Gauß-Manin systems and Frobenius manifolds for linear free divisors

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(joint work with David Mond, Ignacio de Gregorio)

Linear free divisors have been recently introduced by Mond and Buchweitz ([1]) as special examples of free divisors. A reduced divisor $D = h^{-1}(0) \subset V := \mathbb{C}^n$ is called free if the coherent $\mathcal{O}_V$-module $Der(-\log D) := \{\vartheta \in Der_V \mid \vartheta(h) \subset (h)\}$ is $\mathcal{O}_V$-free of rank n. In that case, one can write any basis $\vartheta_1, \ldots, \vartheta_n$ of $Der(-\log D)$ in terms of the coordinate vector fields as $\vartheta_i = \sum_j a_{ij}\partial_{x_i}$, and we call D linear free if all a_{ij} are linear functions on V. The most simple example is the normal crossing divisor given by $h = x_1 \cdot \ldots \cdot x_n$, but many more examples are constructed as discriminants in quiver representation spaces. Linear free diviors are related to the classical theory of prehomogeneous vector spaces, that is, tuples (V, G) such that G acts linearly on V and has a Zariski open orbit. As have been shown in [3], given a linear free divisor D, the group $G_D := \{g \in GL(V) \mid g(D) = D\}$ makes V into a prehomogeneous vector space, where the open orbit is the complement $V \backslash D$. Of particular importance is the case where G_D is reductive, then we call D a reductive linear free divisor.

By work of Sabbah and Douai ([2]), it is known that a universal unfolding space of the Laurent polynomial $\widetilde{f} := x_1 + \ldots + x_{n-1} + \frac{1}{x_1 \cdot \ldots \cdot x_{n-1}}$ carries a Frobenius structure, which is known to be isomorphic to the quantum cohomology ring of $\mathbb{P}^{n-1}$. This Laurent polynomial can be seen as the restriction of the ordinary polynomial $f = x_1 + \ldots + x_n$ to the non-singular fibres of the equation $h = x_1 \cdot \ldots \cdot x_n$. Hence it seems natural to study linear functions on non-singular fibres and on the central fibre of a morphism given by an equation defining a linear free divisor. Let $f \in \mathbb{C}[V]$ be a linear function. We want to study deformations of f modulo coordinate changes preserving the morphism $h : V \to T = \operatorname{Spec} \mathbb{C}[t]$, this is referred to as $\mathcal{R}_h$-equivalence. The corresponding deformation (or Jacobi) algebra is given by

$$\mathcal{T}^1_{\mathcal{R}_h/T}(f) := \frac{\mathcal{O}_V}{df(Der(-\log h))},$$

where $Der(-\log h) := \{\vartheta \in Der_V \mid \vartheta(h) = 0\} \subset Der(-log\, D)$. Actually, we have the direct sum decomposition $Der(-log\, D) = \mathcal{O}_V E \oplus Der(-\log h)$, where $E = \sum_{i=1}^n x_i \partial_{x_i}$. The first result concerns the finiteness of the above family of Jacobian algebras.

Proposition 1 ([5]). *Let D be reductive. Let V^* be the dual space of V, equipped with dual action of G. Then (V^*, G) is again prehomogenous, and the complement D^* of the open orbit in V^* is again linear free. If $f \in V^* \backslash D^*$, then $h_* \mathcal{T}^1_{\mathcal{R}_h/T}(f)$ is $\mathcal{O}_T$-free of rank n. Moreover, the restriction of f to $D_t := h^{-1}(t)$, $t \neq 0$ has n non-degenerate critical points.*

In order to construct Frobenius structures associated to the restrictions $f_{|D_t}$ (and to $f_{|D}$), we study families of Brieskorn lattices. More precisely, define

$$G_0 := \frac{\Omega^{n-1}_{V/T}(\log D)[\theta]}{(\theta d - df\wedge)\Omega^{n-1}_{V/T}(\log D)[\theta]}$$

which is an $\mathcal{O}_T[\theta]$-module, equipped with connection operators $\theta^2\nabla_\theta$ and $\theta t\nabla_t$, i.e., with a connection

$$\nabla : G_0 \longrightarrow G_0 \otimes \theta^{-1}\Omega^1_{\mathbb{C}\times T}((\log\{0\}\times T)\cup(\mathbb{C}\times\{0\})).$$

Here

$$\Omega^\bullet_{V/T}(\log D) := \frac{\Omega^\bullet_V(\log D)}{h^*\Omega^1_T(\log\{0\})\wedge\Omega^{\bullet-1}_V(\log D)}$$

is the relative logarithmic de Rham complex. One of the main results of [5] is the following.

Theorem 2. *Let $f \in V^*\backslash D^*$. Then*

(1) *The restrictions $f_{|D_t}$ are cohomological tame functions in the sense of* [6].
(2) *G_0 is $\mathcal{O}_T[\theta]$-free of rank n.*
(3) *There is a basis $\underline{\omega}$ such that*

$$\nabla(\underline{\omega}) = \underline{\omega}\cdot\left[(A_0\frac{1}{z}+A_\infty)\frac{dz}{z} + (-A_0\frac{1}{z}+A'_\infty)\frac{dt}{nt}\right]$$

where A_0 and A_∞ are constant, $A'_\infty := (0,1,\ldots,n-1) - A_\infty$ and A_∞ is diagonal. These diagonal entries are not necessarily the spectral numbers of the tame functions $f_{|D_t}$ but can be turned into them after some base change which is meromorphic along $0 \in T$, i.e., there is some other basis $\underline{\omega}'$ of $G_0 \otimes_T [\theta, t^{-1}]$, in which the connection also takes the above form and where the diagonal entries of A_∞ are the correct spectral numbers.

As a consequence (by some more arguments concerning the duality theory for these families of Brieskorn lattices), one obtains the following results.

Theorem 3. (1) *The semi-universal unfoldings $(M_t, 0)$ of the tame functions $f_{|D_t}$ can be equipped with the structure of Frobenius manifolds, depending (among other things) on the choice of a primitive and homogenous section of G_0. Any element ω'_i of the above mentioned basis $\underline{\omega}'$ can be chosen as such a form.*
(2) *The germs $(M_t, 0)$ of Frobenius manifolds glue to a germ $(T^*\times\mathbb{C}^{\mu-1}, T^*\times\{0\})$, where $T^* := T\backslash\{0\} = \mathbb{C}^*$.*
(3) *Under some conjecture on the duality theory of G_0, there exists a "limit" Frobenius structure associated to $f_{|D}$, which is constant, i.e., its potential is a polynomial of degree three.*

In order to understand the properties of the duality theory of G_0, it is desirable to have some more concrete informations on the possible values that occur in the matrix A'_∞. It turns out that they are related to the roots of the Bernstein polynomial $b_h(s)$ of h. It has been shown in [4] that for a reductive linear free divisor, these roots (normalized so that they are in the intervall $(-\infty, 1)$) actually lie in $(-1, 1)$, are symmetric around 0 and that $\deg(b_h) = n$. The comparison result with the diagonal entries of A'_∞ can be stated as follows.

Theorem 4 ([7])**.** (1) *The roots of $b_h(s)$ are equal to the diagonal entries of A'_∞.*

(2) *Consider the restriction $G_0(h)/(h)$, which is a free $\mathbb{C}[\theta]$-module of rank n quipped with a connection operator $\theta^2\nabla_\theta$. This object does not depend on the choice of f, has a regular singularity at $\theta = 0$, and the eigenvalues of the residue of ∇_θ on its saturation are, up to multiplication by n, equal to the roots of $b_h(s)$.*

Notice that the second part of this result is an analogue of the classical theorem of Malgrange relating the roots of the Bernstein polynomial of an isolated hypersurface singularity to the residues eigenvalues of the saturation of the Brieskorn lattice.

References

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