\leq\ldots\leq\alpha_\mu$. We have that $\alpha_i=w-\alpha_{\mu+1-i}$ and that α is a spectral number only if $e^{-2\pi i\alpha}$ is an eigenvalue of the monodromy M of H' (in particular, all α_i are real by assumption).

2. *The spectral pairs are a refinement of the spectrum taking into account the weight filtration $W_\bullet(N)$ (Here the restriction of $W_\bullet(N)$ to H^∞_1 is centered around w, and the restriction to $H^\infty_{\neq1}$ is centered around $w-1$). They are given by $\mathrm{Spp}(H,\nabla):=\sum_{\alpha\in\mathbb{Q}}\widetilde{d}(\alpha,k)\cdot(\alpha,k)\in\mathbb{Z}[\mathbb{R}\times\mathbb{Z}]$, where*

$$\widetilde{d}(\alpha,k):=\begin{cases}\dim_\mathbb{C}\mathrm{Gr}_{\widetilde{F}}^{\lfloor w-\alpha\rfloor}\mathrm{Gr}_k^W H^\infty_{e^{-2\pi i\alpha}} & \forall\alpha\notin\mathbb{Z}\\ \dim_\mathbb{C}\mathrm{Gr}_{\widetilde{F}}^{w-\alpha}\mathrm{Gr}_{k+1}^W H^\infty_1 & \forall\alpha\in\mathbb{Z}\end{cases}$$

For a given space $\check{D}_{PMHS}$, and for any element $F^\bullet\in\check{D}_{PMHS}$, one may consider all possible regular singular TERP-structures inducing this filtration via the construction from definition 10, 3. This leads to the following definition.

Definition 14. *[HS08a] Fix $w\in\mathbb{Z}$, a flat bundle $H'\to\mathbb{C}^*$ with real structure $H'_\mathbb{R}\subset H'$, and a flat, $(-1)^w$-symmetric, non-degenerate pairing $P:\mathcal{H}'\otimes j^*\mathcal{H}'\to\mathcal{O}_{\mathbb{C}^*}$, which takes values in $i^w\mathbb{R}$ on $H'_\mathbb{R}$. Write $(H^\infty,H^\infty_\mathbb{R},S)$ for the associated linear algebra data as in definition 10, fix a reference filtration $F_0^\bullet$ on H^∞ and consider the classifying spaces $\check{D}_{PMHS}$ and D_{PMHS}. Define*

$$\begin{aligned}\check{D}_{BL} \quad := \quad & \{\mathcal{H}\subset V^{>-\infty}\,|\,H\to\mathbb{C}\text{ holomorphic vector bundle},H_{|\mathbb{C}^*}=H',\\ & (z^2\nabla_z)(\mathcal{H})\subset\mathcal{H},P(\mathcal{H},\mathcal{H})\subset z^w\mathcal{O}_\mathbb{C}\text{ non-degenerate},\widetilde{F}^\bullet\in\check{D}_{PMHS}\}\end{aligned}$$

Remember that $\widetilde{F}^\bullet$ is the filtration defined on H^∞ by $\mathcal{H}$ (see from definition 10, 3.). Notice also that the condition $\widetilde{F}^\bullet\in\check{D}_{PMHS}$ implies that all elements $\mathcal{H}\in\check{D}_{BL}$ have the same spectral pairs, which are fixed by the choice of $F_0^\bullet$. We write $(\alpha_1,\ldots,\alpha_\mu)$ for the spectral numbers of all elements $\mathcal{H}\in\check{D}_{BL}$, in particular, for all such $\mathcal{H}$ we have $V^{>\alpha_\mu-1}\subset\mathcal{H}\subset V^{\alpha_1}$.

We have a projection map $\check{\alpha}:\check{D}_{BL}\to\check{D}_{PMHS}$ defined by $H\mapsto\widetilde{F}^\bullet$. One of the main results of [Her99] is that $\check{D}_{BL}$ is a complex manifold, and that $\check{\alpha}$ is a locally trivial fibration where the fibres are affine spaces. Again we put $D_{BL}:=\check{\alpha}^{-1}(D_{PMHS})$. The situation can be visualized in the following diagram.

$$\begin{array}{lcl} D_{BL} & \hookrightarrow & \check{D}_{BL}\\ \downarrow\alpha & & \downarrow\check{\alpha}\\ D_{PMHS} & \hookrightarrow & \check{D}_{PMHS}\\ \downarrow\beta & & \downarrow\check{\beta}\\ D_{PHS} & \hookrightarrow & \check{D}_{PHS}\end{array}$$

Note that neither $\check{D}_{BL}$ nor D_{BL} are homogeneous.

By definition of the space $\check{D}_{BL}$, for any family $H \to M$ of regular singular TERP-structures on a simply connected manifold M with constant spectral pairs we obtain a holomorphic map $\phi : M \to \check{D}_{BL}$ by associating to each point x in M the point in $\check{D}_{BL}$ representing the TERP-structure over x, i.e., the restriction of our family to $\mathbb{C} \times \{x\}$.
The following lemma describes some direct consequences of the definition of the spaces $\check{D}_{BL}$ and D_{BL}.

Lemma 15. *[HS08a, sections 3 and 4]*

1. *There is a universal bundle $H \to \mathbb{C} \times \check{D}_{BL}$ of TERP-structures, i.e., the restriction $H_{|\mathbb{C}\times\{x\}}$ is the regular singular TERP-structure represented by the point $x \in \check{D}_{BL}$. We have a connection operator*

$$\nabla : \mathcal{H} \longrightarrow \mathcal{H} \otimes \left(z^{-2}\Omega^1_{\mathbb{C}\times\check{D}_{BL}/\check{D}_{BL}} \oplus z^{-\lfloor \alpha_\mu - \alpha_1 \rfloor}\Omega^1_{\mathbb{C}\times\check{D}_{BL}/\mathbb{C}} \right),$$

   i.e., in general $\mathcal{H}$ does not underly a variation of TERP-structures. The construction described in the beginning of section 2 yields an extension of $\mathcal{H}$ to a locally free $\mathcal{O}_{\mathbb{P}^1}\mathcal{C}^{an}_{\check{D}_{BL}}$-module $\widehat{\mathcal{H}}$, i.e., a real analytic family of twistors, equipped with a polarization as in definition 3.

2. *Define $\check{D}^{pp}_{BL} := \{x \in \check{D}_{BL} \,|\, \widehat{\mathcal{H}}_{|\mathbb{P}^1\times\{x\}}$ is pure polarized $\}$, which is an open submanifold, then the $\mathcal{C}^{an}$-bundle $\pi_*\widehat{\mathcal{H}}_{|\check{D}^{pp}_{BL}}$ carries a positive definite hermitian metric (which is not harmonic in general).*

3. *There is an injective bundle map*

$$\Theta_{\check{D}_{BL}} \hookrightarrow \mathcal{H}om_{\mathcal{O}_{\mathbb{C}\times\check{D}_{BL}}}\left(\mathcal{H}, \frac{z^{-\lfloor \alpha_\mu - \alpha_1 \rfloor}\mathcal{H}}{\mathcal{H}}\right)$$

   defined by $X \longmapsto [s \mapsto \nabla_X(s)]$. This map restricts to $\Theta_{\check{D}^{pp}_{BL}} \hookrightarrow \mathcal{H}om_{\mathcal{O}_{\mathbb{C}\times\check{D}^{pp}_{BL}}}\left(\mathcal{H}, \frac{z^{-\lfloor \alpha_\mu - \alpha_1 \rfloor}\mathcal{H}}{\mathcal{H}}\right)$, and via this inclusion, the tangent bundle $\Theta_{\check{D}^{pp}_{BL}}$ is equipped with a positive definite hermitian metric.

The main result of [HS08a] is then the following.

Theorem 16. *[HS08a, theorem 4.1] There is a coherent subsheaf $\mathcal{T}^{hor}_{\check{D}_{BL}}$ of $\mathcal{T}_{\check{D}_{BL}}$ such that for any variation of TERP-structures on a simply connected space M, we have that $d\phi(\mathcal{T}_M) \subset \phi^*(\mathcal{T}^{hor}_{\check{D}_{BL}})$. In general, $\mathcal{T}^{hor}_{\check{D}_{BL}}$ is not $\mathcal{O}_{\check{D}_{BL}}$-locally free. Write $\mathcal{T}^{hor}_{\check{D}^{pp}_{BL}}$ for the restriction $(\mathcal{T}^{hor}_{\check{D}_{BL}})_{|\check{D}^{pp}_{BL}}$.*
The restriction of the holomorphic sectional curvature $\kappa : T_{\check{D}^{pp}_{BL}}\backslash\{$zero-section$\} \to \mathbb{R}$ to the subspace $T^{hor}_{\check{D}^{pp}_{BL}}\backslash\{$zero section$\}$ (i.e., the complement of the zero section of the linear space associated to the coherent sheaf $\mathcal{T}^{hor}_{\check{D}^{pp}_{BL}}$) is bounded from above by a negative number.

We outline two immediate consequences, which are analogues for the corresponding statements for variations of Hodge structures.

Corollary 17.

1. *[HS08a, proposition 4.3] For a complex manifold M, any horizontal mapping $\phi : M \to \check{D}^{pp}_{BL}$, i.e., such that $d\phi(\mathcal{T}_M) \subset \phi^*(\mathcal{T}^{hor}_{\check{D}^{pp}_{BL}})$ is distance decreasing with respect to the Kobayashi pseudodistance on M and the distance induced by h on $\check{D}^{pp}_{BL}$.*

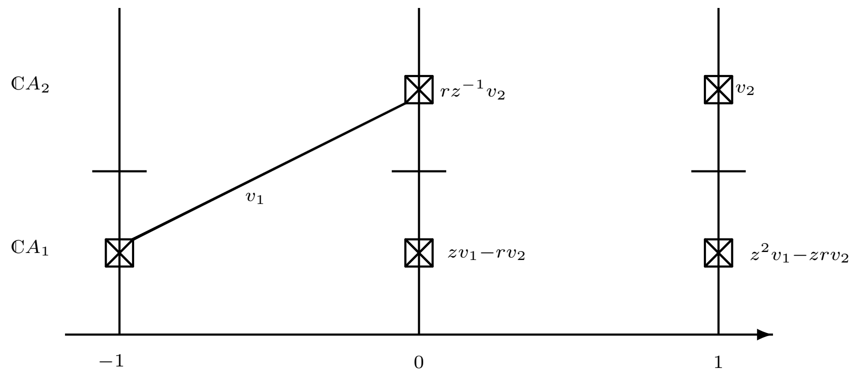
2. *[HS08a, corollary 4.5] Any variation of regular singular pure polarized TERP-structures with constant spectral pairs on $\mathbb{C}^m$ is trivial, i.e., flat in the parameter direction.*

2.3 Limit structures and compatifications

Consider the following example of a classifying space $\check{D}_{BL}$ from the last subsection: Let $H^\infty := \mathbb{C}A_1 \oplus \mathbb{C}A_2$, where $\overline{A}_1 = A_2$, $M(A_i) = A_i$, $w = 0$ and $S(A_i, A_j) = \delta_{i+j,3}$. Define $F_0^\bullet H^\infty$ by:

$$\{0\} = F_0^2 H^\infty \subsetneq F_0^1 H^\infty := \mathbb{C}A_1 = F_0^0 H^\infty \subsetneq F_0^{-1} H^\infty := H^\infty$$

Then $\check{D}_{PMHS} = \{F_0^\bullet, \overline{F}_0^\bullet\}$, and $\check{D}_{BL}$ is a union of two connected components above the two points of $\check{D}_{PMHS}$. Both components are isomorphic to $\mathbb{C} = \text{Spec } \mathbb{C}[r]$, the universal family of the component over $F_0^\bullet$ is $\mathcal{H} := \mathcal{O}_{\mathbb{C}^2}v_1 \oplus \mathcal{O}_{\mathbb{C}^2}v_2$, where $v_1 := z^{-1}A_1 + r \cdot A_2$ and $v_2 := zA_2$. For any $r \in \mathbb{C}$, the TERP-structure $\mathcal{H}_{|\mathbb{C}\times\{r\}}$ has spectrum $(-1, 1)$. The situation is visualized as follows.



We have $\overline{\gamma^*\mathcal{H}} := \mathcal{O}_{(\mathbb{P}^1\setminus\{0\})\times\overline{\mathbb{C}}} v_1' \oplus \mathcal{O}_{(\mathbb{P}^1\setminus\{0\})\times\overline{\mathbb{C}}} v_2'$, where $v_1' = zA_2 + \overline{r}\cdot A_1$ and $v_2' = z^{-1}A_1$. Hence we see that $\widehat{\mathcal{H}}$ is pure for all $r \in \mathbb{C}$ with $|r| \neq 1$ (namely, we have $\widehat{\mathcal{H}}_{|\mathbb{P}^1\times\{r\}} = \mathcal{O}_{\mathbb{P}^1}v_1 \oplus \mathcal{O}_{\mathbb{P}^1}v_1'$ for $|r| \neq 1$), and polarized iff $|r| > 1$. For $|r| = 1$, we have $\widehat{\mathcal{H}} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The definition of the hermitian h metric on $\check{D}_{BL}^{pp} \cong \{r \in \mathbb{C} \,|\, |r| > 1\}$ yields that

$$h(\partial_s, \partial_s) = \frac{1}{s^2\overline{s}^2(1-(s\overline{s})^{-1})^2}$$

where $s := r^{-1}$ is a coordinate on $\check{D}_{BL}^{pp}$ near infinity. We see that for $s \to 0$, $h(\partial_s, \partial_s)$ tends to 1, this implies that the metric space $\check{D}_{BL}^{pp}$ is not complete. The reason for this is that the above family can be completed to a family over $\mathbb{P}^1$: consider the restriction $\mathcal{H}_{|\mathbb{C}\times\mathbb{C}^*}$, then this family is extended by

$$\mathcal{O}_{\mathbb{C}\times(\mathbb{P}^1\setminus\{0\})}(s\cdot z^{-1}A_1 + A_2) \oplus \mathcal{O}_{\mathbb{C}\times(\mathbb{P}^1\setminus\{0\})}A_1,$$

and the restriction to $s = 0$ of this family is $\mathcal{O}_{\mathbb{C}}A_1 \oplus \mathcal{O}_{\mathbb{C}}A_2$ which has spectrum $(0,0)$. Hence, this limit point is missing in the space $\check{D}_{BL}^{pp}$ which therefore cannot be complete for the distance induced by h. Notice that in this example, the subspace D_{PMHS} of $\check{D}_{PMHS}$ is empty: both points $F_0^\bullet$ and $\overline{F}_0^\bullet$ are negative definite pure Hodge structures. One can change the sign of the pairing S in order to make $F_0^\bullet$ and $\overline{F}_0^\bullet$ into pure polarized Hodge structures, however, then the pure polarized part $\check{D}_{BL}^{pp}$ is $\{r \in \mathbb{C} \,|\, |r| < 1\}$. On the other hand, the example given at the end of [HS08a] shows that there are also classifying spaces $\check{D}_{BL}^{pp}$ which are not complete for the hermitian metric h and where the intersection $\check{D}_{BL}^{pp} \cap D_{BL}$ is non-empty.
In order to construct a space with a complete metric, the above reasoning naturally lead us to allow for varying spectral numbers. However, a range for them should be fixed for otherwise the corresponding classifying space has no reason to be finite-dimensional. Hence we make the following definition.

Definition 18. *[HS08b] Fix, as before $H_{\mathbb{R}}^\infty$, $M \in \mathrm{Aut}(H_{\mathbb{R}}^\infty)$, $S : H_{\mathbb{R}}^\infty \times H_{\mathbb{R}}^\infty \to \mathbb{R}$ and $w \in \mathbb{Z}$. Fix also $\alpha_1 \in \mathbb{R}$ such that $e^{-2\pi\alpha_1}$ is an eigenvalue of M and such that $\alpha_1 \leq \frac{w}{2}$. Consider the flat bundle $H' \to \mathbb{C}^*$ corresponding to (H^∞, M) and the Deligne extensions $V^\alpha \to \mathbb{C}$ of H'. Define*

$$\begin{aligned}\mathcal{M}_{BL} \;\; := \;\; & \big\{\mathcal{H} \subset V^{\alpha_1} \,|\, H \to \mathbb{C} \text{ holomorphic vector bundle}, H_{|\mathbb{C}^*} = H', \\ & (z^2\nabla_z)(\mathcal{H}) \subset \mathcal{H}, P(\mathcal{H},\mathcal{H}) \subset z^w\mathcal{O}_{\mathbb{C}} \text{ non-degenerate}\big\}\end{aligned}$$

Notice that the conditions $\mathcal{H} \subset V^{\alpha_1}$ and that P takes values in $z^w\mathcal{O}_{\mathbb{C}}$ on $\mathcal{H}$ and is non-degenerate actually imply that the spectrum of $\mathcal{H}$ is contained in the interval $[\alpha_1, w - \alpha_1]$ (due to the symmetry property $\alpha_i = w - \alpha_{\mu+1-i}$, see definition 13).
With the above definitions at hand, we have the following theorem, which is one of the main results of [HS08b].

Theorem 19. *1. [HS08b, definition-lemma 7.1] $\mathcal{M}_{BL}$ is a projective variety which contains the spaces $\check{D}_{BL}$ (for the various spectral numbers $\{\alpha_1, \ldots, \alpha_\mu\} \subset [\alpha_1, w - \alpha_1]$) as locally closed subspaces. There is a universal locally free $\mathcal{O}_{\mathbb{C}\times\mathcal{M}_{BL}}$-module $\mathcal{H}$ of TERP-structures, which reduces* *to the universal bundle considered in subsection 2.2 when restricted to a subspace $\check{D}_{BL}$. Given a variation of regular singular TERP-structures on a simply connected manifold M, then there is a holomorphic period map $\phi : M \to \mathcal{M}_{BL}$. Here the rational number α_1 used in the construction of $\mathcal{M}_{BL}$ is the smallest element of the generic spectral numbers of the given family (such a minimal spectral number exists).*

2. *[HS08b, section 8] Define $\mathcal{M}_{BL}^{pp} := \left\{ x \in \mathcal{M}_{BL} \,|\, \widehat{\mathcal{H}}_{|\mathbb{P}^1 \times \{x\}} \text{ is pure polarized} \right\}$, then a similar construction as in the case of $\check{D}_{BL}^{pp}$ (lemma 15) endows the tangent sheaf of $\mathcal{M}_{BL}^{pp}$ with a positive definite hermitian metric, which defines a distance function d_h on $\mathcal{M}_{BL}^{pp}$.*

3. *[HS08b, theorem 8.6] The metric space $(\mathcal{M}_{BL}^{pp}, d_h)$ is complete.*

4. *[HS08b, theorem 8.8] Suppose that there is a lattice $H_{\mathbb{Z}}^{\infty} \subset H_{\mathbb{R}}^{\infty}$ with $M \in \mathrm{Aut}(H_{\mathbb{Z}}^{\infty})$. Write $G_{\mathbb{Z}} := \mathrm{Aut}(H_{\mathbb{Z}}^{\infty}, S, M)$, then $G_{\mathbb{Z}}$ acts properly discontinuously on $\mathcal{M}_{BL}^{pp}$, hence, the quotient $\mathcal{M}_{BL}^{pp}/G_{\mathbb{Z}}$ has the structure of a complex space.*

Notice that contrary to $\check{D}_{BL}$ (resp. $\check{D}_{BL}^{pp}$) the variety $\mathcal{M}_{BL}$ (resp. $\mathcal{M}_{BL}^{pp}$) may be singular and even non-reduced. However, one can still work with distance functions defined by a hermitian metric on the tangent sheaf (see, e.g., [Kob05]).
The following corollary is a direct consequence, its proof is analogue to the case of variations of Hodge structures.

Corollary 20. *[HS08b, theorem 9.5] Let X be a complex manifold, $Z \subset X$ a complex space of codimension at least two. Suppose that the complement $Y := X \backslash Z$ is simply connected. Let $(H, H'_{\mathbb{R}}, \nabla, P, w)$ be a variation of regular singular pure polarized TERP-structures on the complement Y which has constant spectral pairs. Then this variation extends to the whole of X, with possibly jumping spectral numbers over Z.*

In applications, one is also interested in understanding extensions over codimension one subvarieties. More precisely, let $1 \leq l \leq n$, $X := \Delta^n$, $Y := (\Delta^*)^l \times \Delta^{n-l}$, $X \backslash Y = \coprod_{i=1,\dots,l} D_i$, and consider a variation of pure polarized regular singular TERP-structures $(H, H'_{\mathbb{R}}, \nabla, P, w)$ on Y. Denote by $M_i \in \mathrm{Aut}(H_{\mathbb{R}}^{\infty})$ the monodromy corresponding to a loop around $\mathbb{C}^* \times D_i \subset \mathbb{C}^* \times X$ and by M_z the monodromy corresponding to a loop around $\{0\} \times Y \subset \mathbb{C} \times Y$. We say that the monodromy respects a lattice if there is a lattice $H_{\mathbb{Z}}^{\infty} \subset H_{\mathbb{R}}^{\infty}$ such that the image of $\gamma : \pi_1(\mathbb{C}^* \times Y) \to \mathrm{Aut}(H_{\mathbb{R}}^{\infty})$ is contained in $\mathrm{Aut}(H_{\mathbb{Z}}^{\infty})$, in that case we put $G_{\mathbb{Z}} := \mathrm{Aut}(H_{\mathbb{Z}}^{\infty}, S, M_z)$.
The following statement is essentially an application of the fundamental results of Mochizuki ([Moc07]) on limit mixed twistor structures, together with a careful discussion on the corresponding limit statements for variation of TERP-structures (see sections 3-5 of [HS08b]).

Theorem 21. *[HS08b, theorem 9.7] Let $(H, H'_{\mathbb{R}}, \nabla, P, w)$ be a variation of regular singular, pure polarized TERP-structures on Y.*

- *If $M_i = Id$ for all $i \in \{1, \dots, k\}$, i.e., if there is a period map $\phi : Y \to \mathcal{M}_{BL}^{pp}$, then this map extends to*
$$\overline{\phi} : X \to \mathcal{M}_{BL}^{pp}$$
and the variation H extends to a variation on X.

- *Suppose that the monodromy respects a lattice. Then we have a locally liftable period map $\phi : Y \to \mathcal{M}_{BL}^{pp}/G_{\mathbb{Z}}$. If all M_i are semi-simple, then ϕ extends holomorphically (not necessarily locally liftable) to*
$$\overline{\phi} : X \to \mathcal{M}_{BL}^{pp}/G_{\mathbb{Z}}.$$

3 Frobenius manifolds and linear free divisors

We start this part by recalling the definition and an alternative description of Frobenius structures on complex manifolds. We also explain briefly two situations in singularity theory where Frobenius structures naturally occur.

Definition 22. *[Dub96, Her02] Let M be a complex manifold. A Frobenius structure on M is given by two tensors $\circ \in (\Omega_M)^{\otimes 2} \otimes_{\mathcal{O}_M} \Theta_M$, $g \in (\Omega_M)^{\otimes 2}$ and two vector fields $E, e \in \Theta_M$ subject to the following relations.*

1. *$\circ$ defines a commutative and associative multiplication on Θ_M with unit e.*
2. *g is bilinear, symmetric and non-degenerate.*
3. *For any $X, Y, Z \in \Theta_M$, $g(X \circ Y, Z) = g(X, Y \circ Z)$.*
4. *g is flat, i.e., locally there are coordinates $t_1, \ldots, t_\mu$ on M such that the matrix of g in the basis $(\partial_{t_1}, \ldots, \partial_{t_\mu})$ is constant.*
5. *Write ∇ for the Levi-Civita connection of g, then the tensor $\nabla\circ$ is totally symmetric.*
6. *$\nabla(e) = 0$.*
7. *$\mathrm{Lie}_E(\circ) = \circ$, $\mathrm{Lie}_E(g) = D \cdot g$ for some $D \in \mathbb{C}$*

Remark: An important consequence of this definition is that the structure $(M, \circ, g, e)$ can be encoded locally by a single holomorphic function, more precisely, let as before $t_1, \ldots, t_\mu$ be local flat coordinates with $e = \partial_{t_1}$, then there is $F \in \mathcal{O}_M$ such that $g(\partial_i \circ \partial_j, \partial_k) = \partial_i\partial_j\partial_k F$, where $\partial_i := \partial_{t_i}$. The associativity of the multiplication $\circ$ is then equivalent to the so called Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation, which says that

$$\sum_{k,l} \partial_i\partial_j\partial_k F \cdot g^{kl} \cdot \partial_l\partial_r\partial_s F = \sum_{k,l} \partial_i\partial_r\partial_k F \cdot g^{kl} \cdot \partial_l\partial_j\partial_s F$$

holds for any $i, j, r, s \in \{1, \ldots, \mu\}$, where (g^{kl}) is the inverse matrix of $(g(\partial_k, \partial_l))_{kl}$ In applications, one uses the following equivalent description of a Frobenius manifold.

Theorem 23. *[Sab07, Her02] Fix an integer $w \in \mathbb{Z}$. Then a Frobenius structure on a complex manifold M can be encoded equivalently be the following set of data.*

1. *a holomorphic vector bundle E on $\mathbb{P}^1 \times M$ such that $\mathrm{rank}(E) = \dim(M)$, which is fibrewise trivial, i.e. $\mathcal{E} = \pi^*\pi_*\mathcal{E}$, (where $\pi : \mathbb{P}^1 \times M \to M$ is the projection) equipped with an integrable connection with a logarithmic pole along $\{\infty\} \times M$ and a pole of type one along $\{0\} \times M$,*
2. *a non-degenerate, $(-1)^w$-symmetric pairing $P : \mathcal{E} \otimes j^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1 \times M}(-w, w)$ (here $j(z, u) = (-z, u)$ and we write $\mathcal{O}_{\mathbb{P}^1 \times M}(a, b)$ for the sheaf of meromorphic functions on $\mathbb{P}^1 \times M$ with a pole of order a along $\{0\} \times M$ and order b along $\{\infty\} \times M$) the restriction of which to $\mathbb{C}^* \times M$ is flat,*
3. *a global section $\xi \in H^0(\mathbb{P}^1 \times M, \mathcal{E})$, whose restriction to $\{\infty\} \times M$ is flat with respect to the residue connection $\nabla^{res} : \mathcal{E}/z^{-1}\mathcal{E} \to \mathcal{E}/z^{-1}\mathcal{E} \otimes \Omega^1_M$ with the following two properties*
   - *(a) The morphism*

     $$\begin{aligned} \Phi_\xi : \mathcal{T}_M &\longrightarrow \mathcal{E}/z\mathcal{E} \cong \pi_*\mathcal{E} \\ X &\longmapsto -[z\nabla_X](\xi) \end{aligned}$$

     is an isomorphism of vector bundles (a section ξ with this property is called primitive),
   - *(b) ξ is an eigenvector of the residue endomorphism $[z^{-1}\nabla_{z^{-1}}] \in \mathcal{E}nd_{\mathcal{O}_M}(\pi_*\mathcal{E}) \cong \mathcal{E}nd_{\mathcal{O}_M}(\mathcal{E}/z^{-1}\mathcal{E})$ (a section with this property is called homogeneous).*

There are two important geometric situations in which Frobenius structures occur in singularity theory, namely, those mentioned in the beginning of section 2: local singularities $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ and tame polynomial functions $f : U \to \mathbb{C}$. The basic recipe to construct Frobenius manifolds is similar in both situations: One considers the Fourier-Laplace transformed Brieskorn lattice (i.e., the bundle denoted by G_0^{an} in theorem 8) of both f and of a semi-universal deformation F of f over a parameter space M, where in the second case there are some subtleties, due to the fact that not all members in this deformation are tame. The main points are to find an extension $\widehat{G}_0^{an}$ of G_0^{an} to a trivial holomorphic $\mathbb{P}^1 \times M$-bundle with a logarithmic pole along infinity and such that $z^{-m}P$ takes values in $\mathcal{O}_M$ on $\pi_* \widehat{G}_0^{an}$, P being the pairing from definition 1. It turns out that it is sufficient to find this extension only at a point of the parameter space, i.e., for the Brieskorn lattice of the function f. Once such an extension is found, it remains to obtain a homogenous and primitive form ξ as above.

Theorem 24 ([Sai81, Sai83, Sai89, Her02]). *Let $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be an isolated hypersurface singularity, and $(M, 0)$ (the germ at zero of) its semi-universal unfolding space. This space carries a Frobenius structure, which depends on the choice of an opposite filtration to the Hodge filtration on the cohomology of the Milnor fibre of the singularity f, and a generator of a certain one dimensional graded piece for this opposite filtration. Up to constant multiplication, this generator is the only primitive and homogenous section. Notice that only the metric g depends on these choices, the multiplication as well as the unit and the Euler field are independently defined. As canonical choice for the opposite filtration is given by Deligne's $I^{p,q}$-decomposition for the mixed Hodge structure.*

In the second case, a weaker result holds. The data 1. to 3. in the above equivalent description of Frobenius structures can be constructed for any tame function, however, the last point is more subtle and for the moment only shown for a restricted class of functions. We summarize these results in the following statement.

Theorem 25 ([Sab06], [DS03]).

1. *Let $f : U \to \mathbb{C}$ be a cohomological tame function. Then any choice of an opposite filtration to the Hodge filtration on H^∞ yields an extension of G_0^{an} to infinity as above. Again a canonical choice is given by the $I^{p,q}$-decomposition of Deligne.*

2. *Let $f : (\mathbb{C}^*)^m \to \mathbb{C}$ be a convenient and non-degenerate Laurent polynomial in the sense of [Kou76]. Then again there is a one-dimensional graded piece of the canonical opposite filtration from 1. such that any generator of this space is a primitive and homogenous global section of the canonical extension of the Brieskorn lattice of f. This one dimensional space is canonical in the sense that it corresponds to the eigenspace of the residue endomorphism $[z^{-1}\nabla_{z^{-1}}]$ for the* smallest *spectral number, i.e., this smallest spectral number has multiplicity one in this case.*

   However, it may happen that there are other primitive and homogenous sections. Any choice of an opposite filtration and a primitive and homogenous section yields a Frobenius structure on a universal unfolding M of f.

Notice that with a more restrictive condition on the Newton boundary, one can also show the existence of a primitive and homogenous section for the case of tame polynomials $f : \mathbb{C}^m \to \mathbb{C}$ (see [Sab06, DS03]).

3.1 Pre-homogenous vector spaces, quiver representations and linear free divisors

A free divisor generalizes in a very natural way a normal crossing divisor. Following Saito ([Sai80]), we call a reduced divisor $D := h^{-1}(0) \subset M$, where M is any n-dimensional complex manifold free iff the coherent $\mathcal{O}_M$-module of *logarithmic vector fields*

$$\Theta(-\log D) := \{\vartheta \in \Theta_M \,|\, (\vartheta)(f) \subset (f)\}$$

is locally free. These divisors appear in many situations, e.g., various kinds of discriminants in deformation spaces are free. In [BM06], the more special class of linear free divisors has been introduced and turned out to be connected to rather different areas: the theory of pre-homogenous vector spaces, and, as a particular class of examples, discriminants of quiver representation spaces.
We give a brief review of the relevant definitions from [BM06], [GMNS09] and [MdGS08].

Definition 26.

1. *Let V be a complex vector space, and G a connected algebraic group acting on V. Then (V, G) is called pre-homogenous iff there is a Zariski open orbit of G in V.*

2. *Let $V := \mathbb{C}^n$, fix coordinates $x_1, \ldots, x_n$, then a reduced algebraic hypersurface $D \subset V$ is called linear free iff it is free in the above sense and if there is a basis $\delta_1, \ldots, \delta_n$ of $\Theta(-\log D)$ such that $\delta_i = \sum_{j=1}^n a_{ij}\partial_{x_j}$ where $a_{ij} \in \mathbb{C}[x_1, \ldots, x_n]_1$ are linear polynomials.*

3. *For a linear free divisor $D \subset V$, let G be the identity component of the algebraic group $G_D := \{g \in \mathrm{Gl}(V) \,|\, g(D) \subset D\}$. Then (V, G) is pre-homogenous, in particular, the complement $V \backslash D$ is an open orbit of G. We call D reductive if G_D is so. A rational function $r \in \mathbb{C}(V)$ is called a semi-invariant if there is a character $\chi_r : G \to \mathbb{C}^*$ such that $g(r) = \chi_r(g) \cdot r$ for all $g \in G$. Obviously, h itself is a semi-invariant.*

Notice that it follows from Saito's criterion ([Sai80, theorem 1.8.ii]) that h is (up to a constant factor) the determinant of the matrix $A \in M(n \times n, \mathbb{C}[V]_1)$ such that $\underline{\delta} = A \cdot \underline{\partial_x}^t$ is a basis of $\Theta(-\log D)$, in particular, it is a homogenous polynomial of degree n.
We give a short list of some examples of linear free divisors.

1. **The normal crossing divisor:** As already mentioned, linear free divisors generalizes divisors with only normal crossing singularities. More precisely, if we put $h := \prod_{i=1}^n x_i \in \mathbb{C}[V]$, then obviously $\Theta(-\log D) := \oplus_{i=1}^n \mathcal{O}_V x_i \partial_{x_i}$, so that D is linear free. Notice however that a normal crossing divisor with less components, i.e., $\widetilde{D} := \widetilde{h}^{-1}(0)$, where $\widetilde{h} := \prod_{j=1}^k x_{i_j}$ with $k < n$ is free, but not linear free, as we then have
$$\Theta(\log \widetilde{D}) = \bigoplus_{j=1}^{k} \mathcal{O}_V \, (x_{i_j} \partial_{x_{i_j}}) \oplus \bigoplus_{i \notin i_1, \ldots, i_k} \mathcal{O}_V \, \partial_{x_i}$$

2. **Quiver representations:** Let $Q := (Q_0, Q_1, h, t)$ be a quiver (i.e., an oriented graph without loops with edge set Q_0, arrow set Q_1 and mappings "head" and "tail" $h, t : Q_1 \to Q_0$) and $\mathbf{d} \in \mathbb{N}^{Q_0}$ be a dimension vector. Consider the space $\mathrm{Rep}(\mathrm{Q}, \mathbf{d}) := \prod_{a \in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{d_{h(a)}}, \mathbb{C}^{d_{t(a)}})$. The "quiver group" $\mathrm{Gl}_{Q,\mathbf{d}} := (\prod_{e \in Q_0} \mathrm{Gl}(d_e, \mathbb{C}))/\mathbb{C}^*$ ($\mathbb{C}^*$ is the subgroup of diagonal matrices with the same entry everywhere) acts by the usual transformation formula on each factor of $\mathrm{Rep}(\mathrm{Q}, \mathbf{d})$. If $\mathbf{d}$ is a real Schur root, i.e. (see, e.g., [BM06]), if there is exactly one orbit of indecomposable representations in $\mathrm{Rep}(\mathrm{Q}, \mathbf{d})$, then $(\mathrm{Rep}(\mathrm{Q}, \mathbf{d}), \mathrm{Gl}_{Q,\mathbf{d}})$ is pre-homogenous. The complement of the open orbit is denoted by D, and if each irreducible component of D contains an open orbit of $\mathrm{Gl}_{Q,\mathbf{d}}$, then D is a linear free divisor. In particular, the latter condition is always satisfied if Q is a Dynkin quiver. As an example of a non-Dynkin quiver giving rise to a linear free divisor, consider the series $\star_r$, given by a quiver Q with r exterior vertices with one arrow each to a common interior vertex, where one attaches 1 as dimension to the exterior vertices and $r - 1$ as dimension to the interior vertex. It is easy to see that in this case $\mathrm{Rep}(\mathrm{Q}, \mathbf{d})$ is identified with the space of $r \times (r-1)$-matrices, on which $(\mathrm{Gl}(r, \mathbb{C}) \times \mathrm{Gl}(r-1, \mathbb{C}))/\mathbb{C}^*$ acts in the natural way.

3. **Two examples of irreducible linear free divisor:**
   (a) [GMNS09, example 1.4.(2)] Consider the space of polynomials of degree 3 in two variables, and inside this space its discriminant, i.e., the hypersurface of polynomials with multiple roots. It can be shown that this is a (reductive) irreducible linear free divisor of degree 4.
   (b) [SK77, proposition 11] Consider the group $G = \mathrm{Sl}(3, \mathbb{C}) \times \mathrm{Gl}(2, \mathbb{C})$ acting on the space $\mathrm{Sym}(3, \mathbb{C}) \times \mathrm{Sym}(3, \mathbb{C})$ of pairs of symmetric 3×3-matrices by
   $$\left(A, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) \longmapsto \big((X, Y) \mapsto \big(A(\alpha X + \beta Y)A^t, A(\gamma X + \delta Y)A^t\big)\big)$$
   The action of this reductive group is pre-homogenous, and the discriminant is a reduced irreducible divisor of degree 12, which is linear free.

4. **A series of non-reductive linear free divisors:** As an example of a non-reductive linear free divisor, consider the space V of symmetric $k \times k$-matrices S, and the group G of upper triangular $k \times k$-matrices B, acting by $B \mapsto (S \mapsto B^t \cdot S \cdot B)$. This action is pre-homogenous, and the complement of the open orbit is a linear free divisor.

For later use we need to discuss a specific extension of the relative de Rham complex of the morphism $h : V \to T = \mathrm{Spec}\ \mathbb{C}[t]$ (recall that h is a reduced equation for the divisor D). More precisely, we call
$$\Omega^\bullet_{V/T}(\log D) := \frac{\Omega^\bullet_V(\log D)}{h^* \Omega^1_T(\log \{0\}) \wedge \Omega^{\bullet-1}_V(\log D)}$$

together with the induced differential the relative logarithmic de Rham complex. (Notice that as we work in the algebraic category in this section, $\Omega_V^\bullet$ means algebraic differential forms). Then the following holds.

Theorem 27. *[MdGS08, theorem 2.7]*

1. *Write* $E = \sum_{i=1}^n x_i \partial_{x_i}$*. There is a direct sum decomposition* $\Theta(-\log D) = \mathcal{O}_V E \oplus \Theta(-\log h)$*, where* $\Theta(-\log h) := \{\theta \in \Theta(-\log D) \,|\, \theta(h) = 0\}$*. Dually, one obtains* $\Omega_V^1(\log D) = \mathcal{O}_V \frac{dh}{h} \oplus \Omega_V^1(\log h)$*, here* $\Omega_V^1(\log h) := \{\alpha \in \Omega_V^1(\log D) \,|\, i_E\alpha = 0\}$*. Then* $\Omega^1_{V/T}(\log D) \cong \Omega^1_V(\log h)$*, and more generally* $\Omega^k_{V/T}(\log D) \cong \bigwedge^k \Omega_V(\log h)$

2. *Let G be reductive. Consider the natural grading induced on* Ω_V^k*,* $\Omega_V^k(\log D)$ *and* $\Omega^k_{V/T}(\log D)$*, then there is a graded isomorphism*

$$H^*((h_*\Omega^\bullet_{V/T}(\log D))_0, d) \otimes_{\mathbb{C}} \mathcal{O}_T \cong H^*(h_*\Omega^\bullet_{V/T}(\log D), d)$$

where the lower index 0 denotes the degree 0 part. In particular, the relative logarithmic de Rham cohomology consists of free $\mathcal{O}_T$-modules of finite rank.

We notice the following consequence.

Corollary 28. *[MdGS08, corollary 2.9] Let D be a reductive linear free divisor with equation h. Then the character χ_h associated to the semi-invariant h is equal to the determinant of the representation* $G \to \mathrm{Gl}(V)$*.*

The main goal of [MdGS08] is to discuss the deformation theory of *linear functions* relative to the morphism $h : V \to T$, and to construct Frobenius structures on these deformation spaces. In order to do that, one has to chose a sufficiently generic linear function f, and this uses the *dual representation* of the group G, as explained in the following lemma.

Lemma 29. *[MdGS08, proposition 3.7] G acts on V^* by the dual action, with dual discriminant $D^* \subset V^*$. If G_D is reductive, then (V^*, D^*) is pre-homogenous. We call a linear form $f \in V^*$ generic with respect to h (or simply generic, if no confusion is possible) if f lies in the open orbit $V^* \backslash D^*$ of the dual action. Notice that if G is not reductive, then it may happen that no generic linear form exist.*
In the reductive case, there is a basis (e_i) of V with corresponding coordinates (x_i) (called unitary) such that G appears as a subgroup of $U(n)$ in these coordinates. Then $D^* = \{h^* = 0\}$, where $h^*(y) := \overline{h(\overline{y})}$, (y_i) being the dual coordinates of (x_i).

In the sequel, we will always denote by h a reduced equation of D, and by f a generic linear form, in particular, we will assume that such an f exists. We write S for the base space of the morphism defined by f. We will be interested in the restriction of f to both the divisor D and a nonsingular fibre $D_t := h^{-1}(t)$ for $t \neq 0$. The latter is a smooth affine variety, and the restriction of f to D_t has critical points. One can also speak about critical points of the restriction f to D, in the stratified sense. Finally, we are interested in the behavior of f at infinity (inside D_t).

Proposition 30. *[MdGS08, proposition 3.5, proposition 3.15] Let D be linear free and f generic. Then*

1. *Write $D_t := h^{-1}(t)$ for $t \neq 0$ for a Milnor fibre of h, then the restriction $f_{|D_t}$ has n distinct non-degenerate critical points.*

2. *There is a Whitney regular stratification of D such that the restriction of f to all open strata is regular. Hence, the only critical point of f on D in the stratified sense is the origin in V.*

3. *For any $t \neq 0$, the restriction $f_{|D_t} : D_t \to \mathbb{C}$ is cohomologically tame in the sense of [Sab06].*

Notice that the first and the second statement are actually used in the proof of the third one.

Proposition 31. *[MdGS08, proposition 3.4, proposition 3.5] Let D be linear free and f generic. Consider the deformation functor of deformations of the (analytic) germ $f : (V, 0) \to (S, 0)$ modulo analytic coordinate changes respecting the morphism h. The tangent space to this functor is given by*

$$\mathcal{T}^1_{\mathcal{R}_h} = \frac{f^*\Theta_S}{df(\Theta(-\log h)) + (h)}$$

We also consider the module

$$\mathcal{T}^1_{\mathcal{R}_h/T} = \frac{f^*\Theta_S}{df(\Theta(-\log h))}$$

Then $h_*\mathcal{T}^1_{\mathcal{R}_h}$ is an n-dimensional $\mathbb{C}$-vector space equal to the fibre over $0 \in T$ of the free rank n $\mathcal{O}_T$-module $h_*\mathcal{T}^1_{\mathcal{R}_h/T}$. A $\mathbb{C}$- resp. $\mathcal{O}_T$-basis of $h_*\mathcal{T}^1_{\mathcal{R}_h}$ resp. $h_*\mathcal{T}^1_{\mathcal{R}_h/T}$ is given by $(f^i)_{i\in\{0,\ldots,n-1\}}$.
Then semi-universal deformation space of the (unobstructed) functor of deformations of f module $\mathcal{R}_h$-equivalence is a smooth n-dimensional germ.
We can also consider a semi-universal deformation space (in the sense of [DS03]) of the restriction $f_{|D_t}$ for any $t \neq 0$. Then the tangent space at 0 of this space is canonically isomorphic to the fibre $(h_*\mathcal{T}^1_{\mathcal{R}_h/T}(f))_{|t}$.

3.2 Gauß-Manin-systems and Frobenius structures

For the later construction of Frobenius structures, we need to consider families of Gauß-Manin-systems, associated to the pair of functions (f,h). These are naturally constructed as the cohomology sheaves of direct images of $\mathcal{D}$-modules. Namely, the following holds.

Definition-Lemma 32. *[MdGS08, section 4], [Sev09, lemma 7]*

1. *Consider the morphism $\Phi := (f,h) : V \to S \times T$, and the cohomology module $M(*D) := \mathcal{H}^0(\Phi_+(\mathcal{O}_V(*D))$. This is a regular holonomic $\mathcal{D}_{S\times T}$-module.*
2. *Denote by $G(*D)$ the partial (with respect to s) localized Fourier-Laplace transform of $M(*D)$, i.e., $G(*D) = M(*D) \otimes_{\mathbb{C}} \mathbb{C}[\partial_s^{-1}]$ as $\mathbb{C}$ vector spaces, equipped with an $\mathbb{C}[\tau, \partial_\tau, t, \partial_t]$-action where t, ∂_t acts as before and where by definition $\tau\cdot := \partial_s$, $\partial_\tau := -s\cdot$. Putting $z := \tau^{-1}$, then $G(*D)$ is $\mathbb{C}[z, \partial_z, t, \partial_t]$-holonomic, with singularities along $(\{0,\infty\} \times T) \cup (\mathbb{C} \times \{0\})$, regular along $z = \infty$ and $\{t = 0, z \neq 0\}$. Moreover $G(*D)$ is localized along $(\{0,\infty\} \times T) \cup (\mathbb{C} \times \{0\})$, i.e., $\mathbb{C}[z, z^{-1}, t, t^{-1}]$-locally free of rank n. We call $G(*D)$ the family of Gauß-Manin systems associated to (f,h).*
3. *Define*

$$G_0(\log D) := \frac{H^0(V, \Omega^{n-1}_{V/T}(\log D))[z]}{(zd - df\wedge)H^0(V, \Omega^{n-2}_{V/T}(\log D))[z]}$$

to be the family of Brieskorn lattices associated to (f,h). $G_0(\log D)$ is a $\mathbb{C}[z,t]$-lattice in $G(*D)$, and comes equipped with a meromorphic connection

$$\nabla : G_0(\log D) \longrightarrow G_0(\log D) \otimes z^{-1}\Omega^1_{\mathbb{C}\times T}\left(\log(\{0\} \times T) \cup (\mathbb{C} \times \{0\})\right).$$

In order to get more concrete information on both $G(*D)$ and the lattice $G_0(\log D)$, we need to find a basis with specific properties. This is done in the following result.

Proposition 33. *[MdGS08, proposition 4.5, corollary 4.12] Let D be any linear free divisor and let f be a generic linear function. Then*

1. *There is a $\mathbb{C}[z,t,t^{-1}]$-basis $\underline{\omega}^{(2)}$ of $G_0(*D) := G(\log D) \otimes_{\mathbb{C}[z,t]} \mathbb{C}[z,t,t^{-1}]$ such that*

$$\nabla_{\partial_z}(\underline{\omega}^{(2)}) = \underline{\omega}^{(2)} \cdot \left(\frac{A_0}{z} + A^{(1)}_\infty\right)\frac{1}{z}$$

where

$$A_0 := \begin{pmatrix} 0 & 0 & \ldots & 0 & c\cdot t \\ 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

for some $c \in \mathbb{C}$ and $A^{(2)}_\infty = \mathrm{diag}(\nu^{(2)}_1, \ldots, \nu^{(2)}_n)$. For any $t \neq 0$, $(\nu^{(2)}_1, \ldots, \nu^{(2)})$ is the spectrum at infinity (in the sense of [Sab06]) of the tame function $f_{|D_t}$, in other words, $\underline{\omega}^{(2)}$ is a good basis in the sense of [Sai89] of the restriction $(G_0(\log D), \nabla)_{|\mathbb{C}\times\{t\}}$.

2. *There is a $\mathbb{C}[z,t]$-basis $\underline{\omega}^{(1)}$ of $G_0(\log D)$ such that*

$$\nabla_{\partial_z}(\underline{\omega}^{(1)}) = \underline{\omega}^{(1)} \cdot \left(\frac{A_0}{z} + A_\infty^{(1)}\right)\frac{1}{z}$$

where A_0 is as before and $A_\infty^{(1)} = \mathrm{diag}(\nu_1^{(1)},\ldots,\nu_n^{(1)})$ and where $(\nu_1^{(1)},\ldots,\nu_1^{(1)})$ is the spectrum at infinity of $(G_0(h),\nabla) := (G_0(\log D),\nabla)_{|\mathbb{C}\times\{0\}}$ (called logarithmic Brieskorn lattice *of h in [Sev09]), that is, $\underline{\omega}^{(1)}$ is a good basis of $(G_0(\log D),\nabla)_{|\mathbb{C}\times\{0\}}$.*

3. *If D is reductive, then moreover*

$$\nabla(\underline{\omega}^{(1)}) = \underline{\omega}^{(1)} \cdot \left[(A_0\frac{1}{z} + A_\infty^{(1)})\frac{dz}{z} + (-A_0\frac{1}{z} + A'_\infty)\frac{dt}{nt}\right] \tag{9}$$

where $A'_\infty := \mathrm{diag}(0,1,\ldots,n-1) - A_\infty^{(1)}$.

Rermark: In the proof of this result, one actually first constructs the basis $\underline{\omega}^{(1)}$ of $G_0(\log D)$, and then this basis can be used to obtain the basis $\underline{\omega}^{(2)}$ in part 1. of the above proposition, which explains the choice of the names for these bases.
A rather easy consequence of part 3. of the above result is the following statement, which links the class of (linear functions on fibrations defined by) linear free divisors to the theory of variations of TERP-structures as discussed in chapter 2.

Corollary 34. *[MdGS08, proposition 4.5 (v)] Let D be reductive. Consider the mapping $u : S' := \mathbb{C}^* \to T' := T\backslash\{0\}$, defined by $t := s^n$, where s is a coordinate on S'. Then the analytic bundle associated to $u^*(G_0(*D),\nabla)$ underlies a* Sabbah orbit *of TERP-structures (see definition 9).*

Using the good bases from above, we can attach Frobenius structures to the various deformation spaces considered in subsection 3.1.

Theorem 35. *[MdGS08, theorem 5.1, theorem 5.7] Let D be a linear free divisor, and f generic. Then the following holds.*

1. *Fix any $t \in T\backslash\{0\}$, and write M_t for a semi-universal unfolding space of $f_{|D_t}$. Suppose that the minimal spectral number $\min_{i\in\{1,\ldots,n\}}(\nu_i^{(2)})$ is of multiplicity one. Then*
   - *(a) We have that $z^{-n-1}P(\omega_i^{(2)},\omega_j^{(2)}) \in \mathbb{C}$, i.e, the extension $\oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1\times\{t\}}\omega_i^{(2)}$ of $(G_0(*D))_{|\mathbb{C}\times\{t\}}$ is compatible with P as required in the description before theorem 24.*
   - *(b) Any of the sections $\omega_i^{(2)}$ is homogenous and primitive and yields a Frobenius structure on M_t.*
2. *In general (i.e., without any hypotheses on the minimal spectral number), any section $\omega_i^{(2)}$ such that $\nu_i^{(2)} = \min_{j\in\{1,\ldots,n\}}(\nu_j^{(2)})$ is homogenous and primitive for the canonical extension of $G_0(*D)_{|\mathbb{C}\times\{t\}}$ referred to in theorem 25, 2., and hence yields a Frobenius structure on M_t (possibly different from the one in 1.).*
3. *Let D be reductive. Under an additional assumption on the behavior of P when $t \to 0$, which is conjectured to be true in all cases (see [MdGS08, conjecture 5.5]), there is a Frobenius structure on the $\mathcal{R}_h$-semi-universal deformation space attached to the restriction $f_{|D}$. It is constant, i.e., its potential is a polynomial of degree three when expressed in flat coordinates.*

Remark: As we have seen in the beginning of this section, the normal crossing divisor is the simplest example of a linear free divisor. In this case, the Frobenius structures form the last theorem are known and relevant in *Mirror symmetry*, more precisely, consider the total cohomology space $H^*(\mathbb{P}^{n-1},\mathbb{C})$ of the $n-1$-dimensional projective space. Chose the basis $\eta_i := (c_1(\mathcal{O}_{\mathbb{P}}^{n-1}))^i$ of $H^*(\mathbb{P}^{n-1},\mathbb{C})$ with corresponding coordinates r_i and consider the projection mapping

$$\begin{array}{rcl} p : H^*(\mathbb{P}^{n-1},\mathbb{C}) & \longrightarrow & \widetilde{H} := H^*(\mathbb{P}^{n-1},\mathbb{C})/H^2(\mathbb{P}^{n-1},\mathbb{Z}) \cong H^0(\mathbb{P}^{n-1},\mathbb{C}) \times \mathbb{C}^* \times \bigoplus_{i>2} H^{2i}(\mathbb{P}^{n-1},\mathbb{C}) \\ & & = \mathrm{Spec}\ \mathbb{C}[r_0,q,q^{-1},r_{>1}] \subset \widehat{H} := \mathrm{Spec}\ \mathbb{C}[r_0,q,r_{>1}] \\ (r_0,\ldots,r_{n-1}) & \longmapsto & (r_0, q := e^{r_1}, r_2,\ldots,r_{n-1}) \end{array}$$

Then it is well known that for any point $x \in \widetilde{H}$, the germ $(\widetilde{H}, x)$ carries a Frobenius structure, defined by quantum multiplication (see, e.g., [FP97], [Aud] or [Gue08]). Mirror symmetry for the projective space now states that this Frobenius structure is exactly the one from theorem 35, 1. from above for the point $t := q$, if one chooses $\omega_1^{(2)}$ as primitive and homogenous section. The proof consists of a computation of the *small quantum cohomology* of $\mathbb{P}^{n-1}$, which, when expressed as the *Givental connection* is exactly the connection on the bundle $G_0(\log D)$ from proposition 33, 3. Notice that the mirror correspondence gives a stronger result than just an equivalence of germs of Frobenius structures in this case: As explained in [MdGS08, section 5.3], the different germs $(M_t, 0)$ glue to a Frobenius manifold with a *logarithmic pole* along $t = 0$ (in the sense of [Rei09]). The same logarithmic structure is obtained on the space $\widehat{H}$, along the divisor $\widehat{H} \backslash \widetilde{H}$, due to the particular form of the Gromov-Witten potential, more precisely, due to the divisor axiom for Gromov-Witten invariants.

3.3 Bernstein polynomials

The Bernstein polynomial is a classical invariant attached to any polynomial or holomorphic function. The following is due to Bernstein ([Ber72]) in the algebraic case, and to Björk [Bjö79] in the analytic case.

Theorem 36. *Let h be an element of $\mathbb{C}\{x_1, \ldots, x_n\}$ resp. $\mathbb{C}[x_1, \ldots, x_n]$. Then there is $P(x_i, \partial_{x_i}, s) \in \mathcal{D}[s]$ and $B(s) \in \mathbb{C}[s]$ such that*

$$P(x_i, \partial_{x_i}, s) h^{s+1} = B(s) h^s.$$

where $\mathcal{D}$ is the ring of germs of holomorphic differential operators at the origin of $\mathbb{C}^n$ resp. of algebraic differential operators on $\mathbb{C}^n$.
All polynomials $B(s) \in \mathbb{C}[s]$ having this property form an ideal in $\mathbb{C}[s]$, and we denote by $b_h(s)$ the unitary generator of this ideal. $b_h(s)$ is called the Bernstein polynomial of h.

If h defines a linear free divisor, then the theory of pre-homogenous vector spaces shows that the functional equation defining $b_h(s)$ is of a particular type.

Theorem 37 ([SK77], [Gyo91], [GS08])**.** *Let $D = h^{-1}(0)$ be a reductive linear free divisor, then the operator P appearing in Bernstein's functional equation is given by $P := h^*(\partial_{x_1}, \ldots, \partial_{x_1})$ (remember that $h^*(\underline{y}) = \overline{h(\overline{\underline{y}})}$, where x_i are the unitary coordinates and y_i are their duals). In particular, it is an element of $\mathbb{C}\langle \partial_{x_1}, \ldots, \partial_{x_n} \rangle$. Moreover, the degree of $b_h(s)$ is equal to n and the roots of $b_h(s)$ are contained in the open interval $(-2, 0)$ and are symmetric around -1. In particular, -1 is the only integer root.*

This result describes the Bernstein polynomial of reductive linear free divisors quite precisely, however, it is almost impossible to calculate these polynomials explicitly in examples. The following theorem offers a way to do these calculations, moreover, it yields a variant of a classical result of Malgrange which relates the Bernstein polynomial and the spectral numbers for isolated hypersurface singularities.

Theorem 38. *[Sev09, theorem 13, theorem 16] Let D be a reductive linear free divisor with defining equation h and choose a generic linear function f. Then*

1. *The restriction $G_0(\log D)_{|z=1}$, i.e., the top cohomology $\mathcal{H}^{n-1}(h_* \Omega^\bullet_{V/T}(\log D), d - df\wedge)$ of the twisted relative logarithmic de Rham complex is a free $\mathcal{O}_T$-module of rank n, equipped with a meromorphic connection ∇ with logarithmic pole along $0 \in T$. Write $b_\nabla(s)$ for the spectral polynomial at $t = 0$, i.e., $b_\nabla(s) = \prod_{i=1}^n (s - \alpha_i)$, where $(\alpha_1, \ldots, \alpha_n)$ are the eigenvalues of the residue endomorphism of ∇ at zero. Then we have $b_\nabla(s) = b_h(s-1)$.*

2. *Consider the logarithmic Brieskorn lattice $G_0(h) := (G_0(\log D)_{|t=0}, \nabla)$ (Notice that $G_0(h)$ does not depend on the choice of f in $V^* \backslash D^*$). Then ∇ has a* regular singularity *on $G_0(h)$ at $z = 0$. Consider the saturation $\widetilde{G}_0(h) := \sum_{k \geq 0} (\nabla_{z\partial_z})^k G_0(h)$, which has a logarithmic pole at $z = 0$. Let $b_{\widetilde{G}_0(h)}(s)$ be the minimal polynomial of the residue endomorphism of ∇_z on $\widetilde{G}_0(h)$. Then $b_{\widetilde{G}_0(h)}(ns + 1) = b_h(s)$.*

Using this result, and some of the computations from [MdGS08], we obtain the Bernstein polynomials for some of the examples of linear free divisors mentioned at the beginning of this section (see table 1). We finish by outlining an observation and a precise conjecture on both the Bernstein polynomial for a linear free divisor and the group action on the abient space.

linear free divisor	**Bernstein polynomial of h**
A_n - quiver	$(s+1)^n$
D_m - quiver	$\left(s+\frac{4}{3}\right)^{m-3}\cdot(s+1)^{2m-4}\cdot\left(s+\frac{2}{3}\right)^{m-3}$
E_6 - quiver	$(s+\frac{7}{5})\cdot(s+\frac{4}{3})^4\cdot(s+\frac{6}{5})\cdot(s+1)^{10}\cdot(s+\frac{4}{5})\cdot(s+\frac{2}{3})^4\cdot(s+\frac{3}{5})$
$\star_m$ - quiver	$\prod_{l=0}^{m-3}\left(s+\frac{2(m-1)-l}{m}\right)^{l+1}\cdot(s+1)^{2(m-1)}\cdot\prod_{l=0}^{m-3}\left(s+\frac{m-1-l}{m}\right)^{m-l-2}$
discriminant in $S^3((\mathbb{C}^2)^*)$	$\left(s+\frac{7}{6}\right)\cdot(s+1)^2\cdot\left(s+\frac{5}{6}\right)$
discriminant of $\mathrm{Sl}(3,\mathbb{C})\times\mathrm{Gl}(2,\mathbb{C})$ action on $\mathrm{Sym}(3,\mathbb{C})\times\mathrm{Sym}(3,\mathbb{C})$	$\left(s+\frac{5}{4}\right)^2\cdot\left(s+\frac{7}{6}\right)^2\cdot(s+1)^4\cdot\left(s+\frac{5}{6}\right)^2\cdot\left(s+\frac{3}{4}\right)^2$

Table 1: Bernstein polynomials for some examples of reductive linear free divisors

Conjecture 39. *Let G be reductive, (V,G) be pre-homogenous and suppose that the discriminant $D\subset V$ is a linear free divisor with defining equation h. Denote by $T\subset G$ a maximal torus. Then there is a T-invariant linear subspace $W_T\subset V$, with $W\not\subset D$ and $\dim(T)=\dim(W_T)$ (hence, of minimal dimension). It then follows that the restriction $h_{|W_T}$ is a monomial. Moreover, the multiplicity of -1 of $b_h(s)$ equals $\dim(T)$.*

Remark: The interest of the above conjecture is motivated by comparing the situation studied in [MdGS08] to the one from mirror symmetry for weighted projective spaces, hence, by generalizing the observation from the end of subsection 3.2. As discussed in detail in [DS04] and [Man08], the mirror of the orbifold quantum cohomology of the weighted projective space $\mathbb{P}(w_0,w_1,\ldots,w_n)$ is given by the restriction of a generic linear polynomial to the (non-singular) fibres of the morphism $g:\mathbb{C}^n\to\mathbb{C}$ given by $g=\prod_{i=0}^n x_i^{w_i}$. Now orbifold quantum cohomology decomposes into several pieces, the so-called twisted sectors. On the singularity side, these sectors are visible in the spectrum at infinity of the tame function (the restriction of the linear function to a Milnor fibre of g). A similar structure of the spectral numbers is observed for linear free divisors, and in particular, it would follow from the above conjecture that the multiplicity of the root -1 of b_h is exactly the dimension of the cohomology of the untwisted sector of the (various) weighted projective spaces appearing as mirrors of the monomials obtained by restricting h to torus invariant subspaces. One may also ask how the the other roots of b_h resp. the corresponding spectral numbers behave when h is restricted to such a torus invariant subspace of minimal dimension, or whether there is any deeper relation between the Frobenius structures for a linear function f on a Milnor fibre of the divisor D and the Frobenius structure of $f_{|W_T}$ restricted to the Milnor fibre of the non-reduced divisor $g^{-1}(0)\subset W_T$, i.e., the mirror of the corresponding weighted projective space.

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