Rigid and Complete Intersection Lagrangian Singularities

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Abstract

In this article we prove a rigidity theorem for lagrangian singularities by studying the local cohomology of the lagrangian de Rham complex that was introduced in [SvS03]. The result can be applied to show the rigidity of all open swallowtails of dimension ≥ 2. In the case of lagrangian complete intersection singularities the lagrangian de Rham complex turns out to be perverse. We also show that lagrangian complete intersections in dimension greater than two cannot be regular in codimension one.

1 Introduction

Since the work of Arnold and his school ([Arn82], [Arn83] and [Giv88]), singular lagrangian subvarieties in symplectic manifolds have become increasingly important in different areas of mathematics. Arnold and Givental mainly studied lagrangian projections and calculated normal forms for these objects starting from the correspondence between such projections from *smooth* lagrangian germs to the base and generating families. This does not, however, include the study of deformation spaces which allows the lagrangian singularity itself to deform. In [SvS03] we considered the deformation problem for a lagrangian singularity $(L, 0) \subset (\mathbb{C}^{2n}, 0)$ given by the deformation functor $LagDef^{loc}_{L,0}$ associating to a base space S the set of isomorphism classes of flat families $\mathcal{L} \to S$ sitting inside $\mathbb{C}^{2n} \times S$ with the property that each fibre $\mathcal{L}_s$ for $s \in S$ is lagrangian in $\mathbb{C}^{2n} \times \{s\}$. Similarly, one might define a corresponding functor $LagDef_L$ for an analytic lagrangian subspace L inside a symplectic manifold M. The main result of the quoted paper is a description of the tangent space of this functor using the so-called lagrangian de Rham complex. We recall this construction in section 2 below.

In this paper we investigate some further properties of this complex. We derive an inductive principle which can be used to prove vanishing of the cohomology of the lagrangian de Rham complex. This yields rigidity theorems for certain lagrangian singularities of dimension higher than two and is similar in spirit to the result of Schlessinger [Sch71] allowing to conclude that quotient singularities which are regular in codimension two are rigid. In [SvS03], we also developed a constructive method to calculate deformation spaces, but this was limited to lagrangian surfaces. Therefore the results here are complementary to our first paper, in that they extend the class of examples for which deformations can be studied. On the other hand, the explicit calculations from [SvS03] are used to make the induction principle work.

The essential ingredients used in this article are the special behavior of lagrangian deformations with respect to the canonical stratification of a singularity and the local cohomology of the lagrangian de Rham complex. One particular example of lagrangian singularities to which our method applies are the so-called open swallowtails. We show that they are all rigid.

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The local cohomology sheaves of the lagrangian de Rham complex also play a role in deciding whether it is perverse. We show here that lagrangian complete intersections have perverse lagrangian de Rham complex. In this case, there is (via the Riemann-Hilbert correspondence) a single $\mathcal{D}$-module associated to the lagrangian de Rham complex. This is consistent with an abstract construction of this complex described in [Sev03].
A last result contained in this paper is concerned with the codimension of the singular locus for lagrangian complete intersections. We show that if such a singularity is regular in codimension two, the tangent module is free. So the space is smooth for all cases where the Zariski-Lipman problem is solved in the affirmative, in particular in the quasi-homogeneous case and the case where the space is regular in codimension two.
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2 The lagrangian de Rham complex

We recall in this section the construction from [SvS03] of a sheaf complex associated to any Lagrangian variety. The relationship of Lie algebroids and lagrangian singularities is described in detail in [Sev03].

Definition 1. *Let $L \subset \mathbb{C}^{2n}$ be a lagrangian subvariety with defining ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathbb{C}^{2n}}$. Denote by $\mathcal{O}_L := \mathcal{O}_{\mathbb{C}^{2n}}/\mathcal{I}$ the structure sheaf of L. The module $\mathcal{I}/\mathcal{I}^2$ is the conormal module and has a structure of a Lie algebroid over $\mathcal{O}_L$, i.e., there are operations*

$$\{\,,\,\} : \mathcal{I}/\mathcal{I}^2 \times \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{I}/\mathcal{I}^2, \qquad \{\,,\,\} : \mathcal{I}/\mathcal{I}^2 \times \mathcal{O}_L \longrightarrow \mathcal{O}_L$$

Define a sheaf complex $(\mathcal{C}^\bullet_L, \delta)$, the lagrangian de Rham complex by

$$\mathcal{C}^p_L := \mathcal{H}om_{\mathcal{O}_L}\left(\bigwedge^p \mathcal{I}/\mathcal{I}^2, \mathcal{O}_L\right)$$

and $\delta : \mathcal{C}^p_L \to \mathcal{C}^{p+1}_L$ with

$$\begin{array}{l}(\delta(\phi))(h_1 \wedge \ldots \wedge h_{p+1}) := \\ \sum_{i=1}^{p+1} (-1)^i \left\{h_i, \phi\left(h_1 \wedge \ldots \wedge \widehat{h}_i \wedge \ldots h_{p+1}\right)\right\} \\ + \sum\limits_{1 \le i < j \le p+1} (-1)^{i+j-1} \phi\left(\{h_i, h_j\} \wedge h_1 \wedge \ldots \wedge \widehat{h}_i \wedge \ldots \wedge \widehat{h}_j \wedge \ldots \wedge h_{p+1}\right)\end{array}$$

We quote the main results from [SvS03] and [Sev03] concerning the lagrangian de Rham complex. The first one relates $\mathcal{C}^\bullet_L$ to the deformation theory of L.

Theorem 2. *Consider the first three cohomology sheaves of $\mathcal{C}^\bullet_L$. Then*

1. $\mathcal{H}^0(\mathcal{C}^\bullet_L) = \mathbb{C}_L$
2. *$\mathcal{H}^1(\mathcal{C}^\bullet_L)$ is the sheaf of first order flat lagrangian deformations. This means that at every point $p \in L$, the tangent space of the functor $LagDef^{loc}_{L,p}$ is $H^1(\mathcal{C}^\bullet_{L,p})$.*
3. *Let $(L, 0)$ be either a complete intersection or Cohen-Macaulay of codimension two. Suppose moreover that $\mathcal{H}^2(\mathcal{C}^\bullet_L) = 0$. Then the functor $LagDef^{loc}_{L,0}$ is unobstructed.*

By the theory of Schlessinger, it is of obvious importance to know whether the cohomology of the lagrangian de Rham complex is finite. This is answered by the following result.

Theorem 3. *Consider the canonical stratification of L by embedding dimension, i.e., let $S_k^L := \{p \in L \,|\, edim_p(L) = 2n - k\}$, where $k \in \{0, \ldots, n\}$. Suppose that "Condition P" holds, that is, $\dim(S_k^L) \leq k$ for all k. Then the cohomology sheaves $\mathcal{H}^p(\mathcal{C}_L^\bullet)$ are constructible with respect to the canonical stratification. In particular, for a germ $(L, 0)$, $H^1(\mathcal{C}_{L,0}^\bullet)$ is a finite dimensional vector space. Therefore, there is a formally semi-universal deformation with respect to $LagDef_{L,0}^{loc}$.*

3 The rigidity theorem

In this section, we state and prove our main theorem. The technical tool used is the local cohomology of a sheaf, that is, the derived functor of the functor $\Gamma_T(X, -)$ of sections of a sheaf $\mathcal{F}$ over a space X with support in a closed subspace T. Let us start with some preliminary lemmas. In what follows we consider a lagrangian subvariety $X \subset \mathbb{C}^{2n}$ which is not necessarily Stein or contractible. $T \subset X$, $T \neq X$ is always a closed analytic subspace.

Lemma 4. *Denote by $\delta\mathcal{O}_X \subset \mathcal{N}_X$ the image (sheaf) of the differential*

$$\delta : \mathcal{C}_X^0 = \mathcal{O}_X \longrightarrow \mathcal{C}_X^1 = \mathcal{N}_X$$

Then we have

$$H_T^0(\mathcal{H}^1(\mathcal{C}_X^\bullet)) = Ker\left(H_T^1(\delta\mathcal{O}_X) \to H_T^1(\mathcal{N}_X)\right)$$

Proof. Consider the first three terms of the sheaf complex $\mathcal{C}_X^\bullet$ associated to the lagrangian subvariety $X \subset \mathbb{C}^{2n}$. It reads

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{N}_X \longrightarrow \mathcal{C}_X^2$$

We know that $\mathcal{H}^0(\mathcal{C}_X^\bullet) = \mathcal{K}er(\mathcal{O}_X \to \mathcal{N}_X) = \mathbb{C}_X$. By splitting into short exact sequences, we obtain

$$0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{O}_X \longrightarrow \delta\mathcal{O}_X \longrightarrow 0$$

$$0 \longrightarrow \delta\mathcal{O}_X \longrightarrow \mathcal{K} \longrightarrow \mathcal{H}^1(\mathcal{C}_X^\bullet) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{N}_X \longrightarrow \delta\mathcal{N}_X \longrightarrow 0$$

Here $\mathcal{K} = \mathcal{K}er(\mathcal{N}_X \to \mathcal{C}_X^2)$ and $\delta\mathcal{N}_X = \mathcal{I}m(\mathcal{N}_X \to \mathcal{C}_X^2)$. Now we can apply the functor $H_T^\bullet(-)$ to each of these sequences. This gives three long exact sequences of local cohomology sheaves. However, we know in advance that sheaves of type $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$ are torsion free, so in particular $H_T^0(\mathcal{C}_X^i) = 0$ for all i. Moreover, $\mathbb{C}_X$, $\delta\mathcal{O}_X$, $\mathcal{K}$ and $\delta\mathcal{N}_X$ are subsheaves of $\mathcal{O}_X$, $\mathcal{N}_X$ resp. $\mathcal{C}_X^2$, so for them the group $H_T^0(-)$ also vanishes. We obtain exact sequences

$$0 \longrightarrow H_T^1(\mathbb{C}_X) \longrightarrow H_T^1(\mathcal{O}_X) \longrightarrow H_T^1(\delta\mathcal{O}_X) \longrightarrow H_T^2(\mathbb{C}_X)$$

$$0 \longrightarrow H_T^0(\mathcal{H}^1(\mathcal{C}_X^\bullet)) \longrightarrow H_T^1(\delta\mathcal{O}_X) \longrightarrow H_T^1(\mathcal{K})$$

$$0 \longrightarrow H_T^1(\mathcal{K}) \longrightarrow H_T^1(\mathcal{N}_X) \longrightarrow H_T^1(\delta\mathcal{N}_X)$$

Combining the last two sequences yields the desired formula. The first sequence will be used later. □

We need to investigate further the local cohomology of the sheaf $\mathcal{H}^1(\mathcal{C}_X^\bullet)$.

Lemma 5. *There is an exact sequence*

$$0 \longrightarrow H^0(X, \delta\mathcal{O}_X) \longrightarrow H^0(X \backslash T, \delta\mathcal{O}_X) \longrightarrow H_T^1(\delta\mathcal{O}_X)$$

If X is Stein and contractible (e.g., a representative of a germ $(X, 0)$), then the last arrow in the above sequence is surjective.

Proof. Consider the following basic sequence in local cohomology (see [Gro67]: Let $\mathcal{F}$ be a sheaf on a topological space Y and T any closed subspace, then:

$$0 \to H^0_T(\mathcal{F}) \to H^0(Y, \mathcal{F}) \to H^0(Y \backslash T, \mathcal{F}) \to H^1_T(\mathcal{F}) \to H^1(Y, \mathcal{F}) \to \dots \qquad (1)$$

For $Y = X \subset \mathbb{C}^{2n}$ and $\mathcal{F} = \delta\mathcal{O}_X$, we know that $H^0_T(\delta\mathcal{O}_X) = 0$. This gives the sequence in the general case. Moreover, we can apply the usual cohomology functor to the sequence

$$0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{O}_X \longrightarrow \delta\mathcal{O}_X \longrightarrow 0$$

yielding

$$\dots \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \delta\mathcal{O}_X) \longrightarrow H^2(X, \mathbb{C}_X) \longrightarrow \dots$$

In case that X is contractible ($H^2(X, \mathbb{C}_X) = 0$) and Stein ($H^1(X, \mathcal{O}_X) = 0$) the term $H^1(X, \delta\mathcal{O}_X)$ vanishes. □

These last two results tell us how to understand sections of the cohomology sheaf $\mathcal{H}^1(\mathcal{C}^\bullet_X)$ with support in a subspace T, that is, deformations which do not deform the space $X \backslash T$: these are elements of $H^1_T(\delta\mathcal{O}_X)$, thus, sections of $\delta\mathcal{O}_X$ over $X \backslash T$ which do not extend over T. If we consider the case $T = Sing(X)$, this means that a deformation is trivial iff the hamiltonian vector field which trivializes it on the regular part (because $\mathcal{H}^1(\mathcal{C}^\bullet_X)$ is zero on X_{reg}) extends over the whole of X.

Theorem 6. *Let $L \subset \mathbb{C}^{2n}$ be a representative of a lagrangian singularity $(L, 0) \subset (\mathbb{C}^{2n}, 0)$ satisfying Condition P. Denote by $S \subset L$ the singular locus. Let $T \subset S$ be a closed analytic subspace in L contained in the singular locus. Suppose that*

1. $H^1_T(\delta\mathcal{O}_L) = 0$
2. $H^0(L^*, \mathcal{H}^1(\mathcal{C}^\bullet_{L^*})) = 0$*, where* $L^* := L \backslash T$*.*

Then $H^1(\mathcal{C}_{L,0}) = 0$, i.e., L is rigid under lagrangian deformations.

Proof. Denote by S^* the singular locus of L^*, obviously, $S^* := S \backslash T$. Note that $L_{reg} = L^* \backslash S^*$ because of $T \subset S$. From lemma 5, applied to the spaces L and L^*, we obtain the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(L, \delta\mathcal{O}_L) & \longrightarrow & H^0(L_{reg}, \delta\mathcal{O}_L) & \longrightarrow & H^1_S(\delta\mathcal{O}_L) & \longrightarrow & 0 \\
 & & \big\downarrow \alpha & & \big\| & & \big\downarrow \beta & & \\
0 & \longrightarrow & H^0(L^*, \delta\mathcal{O}_L) & \longrightarrow & H^0(L_{reg}, \delta\mathcal{O}_L) & \longrightarrow & H^1_{S^*}(\delta\mathcal{O}_{L^*}) & &
\end{array}$$

Here α is the restriction map and β is the induced map. Moreover, a class $c \in H^0(\mathcal{H}^1(\mathcal{C}^\bullet_L)) = H^0_S(\mathcal{H}^1(\mathcal{C}^\bullet_L))$ corresponding to a flat lagrangian deformation of $L \subset \mathbb{C}^{2n}$ is represented by lemma 4 by a class (denoted by the same letter) $c \in H^1_S(\delta\mathcal{O}_L)$ which goes to zero in $H^1_S(\mathcal{N}_L)$. The same diagram, with the sheaf $\delta\mathcal{O}_L$ replaced by $\mathcal{N}_L$ shows that $\beta(c)$ goes to zero in $H^1_{S^*}(\mathcal{N}_L)$. By lemma 4 we also know that

$$H^0(\mathcal{H}^1(\mathcal{C}^\bullet_{L^*})) = H^0_{S^*}(\mathcal{H}^1(\mathcal{C}^\bullet_{L^*})) = Ker\left(H^1_{S^*}(\delta\mathcal{O}_L) \to H^1_{S^*}(\mathcal{N}_L)\right)$$

which vanishes by the second hypothesis. So we get that $\beta(c) = 0$, this means that there is a section $\widetilde{c}$ extending c over L^*.
We can apply lemma 5 again, this time to the pair (L, T), yielding the sequence

$$0 \longrightarrow H^0(L, \delta\mathcal{O}_L) \longrightarrow H^0(L^*, \delta\mathcal{O}_L) \longrightarrow H^1_T(\delta\mathcal{O}_L)$$

From the first hypothesis, we obtain that $\widetilde{c}$ extends to the whole of L, which implies immediately that the original class c in $H^1_S(\delta\mathcal{O}_L)$ is zero. Therefore, L is infinitesimal rigid. □

Using lemma 4, the first condition implies in particular that $H^0_T(\mathcal{H}^1(\mathcal{C}^\bullet_L)) = 0$, that is, there are no deformations deforming only T. This is of course weaker than the vanishing of $H^1_T(\delta\mathcal{O}_L)$ but still sufficient: By the same argument as above, we see that the class $\widetilde{c} \in H^1_T(\delta\mathcal{O}_L)$ maps to zero in $H^1_T(\mathcal{N}_L)$ thus defining an element in $H^0_T(\mathcal{H}^1(\mathcal{C}^\bullet_L))$. But in applications, we will rather prove that $H^1_T(\delta\mathcal{O}_L) = 0$, therefore, it is more natural to impose this condition than the vanishing of $H^0_T(\mathcal{H}^1(\mathcal{C}^\bullet_L))$.
In order to make use of these result, we have to find conditions that give $H^1_T(\delta\mathcal{O}_L) = 0$ and $H^0(\mathcal{H}^1(L^*, \mathcal{C}^\bullet_{L*})) = 0$. We start with the first group. It sits in the exact sequence

$$\ldots \longrightarrow H^1_T(\mathcal{O}_L) \longrightarrow H^1_T(\delta\mathcal{O}_L) \longrightarrow H^2_T(\mathbb{C}_L) \longrightarrow \ldots$$

so a sufficient condition is the vanishing of the groups $H^1_T(\mathcal{O}_L)$ and $H^2_T(\mathbb{C}_L)$. Obviously, $H^1_T(\mathcal{O}_L)$ is of analytic and $H^2_T(\mathbb{C}_L)$ of topological nature.

Lemma 7. *Let* $\dim(L) \geq 2$ *and* T *be a closed subspace such that* $\operatorname{depth}(\mathcal{O}_{L,0}) \geq 2 + \dim(T)$*. Then* $H^1_T(\mathcal{O}_L) = 0$.

Proof. The well-known relation between local cohomology and *Ext* leads to the statement that $H^p_T(\mathcal{F}) = 0$ is equivalent to $Ext^p_{\mathcal{O}_L}(\mathcal{G}, \mathcal{F}) = 0$ for any sheaf G with $supp(\mathcal{G}) \subset T$, see [Gro67], proposition 3.7. By the lemma of Ischebeck ([Mat89]), $Ext^p_{\mathcal{O}_L}(\mathcal{G}, \mathcal{F}) = 0$ for all $p < depth(\mathcal{F}) - \dim(supp(\mathcal{G}))$. So for $\mathcal{F} = \mathcal{O}_L$ and $supp(G) \subset T$ we obtain that $H^1_T(\mathcal{O}_L) = 0$. □

The next step is to investigate the topological group $H^2_T(\mathbb{C}_L)$. First it follows from the sequence 1 that in case that L is contractible (e.g, for a representative of a germ $(L, 0)$), we have $H^2_T(\mathbb{C}_L) = H^1(L\backslash T, \mathbb{C}_L)$. The following lemma lists some cases where the first homology of $L\backslash T$ is zero.

Lemma 8. *We consider a general situation of a germ* $(X, 0)$ *of a complex space.*

1. *Consider the normalization*

$$n : \widetilde{X} \longrightarrow X$$

   and suppose that $\widetilde{X}$ *is smooth. Let* T *a subspace of codimension at least two such that* n *induces an homeomorphism from* $\widetilde{X}\backslash\widetilde{T}$ *to* $X\backslash T$*, where* $\widetilde{T} := n^{-1}(T)$*. Then* $H^1(X\backslash T, \mathbb{C})$ *vanishes.*

2. *Let* $(X, 0)$ *be a rational normal surface singularity and* $T = Sing(X) = \{0\}$*. Then we also have* $H^1(X\backslash T, \mathbb{C}) = 0$.

3. *Suppose that* X *is a complete intersection and* T *a closed subspace of codimension at least three which contains* $Sing(X)$*, then* $H^1(X\backslash T, \mathbb{C}) = 0$.

Proof.

1. This is obvious since $\widetilde{X}\backslash\widetilde{T}$ is simply connected and homeomorphic to $X\backslash T$.

2. It is known that the link M of $(X, 0)$ is a deformation retract of $X\backslash T$. On the other hand, for rational singularities the group $H^1(M, \mathbb{Z})$ is torsion (see, e.g., [Bri68]), so that $H^1(X\backslash T, \mathbb{C})$ is zero.

3. This can be found in [Gre75] or [Loo84]. We sketch the argument: First it follows from a result on the depth of the modules of differential forms on X that

$$H^1(\Omega^\bullet_{X,0}, d) \cong H^1(\Gamma(X\backslash T, \Omega^\bullet_{X\backslash T}), d)$$

   The same reasoning shows (using also the two spectral sequences for the hypercohomology of a sheaf complex) that $H^1(\Gamma(X\backslash T, \Omega^\bullet_{X\backslash T}), d) \cong H^1(X\backslash T, \mathbb{C})$. By an analytic argument, one can show that the de Rham complex of X is exact in degree one. This yields immediately that $H^1(X\backslash T, \mathbb{C}) = 0$.

□

Combining the last two lemmas, we get conditions for $H^1_T(\delta\mathcal{O}_L)$ to be zero. Whenever this is the case, a lagrangian deformation of the germ $(L, 0)$ comes (if it exists) from a deformation of a transversal slice at a point $p \in L\backslash T$. If we know that there are no such deformations, we can conclude that L is rigid. This enables us for example to show that any lagrangian rational triple point in $\mathbb{C}^4$ is rigid. As a further consequence, we obtain from the third part of the last lemma that lagrangian complete intersection singularities L with $\mathrm{codim}(Sing(L)) > 2$ are rigid. However, as we will see in the last section, such objects simply do not exist.

4 Applications

We will use the theorem from the last section to prove rigidity under lagrangian deformations of a number of examples including the so-called open swallowtails. Givental introduces these varieties in [Giv95] as subvarieties of certain jet spaces in order to obtain normal form results for systems of partial differential equations. All examples studied in that paper are obtained using *generating functions* of special type. Recall that for any function germ F defined on a product of two smooth spaces $B \times X$ such that the restriction f of F to $\{0\} \times X$ defines a function germ with isolated critical points, one can define (choosing coordinates $(x_1, \ldots, x_k)$ on X and $(q_1, \ldots, q_n)$ on B)

$$Lag(F) := \{(p, q) \in T^*B \,|\, \exists x \in X \ (\partial_{x_i} F)(x, q) = 0 \ ; \ \ p_i = \partial_{q_i} F \ ; \ \ \forall i\} \subset T^*B$$

It is well known that $Lag(F)$ is a lagrangian subvariety in T^*B. Moreover, the generating function also gives rise to a legendrian variety in $\mathbb{C}^{2n+1}$ (with coordinates $(u, \mathbf{p}, \mathbf{q})$ and the standard contact structure $u - \mathbf{p}\, d\mathbf{q}$), simply by setting $u = F(x, q)$. The front of the lagrangian resp. legendrian variety is the image of the projection to the $(u, \mathbf{q})$-space.
On the other hand, the space of polynomials

$$\mathcal{P}_{2n+1} = \left\{ t^{2n+1} + \frac{a_1}{(2n-1)!} t^{2n-1} + \frac{a_2}{(2n-2)!} t^{2n-2} + \ldots + a_{2n} \right\}$$

carries a natural symplectic structure related to the representation theory of sl_2. The subvariety consisting of all polynomials having a root of multiplicity at least $n + 1$ is lagrangian and appears as generic singularity of the so-called "obstacle problem" ([Giv88]). It is called n-dimensional open swallowtail and was denoted Σ_n in [SvS03]. We will see that it can be described using generating functions. More precisely, let $g_n(x, q) := x^{n+1} + q_1 x^{n-1} + \ldots + q_n$ and set $F_{n,k}(x, q) := \int_0^x g_n(s, q)^{k+1} ds$. Denote by $\Sigma_{n,k}$ the lagrangian subspace $Lag(F_{n,k}) \subset \mathbb{C}^{2n}$ and by $\Lambda_{n,k}$ its front. The following lemma, extracted from [Giv88] and [Giv95], describes the geometry of the singularities $\Sigma_{n,k}$ (and of its front $\Lambda_{n,k}$). Some of these facts are needed later to apply our rigidity theorem.

Lemma 9. *1. Denote by $\mathcal{P}_{m,n}$ the space of polynomials of degree $(k + 1)(n + 1) + 1$ with fixed highest coefficient, sum of roots equal to zero and $n + 1$ critical points of multiplicity $k + 1$, i.e., all polynomials of the form*

$$p_{q_1,\ldots,q_n,u}(s) = \int_0^x g_n(s, q_1, \ldots, q_n)^{k+1} ds - u$$

The front $\Lambda_{n,k}$ of the lagrangian singularity $\Sigma_{n,k}$ is isomorphic to the hypersurface of polynomials in $\mathcal{P}_{n,k}$ with multiple roots (such a root has automatically multiplicity at least $k+2$).

2. A smooth normalization of $\Sigma_{n,k}$ is given by the map

$$\begin{array}{rcl} n \,:\, (\mathbb{C}^n, 0) & \longrightarrow & (\Sigma_{n,k}, 0) \\ (x, q_1, \ldots, q_{n-1}) & \longmapsto & (q_1, \ldots, q_n, p_1, \ldots, p_n) \end{array}$$

here $q_n = x^{n+1} + \sum_{i=1}^{n-1} q_i x^{n-i}$, $p_i := \partial_{q_i} F_{n,k}$.

3. *The variety* $\Sigma_{n,1}$ *is isomorphic to the* n*-dimensional open swallowtail* Σ_n.

4. $(\Sigma_{n,k}, 0)$ *is Cohen-Macaulay.*

Proof. 1. This is almost a tautology: The front $\Lambda_{n,k}$ is the graph of the generating function $F_{n,k}$, seen as a multi-valued function (with $n+1$-sheets) on the base B. For any point $\mathbf{q} = (q_1, \dots, q_n) \in B$, let $\lambda_1, \dots, \lambda_{n+1}$ be the zeros of g_n^{k+1}. Then the $n+1$ points of $\Lambda_{n,m}$ lying over $\mathbf{q}$ correspond to the elements $p_{(\mathbf{q},u)} \in \mathcal{P}_{n,k}$ with $u = F(\lambda_i, \mathbf{q})$. Obviously, λ_i is a zero of $p_{(\mathbf{q},u)}$ and of its derivative, so belongs to the discriminant in $\mathcal{P}_{n,k}$.

2. The map n is generically one to one and therefore the normalization.

3. We will see that $\mathcal{O}_{\Sigma_n,0}$ and $\mathcal{O}_{\Sigma_{n,1},0}$ can be identified as subalgebras of their respective (smooth) normalization. Following [Giv88], the normalization of Σ_n is given by the following map

$$\begin{array}{rcl} \varphi : \widetilde{\Sigma}_n \cong \mathbb{C}^n & \longrightarrow & \Sigma_n \subset \mathcal{P}_{2n+1} \\ (x, a_1, \dots, a_{n-1}) & \longmapsto & (t-x)^{n+1} \cdot (t^n + b_1 t^{n-1} + \dots + b_{n-1}) \end{array}$$

where $b_i \in \mathcal{O}_{\widetilde{\Sigma}_n,0}$ are chosen such that the coefficient of t^{2n+1-i} in the polynomial $\varphi(x, \mathbf{a})$ is precisely $a_i/(2n+1-i)!$ for $i = 1, \dots, n-1$ (in particular, $b_1 = (n+1)t$). Then we get

$$\mathcal{O}_{\Sigma_n,0} = \left\{ f \in \mathbb{C}\{t, a_1, \dots, a_{n-1}\} \mid f = \int_0^x Q(s, \mathbf{a}) F_n(s, \mathbf{a}) ds + C(\mathbf{a}) \right\}$$

On the other hand, it is shown in [Giv95] that

$$\mathcal{O}_{\Sigma_{n,k},0} = \left\{ f \in \mathbb{C}\{x, q_1, \dots, q_{n-1}\} \mid f = \int_0^x \Phi(s, \mathbf{q}) g_n(s, q)^k ds + Q(\mathbf{q}) \right\}$$

So $\mathcal{O}_{\Sigma_{n,1},0} \cong \mathcal{O}_{\Sigma_n,0}$.

4. One has to show that the finite analytic mapping $(\Sigma_{n,k}, 0) \to (B, 0)$ makes $\Sigma_{n,k}$ into a **free** $\mathcal{O}_{B,0}$-module of rank $n+1$. This is done in [Giv95] (for $k = 1$, this map is simply n-fold differentiation). Then the statement follows.

□

From the first point of the lemma, we deduce

Lemma 10. *Let*

$$\{0\} \subset \Sigma_{n,k}^{(1)} \subset \dots \subset \Sigma_{n,k}^{(n-1)} \subset \Sigma_{n,k}^{(n)} = \Sigma_{n,k}$$

be the canonical stratification with $\dim(\Sigma_{n,k}^{(k)}) = k$ *(Condition P). Let* $p \in \Sigma_{n,k}^{(i)} \setminus \Sigma_{n,k}^{(i-1)}$, *then we have* $(\Sigma_{n,k}, p) \cong (\Sigma_{n-i,k}, 0) \times (\mathbb{C}^i, 0)$.

Proof. That $\Sigma_{k,n}$ locally decomposes into a product of a lagrangian variety and a smooth germ is a general fact (this is the essential ingredient in the proof of theorem 3, see [SvS03] and [Sev03]). We only need to show that the transversal section is precisely $(\Sigma_{n-i,k}, 0)$. First it is obviously sufficient to do the case $i = n-1$. For this, we will show that the transversal singularity of the front $\Lambda_{n,k}$ is $\Lambda_{n-1,k}$. This follows directly from the description of the front given as discriminant in the polynomial space $\mathcal{P}_{n,k}$. A general polynomial P in this space can be written in the form

$$\int_0^x (s-\lambda)^{k+1} (s-\mu)^{k+1} (s^{n-1} + (\lambda+\mu) s^{n-2} + q_1' s^{n-3} \dots + q_{n-2}')^{k+1} ds$$

with the additional condition that there is a common zero of P and its derivative. If $\lambda = \mu$, then the polynomial P represents a point $\widetilde{p} \in \Lambda_{n,k}$ corresponding to the point $p \in \Sigma_{n,k}$ from above. A transversal section at $\widetilde{p}$ is given (in appropriate local coordinates) by setting $\lambda =$ const and by translating the argument. Therefore, in a neighborhood of $\widetilde{p}$ a point of such a transversal section is represented as $\int_0^x (s-\mu)^{k+1}(s^{n-1} + \mu s^{n-2} + \widetilde{q}_1 s^{n-3} \ldots + \widetilde{q}_{n-2})^{k+1} ds$, that is, corresponds to a point in $\Lambda_{n-1,k} \subset \mathcal{P}_{n-1,k}$. □

In [SvS03], an algorithm to calculate $H^1(\mathcal{C}^\bullet_{L,0})$ for quasi-homogenous lagrangian surface singularities was described. For the spaces $\Sigma_{2,k}$ one obtains by computer calculation

Lemma 11. $H^1(\mathcal{C}^\bullet_{\Sigma_{2,k},0}) = 0$ *for* $k = 2, 3, 4, 5$. *In these cases, as in the examples studied in [SvS03] the spectral numbers of the local system* $\mathcal{H}^1(\mathcal{C}^\bullet_L)_{|Sing(L)}$ *(for* $L = \Sigma_{2,k}$*) have a symmetry property.*

For higher k the computation is possible in the same way and limited only by computer power. Conjecturally, all $\Sigma_{2,k}$ are rigid. For all k such that $\Sigma_{2,k}$ is rigid, we can use theorem 6 to obtain.

Theorem 12. *Suppose that for fixed k, the lagrangian singularity* $(\Sigma_{2,k}, 0) \subset (\mathbb{C}^4, 0)$ *is rigid. Then for all* $n > 2$ $(\Sigma_{n,k}, 0) \subset (\mathbb{C}^{2n}, 0)$ *is rigid.*

Corollary 13. *All open swallowtails of dimension greater one are rigid lagrangian singularities.*

Proof of the theorem. We do induction on n. For $n = 2$, we are done by hypothesis. Otherwise, we know that for $p \in \Sigma^{(1)}_{n,k}\backslash\{0\}$, there is a decomposition $(\Sigma_{n,k}, p) \cong (\Sigma_{n-1,k}, 0) \times (\mathbb{C}, 0)$ and moreover, $H^1(\mathcal{C}^\bullet_{\Sigma_{n,k},p}) \cong H^1(\mathcal{C}^\bullet_{\Sigma_{n-1,k},0})$. This last group is zero by the induction hypothesis. This implies that for $T = \{0\} \subset \Sigma_{n,k}$, we have $H^0(\Sigma_{n,k}\backslash T, \mathcal{H}^1(\mathcal{C}^\bullet_{\Sigma_{n,k}\backslash T})) = 0$. The second point we need to check in order to apply theorem 6 is the vanishing of $H^1_T(\delta\mathcal{O}_{\Sigma_{n,k}})$. We use lemma 7: We need that T is of codimension at least two and that $\mathrm{depth}(\mathcal{O}_L) > \dim(T) + 2$ which is obviously satisfied in view of the last point of lemma 9. Moreover, the second statement of this lemma gives smoothness of the normalization of $(\Sigma_{n,k}, 0)$, so that the second (topological) condition of lemma 7 is also satisfied. Therefore, $H^1_T(\delta\mathcal{O}_{\Sigma_{n,k}}) = 0$. Now we can apply theorem 6, which proves rigidity of $\Sigma_{n,k}$. □

Remark: As explained in detail in [Sev03], the base space of a semi-universal unfolding of a hypersurface singularity with non-degenerate intersection form carries a natural symplectic structure provided that its Milnor number is even. This applies in particular to irreducible plane curve singularities. There is a distinguished subspace called δ-constant strata, corresponding to deformations of the curve coming from deformations of its normalization (more precisely, the normalization of the δ-constant stratum is the semi-universal deformation of the normalization of the curve, see [Tei80], [DH88] or [FGvS99]). It follows from [VG82] that the δ-constant stratum is a lagrangian singularity. For the A_n-series, with n even, we get precisely the open swallowtails $\Sigma_{n,1}$. This leads us to the following conjecture.

Conjecture 14. *Let* $(C, 0) \subset (\mathbb{C}^2, 0)$ *be an irreducible plane curve singularity, defined by a holomorphic function* $f \in \mathcal{O}_{\mathbb{C}^2,0}$. *Let* $F \in \mathcal{O}_{\mathbb{C}^2\times\mathbb{C}^\mu,0}$ *be its semi-universal unfolding. Then the germ of the* δ*-constant stratum* $(B^\delta, 0) \subset (\mathbb{C}^\mu, 0)$ *is a rigid lagrangian singularity.*

Let us remark that the only missing piece in the proof of this conjecture is the last point of lemma 9: It is not known in general whether B^δ is Cohen-Macaulay.

5 Lagrangian complete intersections

The perversity condition mentioned in the introduction involves the study of the local cohomology of the lagrangian de Rham complex. For that reason, it is quite natural to include it here. It turns out that a positive answer to this problem is possible in the case of lagrangian complete intersections. Let us first recall what it means for a complex to be perverse. Consider a, say, complex space X of dimension n and a sheaf complex $\mathcal{K}^\bullet$ on X (we suppose for simplicity that it is concentrated in non-negative degrees). Then there are two condition, called first and second perversity conditions. The first one states that

$$\dim supp(\mathcal{H}^i(\mathcal{K}^\bullet)) \leq n - i$$

for all $i \leq 0$. The second one (also called co-support condition) involves the derived functor $\mathbb{R}\Gamma_T$ (seen as functor in the derived category), where T is a closed analytic subspace in X. It states that

$$\dim supp(\mathbb{R}^q\Gamma_T(\mathcal{K}^\bullet)) < \dim(T)$$

for any such T and for all $i \in \{0, \ldots, n - \dim(T) - 1\}$. We also recall the spectral sequence with E_2-term $\mathcal{H}^p(\mathcal{H}^q_T(\mathcal{K}^\bullet))$ which converges to $\mathbb{R}^{p+q}\Gamma_T(\mathcal{K}^\bullet)$. Now consider the case $\mathcal{K}^\bullet = \mathcal{C}^\bullet$.

Theorem 15. *Let $L \subset \mathbb{C}^{2n}$ be a representative of a lagrangian complete intersection singularity. Then the complex $\mathcal{C}^\bullet_L$ is perverse.*

Proof. The first condition is easily verified using the decomposition of a lagrangian variety around a point of non-maximal embedding dimension (this has already been done in [SvS03]).
Consider the above spectral sequence. L is a complete intersection, therefore, the conormal module and hence the modules $\mathcal{C}^p_L$ are locally free. In particular, $\text{depth}(\mathcal{C}^p_L) = n$. By the lemma of Ischebeck (see the proof of lemma 7), we have that $\mathcal{H}^q_T(\mathcal{C}^p_L) = 0$ for all $q < n - \dim(T)$. This implies the vanishing of the corresponding local hypercohomology $\mathbb{R}^q\Gamma_T(\mathcal{C}^\bullet_L)$, as required. □

Corollary 16. *Let L be lagrangian with* $\dim(L) \leq 3$ *and* $\text{depth}(L) \geq 2$. *Then $\mathcal{C}^\bullet_L$ is perverse.*

Proof. The proof of the last theorem shows that whenever we have a vanishing of $\mathcal{H}^q(\mathcal{C}^p_L)$, we get vanishing of the hypercohomology. But there is a general statement (see, e.g., [Sch71]) that for a space X of depth at least two, sheaves of type $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ are of depth at least two. Up to dimension three, this vanishing result is sufficient for the co-support condition to be satisfied. □

The natural question whether there exist examples of lagrangian singularities with non-perverse lagrangian de Rham complex is still open. If one looks at the open swallowtail $\Sigma_4 \subset \mathcal{P}_9 \cong \mathbb{C}^8$, it would be sufficient to have $\text{depth}(\mathcal{N}_{\Sigma_2}) = 2$ in order to get a counterexample, but we were not able to compute this depth.
We remark that the co-support condition simplifies due to the decomposition principle as follows: Let $\{0\} = L^{(0)} \subset L^{(1)} \subset \ldots \subset L^{(n)} = L$ be the canonical stratification. Then it is sufficient to show the co-support condition only for subspaces $T = L^{(i)}$. Moreover, if we can show that for all i and all $q \in \{0, \ldots, n - i - 1\}$,

$$supp(\mathbb{R}^q\Gamma_{L^{(i)}}(\mathcal{C}^\bullet_L)) \subset L^{(i-1)}$$

(where $L^{(-1)} := \emptyset$), then we are done by "Condition P". This amounts to show that for $p \in L^{(i)} \backslash L^{(i-1)}$, the stalk $\mathbb{R}^q\Gamma_{L^{(i)}}(\mathcal{C}^\bullet_L)_p$ is zero. But we know that $(L, p) \cong (L', p') \times (\mathbb{C}^i, 0)$ with $p' \in L^{(0)}$ and that $\mathcal{C}^\bullet_{L,p}$ is quasi-isomorphic to $\pi^{-1}\mathcal{C}^\bullet_{L',p'}$ (with $\pi : (L, p) \to (L', p')$ the projection). Therefore

$$\mathbb{R}^q\Gamma_{L^{(i)}}(\mathcal{C}^\bullet_L)_p = \mathbb{R}^q\Gamma_{L^{(i)}}(\pi^{-1}\mathcal{C}^\bullet_{L'})_p = \mathbb{R}^q\Gamma_{L'^{(0)}}(\mathcal{C}^\bullet_{L'})_{p'}$$

So if we know for a class of lagrangian singularities that the transversal slices also belongs to this class (as, e.g., for complete intersections), it suffices to show that $\mathbb{R}^q\Gamma_{\{0\}}(\mathcal{C}^\bullet_L) = 0$ for all $0 \leq q < n$ for all L in this class.
We add here a statement giving a partial answer to a question on the singular locus of lagrangian complete intersections.

Theorem 17. *Let* $(L,0) \subset (\mathbb{C}^{2n},0)$ *be a lagrangian complete intersection singularity such that* $\operatorname{codim}(\mathit{Sing}(L)) \geq 2$*. Then the tangent module* $\Theta_{L,0}$ *is free.*

Proof of the theorem. Let $I \subset \mathcal{O}_{\mathbb{C}^{2n},0}$ be the defining ideal of $(L,0)$. From [SvS03], we have the following diagram

$$\begin{array}{ccccccccc} & & I/I^2 & \longrightarrow & \Omega^1_{\mathbb{C}^{2n},0}\otimes\mathcal{O}_{L,0} & \longrightarrow & \Omega^1_{L,0} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\widetilde{\alpha}} & & \\ 0 & \longrightarrow & \Theta_{L,0} & \longrightarrow & \Theta_{\mathbb{C}^{2n},0}\otimes\mathcal{O}_{L,0} & \longrightarrow & N_{L,0} & \longrightarrow & T^1_{L,0} \longrightarrow 0 \end{array}$$

where α and $\widetilde{\alpha}$ are isomorphisms on L_{reg}. By the snake lemma, $\mathit{Coker}(\alpha) \cong \mathit{Ker}(\widetilde{\alpha})$. We know that $N_{L,0}$ is torsion free and that the kernel of $\widetilde{\alpha}$ is concentrated on the non-smooth locus, hence, $\mathit{Coker}(\alpha) \cong \mathit{Tors}(\Omega^1_{L,0})$. Now if L is a complete intersection then it follows from [Gre75] that $\Omega^p_{L,0}$ is torsion free for all $p < \operatorname{codim}(\mathit{Sing}(L))$, in particular, $\Omega^1_{L,0}$ is torsion free under the hypotheses of the theorem. This shows that α is surjective. For a complete intersection, I/I^2 is free and $I/I^2 \to \Omega^1_{\mathbb{C}^{2n},0}\otimes\mathcal{O}_{L,0}$ is injective. Therefore, $\Theta_{L,0} \cong I/I^2$ is free. □

From the freeness of $\Theta_{L,0}$ one would like to conclude that $(L,0)$ is in fact smooth. This is the celebrated *Zariski-Lipman*-conjecture.

Let R be an analytic $\mathbb{C}$-algebra such that the R-module
$\Theta_R := \mathit{Der}_{\mathbb{C}}(R,R)$ is free. Then R is smooth.

This conjecture is proved in a number of cases. The first case is the graded one, due to Platte [Pla78], starting from a proof in the algebraic case by Hochster, [Hoc75], [Hoc77].

Lemma 18. *Let A be a positively graded analytic algebra, that is, there is $E \in \Theta_A$ such that the maximal ideal $\mathbf{m}_A$ is generated by elements x_i with $E(x_i) = w(i)$, where $w(i) \in \mathbb{N}_{>0}$. If Θ_A is a free A-module, then A is regular.*

If R is not graded, one has to use rather different techniques. The following lemma ([SS85]) relates the Zariski-Lipman conjecture with the question of extendability of differential forms on R to its resolution.

Lemma 19. *Let $(X,0)$ the germ of an analytic space X. Consider a resolution $\pi : \widetilde{X} \to X$ with $\pi_*\Theta_{\widetilde{X}} \cong \Theta_X$. Let $U := X \backslash \mathit{Sing}(X)$. If the natural morphism $\Omega_{\widetilde{X}} \to \pi^*\Omega_U$ is surjective, then X is smooth if Θ_X is locally free.*

Proof. The idea is simply that a basis $\theta_1, \ldots, \theta_n$ of Θ_X gives rise to vector fields on $\widetilde{\theta}_1, \ldots, \widetilde{\theta}_n$ on $\widetilde{X}$ tangent to the exceptional locus E of the resolution. On $\widetilde{X}\backslash E$, there are independent forms $\alpha_1, \ldots, \alpha_n$ dual to these vector fields which extends over E. This is a contradiction, as for any point $p \in E$, the vectors $\widetilde{\theta}_i(p)$ cannot be linearly independent, because $\dim(E) < n$, unless X is smooth. □

In the quoted paper, the extendability of differential p-forms on isolated singularities is studied and the authors prove that any p-form with $p < \dim(R)-1$ is extendible. Flenner ([Fle88]) showed that more generally, for any space X, a p-form on $X\backslash \mathit{Sing}(X)$ with $p < \operatorname{codim}(\mathit{Sing}(X)) - 1$ extends to a resolution $\widetilde{X}$ of X. Therefore, one has

Corollary 20. *Let R be any analytic algebra such that* $\operatorname{codim}(\mathit{Sing}(R)) \geq 3$*. Then the Zariski-Lipman conjecture is true.*

One of the sources of lagrangian singularities are Frobenius manifolds, where they arise as *spectral covers* of the multiplication on the tangent bundle. It was asked in [Her02], chapter 14, if there exist an isolated Gorenstein, hence complete intersection, lagrangian surface singularity. In the quasi-homogeneous case this is excluded by our theorem. The case of a non-quasi-homogenous lagrangian isolated complete intersection surface singularity remains open, because the Zariski-Lipman conjecture is unproven in this key case.

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