D-modules and isolated hypersurfaces singularities

Main object of study;

$f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ germ of holomorphic function with isolated critical point.

Topics: - singular Gauß-Manin connection
- Brieskorn lattices, Gauß-Manin systems
- mixed Hodge structures and spectral numbers
- Period mappings
- non-commutative Hodge structures

(overview!)

1. Milnor fibration and Gauß-Manin connection

given $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, consider good representative $f : X \to S$, where $S := \{|t| < \delta\} \subset \mathbb{C}$,

$X = \{|x| < \varepsilon\} \cap f^{-1}(S)$. $S' := S \setminus \{0\}$; $X' := f^{-1}(S')$

$f_{|X'} : X' \to S'$ is smooth.

<u>Milnor:</u> ε and δ sufficiently small:

$f' := f_{|X'} : X' \to S'$ is a locally trivial C^∞-fibration, and $X_t := f^{-1}(t) \simeq \bigvee_{i=1}^{\mu} S^n$

where $\mu := \dim_{\mathbb{C}} \dfrac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{(\partial_{x_1} f, \ldots, \partial_{x_{n+1}} f)}$; $x_1 \ldots x_{n+1}$

coordinates on $\mathbb{C}^{n+1}$.

$$\rightsquigarrow H^i(X_t, \mathbb{C}) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i \notin \{0, n\} \\ \mathbb{C}^{\mu} & i = n \end{cases}$$

<u>relative version:</u> $\mathcal{L} := R^n f_* \mathbb{C}_{X'}$ is a locally constant sheaf (local system) of μ-dim. $\mathbb{C}$-vector spaces. Define

$\mathcal{H}' := \mathcal{L} \underset{\mathbb{C}_{S'}}{\otimes} \mathcal{O}_{S'}$, then $\mathcal{H}'$ is $\mathcal{O}_{S'}$-locally free and comes equipped with a connection, namely $\nabla : \mathcal{H}' \to \mathcal{H}' \underset{\mathcal{O}_{S'}}{\otimes} \Omega^1_{S'}$ is defined by $\nabla(\ell \otimes f) := \ell \otimes df$ for any local section $\ell \in \mathcal{L}$ and $f \in \mathcal{O}_{S'}$. Call $(\mathcal{H}', \nabla)$ the topological Gauß-Manin connection of f.

<u>Question:</u> How to calculate monodromy eigenvalues?

<u>Brieskorn:</u> extend $\mathcal{H}'$ over S!

<u>Tool:</u> differential forms and (relative) de Rham complexes

<u>recall</u>: relative Poincaré lemma $\Longrightarrow$

$$f^{-1}\mathcal{O}_{S'} \xrightarrow[\text{quism}]{\simeq} \Omega^\bullet_{X'/S'}$$

where $\Omega^p_{X'/S'} := \Omega^p_{X'} / f^*\Omega^1_{S'} \wedge \Omega^{p-1}_{X'}$

(valid for any smooth holomorphic map)

<u>consequence</u>: $\mathbb{R}^p f_* \Omega^\bullet_{X'/S'} = R^p f_* f^{-1}\mathcal{O}_{S'}$

$$\underset{\substack{\uparrow \\ \text{projection formula}}}{=} \left(R^p f_* \mathbb{C}_{X'}\right) \otimes_{\mathbb{C}_{S'}} \mathcal{O}_{S'}$$

hence $\mathbb{R}^n f_* \Omega^\bullet_{X'/S'} = \mathcal{H}'$. This indicates how to extend $\mathcal{H}'$ over S: Put

$$\mathcal{H} := \mathbb{R}^n f_* \Omega^\bullet_{X/S} \rightsquigarrow \mathcal{O}_S\text{-module}$$

2 questions:

1.) is $\mathcal{H}$ $\mathcal{O}_S$-coherent? locally free?

2.) How to extend ∇ from $\mathcal{H}'$ to $\mathcal{H}$

1.) Brieskorn's construction: $f \underset{R}{\sim} p \in \mathbb{C}[\underline{x}]$

$\rightsquigarrow \exists$ Y smooth and $F: Y \to S$ projektiv

with $F_{|F^{-1}(S')}$ smooth and $\operatorname{Sing} F^{-1}(0) = \{x\}$

s.t. $\left[(F,x):(Y,x) \to (S,0)\right] \sim \left[f:(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)\right]$

Then $\mathbb{R}^p F_* \Omega^\bullet_{Y/S}$ is $\mathcal{O}_S$-coherent

(Grauert's theorem) $\underset{\text{Brieskorn}}{\Longrightarrow}$ $\mathcal{H}$ is $\mathcal{O}_S$-coherent

use: $\mathbb{R} f_! \Omega^\bullet_{Z/S}$ where $Z := Y \setminus \overline{X}$, where

$\overline{X} :=$ closure of X in Y, notice that $f: Z \to S$ is trivial.

<u>remark:</u> Put $\Omega^\bullet_f := \Omega^\bullet_{\mathbb{C}^{n+1},0} / f^*\Omega^1_{S,0} \wedge \Omega^{\bullet-1}_{\mathbb{C}^{n+1},0}$

then $\forall p \in \mathbb{N}$ there is an exact sequence:

$$0 \to R^p f_* \mathbb{C}_X \otimes \mathcal{O}_S \to \mathbb{R}^p f_* \Omega^\bullet_{X/S} \to f_* \mathcal{H}^p \Omega^\bullet_{X/S} \to 0$$

$$\& \left[\mathcal{H}^p f_* \Omega^\bullet_{X/S}\right]_0 = f_* \mathcal{H}^p \Omega^\bullet_{X/S} = H^p(\Omega^\bullet_f, d)$$

(skyscraper sheaf)

<u>Pf:</u> 2 spectral sequences

$${}'E_2^{k,l} := \mathcal{H}^k\left(R^l f_* \Omega^\bullet_{X/S}, d\right) \Rightarrow \mathbb{R}^{k+l} \Omega^\bullet_{X/S}$$

$${}''E_2^{k,l} := R^k f_*\left(\mathcal{H}^l(\Omega^\bullet_{X/S}, d)\right) \Longrightarrow \mathbb{R}^{k+l} \Omega^\bullet_{X/S}$$

<u>${}'E_\bullet^{k,l}$:</u> f is Stein $\Rightarrow R^l f_* \Omega^\bullet_{X/S} = 0 \quad \forall l > 0$

$\Rightarrow {}'E_\bullet^{k,l}$ degenerates $\Rightarrow \mathbb{R}^k f_* \Omega^\bullet_{X/S} = \mathcal{H}^k f_* \Omega^\bullet_{X/S}$

<u>${}''E_\bullet^{k,l}$:</u> Crit $f = \{x\} \Rightarrow \operatorname{Supp} \mathcal{H}^l \Omega^\bullet_{X/S} \subset \{x\} \quad \forall l > 0$

$\Rightarrow R^k f_* \mathcal{H}^l \Omega^\bullet_{X/S} = 0 \quad \forall k, l > 0; \quad {}''E_2^{0,l} = f_* \mathcal{H}^l \Omega^\bullet_{X/S}$

$${}''E_2^{k,0} = R^k f_* f^{-1}\mathcal{O}_S = R^k f_* \mathbb{C}_X \otimes_{\mathbb{C}_S} \mathcal{O}_S$$

Sebastiani: $\mathcal{H} = \mathbb{R}^n f_* \Omega^\bullet_{X/S}$ is $\mathcal{O}_S$-free

(and, as we will see later: $\mathbb{R}^p f_* \Omega^\bullet_{X/S} = 0 \ \forall p \neq 0, n$)

2.) recall: $\operatorname{Crit} f = V(\partial_{x_1} f, \dots, \partial_{x_{n+1}} f) = \{x\} \subset X$ is a complete intersection, i.e., $\partial_{x_1} f, \dots, \partial_{x_n} f$ reg. sequence in $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$;

$$H^p(\Omega^\bullet_X, \wedge df) = \begin{cases} 0 & \forall p \neq n+1 \\ \mathcal{O}_{\mathbb{C}^{n+1},0} / J_f \cdot \mathrm{vol} & p = n+1 \end{cases} ; \ \mathrm{vol} = dx_1 \wedge \dots \wedge dx_n$$

Consider the morphism

$$f^* \Omega^1_S \otimes_{\mathcal{O}_X} \Omega^{p-1}_{X/S} \xrightarrow{\alpha} \Omega^p_X$$

$$\alpha \otimes \beta \longmapsto \alpha \wedge \beta$$

α injective $\forall p \leq n+1$. Hence put

$$\overline{\Omega}^p_{X/S} := \begin{cases} \Omega^p_{X/S} & \forall 0 \leq p \leq n \\ 0 & \text{else} \end{cases}$$

Then we have the exact sequence

$$0 \to f^*\Omega^1_S \otimes \Omega^{\bullet-1}_{X/S} \xrightarrow{\alpha} \Omega^\bullet_X \longrightarrow \Omega^\bullet_{X/S} \to 0$$

and the long exact hypercohomology sequence:

$$\ldots \to \mathcal{H}^p f_* \Omega^\bullet_X \to \mathcal{H}^p f_* \Omega^\bullet_{X/S} \xrightarrow{\delta} \Omega^1_S \otimes \mathcal{H}^p f_* \Omega^\bullet_{X/S} \to \ldots \quad (*)$$

for $p \in \{1, \ldots, n-1\}$ and

$$\ldots \to \mathcal{H}^n f_* \Omega^\bullet_X \to \mathcal{H}^n f_* \Omega^\bullet_{X/S} \xrightarrow{\delta} \Omega^1_S \otimes {}'\mathcal{H} \to \ldots \quad (**)$$

where

$${}'\mathcal{H} := \frac{f_* \Omega^n_{X/S}}{d(f_* \Omega^{n-1}_{X/S})} = \frac{f_* \Omega^n_X}{d(f_* \Omega^{n-1}_X) + df \wedge f_* \Omega^{n-1}_X} \, ; \quad {}'\mathcal{H}_{|S'} \cong \mathcal{H}_{|S'} \text{ , as } \Big(d : \Omega^n_{X'/S'} \to \underbrace{\Omega^{n+1}_{X'/S'}}_{0}\Big) = 0$$

At the stalk level, we have $\nabla_t := \iota_{\partial_t} \circ \delta : H^p(\Omega^\bullet_f) \circlearrowleft$ resp. $\nabla_t : H^n(\Omega^\bullet_f) \to {}'\mathcal{H}_0$ defined by

$\nabla_t([\omega]) := [\eta]$ where $\eta \in \Omega^n_X$ is such that $d\omega = df \wedge \eta$. The operator $\nabla_t : \mathcal{H} \to \Omega^1_S \otimes {}'\mathcal{H}$ is called the singular Gauß-Manin connection of $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$

important properties:

a) $\nabla_t : H^p(\Omega^{\bullet}_f) \to H^p(\Omega^{\bullet}_f)$ $\forall p \in \{1, \dots, n-1\}$ and

$\nabla_t : \mathcal{H}_0 \to \mathcal{H}_0$ are isomorphisms

b) $\mathcal{H}^p f_* \Omega^{\bullet}_{X/S} = 0 \quad \forall\, 0 < p < n$

c) The restriction $\nabla_t : \mathcal{H}^n f_* \Omega^{\bullet}_{X'/S'} \circlearrowleft$ coincides with the topological GM-conn.

d) Application: without using Milnor's theorem, we have: $h^p(X_t, \mathbb{C}) = 0$, $\forall p \in \{1, \dots, n-1\}$

Proof: a) follows from sequences (*) and (**); $\mathcal{H}^p(f_* \Omega^{\bullet}_X) = R^p f_* \Omega^{\bullet}_X = R^p f_* \underline{\mathbb{C}}_X$ so that $(\mathcal{H}^p f_* \Omega^{\bullet}_X)_0 = 0$

b) $\forall p \in \{1,\dots,n-1\}$: $\mathcal{H}^p f_* \Omega^\bullet_{X/S}$ is $\mathcal{O}_S$-coherent and equipped with connection ∇_t, hence free and $\mathcal{H}^p f_* \Omega^\bullet_{X/S} \simeq \operatorname{Ker}\nabla_t \otimes_{\mathbb{C}_S} \mathcal{O}_S$, but at 0 ∇_t is an isomorphism $\Rightarrow [\mathcal{H}^p f_* \Omega^\bullet_{X/S}]_0 = 0 \overset{\text{free}}{\Longrightarrow} \mathcal{H}^p f_* \Omega^\bullet_{X/S} = 0$

c) it suffices to show that $\operatorname{Ker}\nabla_t : \mathcal{H}_{|S'} \to \mathcal{H}_{|S'}$ equals $R^n f_* \mathbb{C}_{X'}$. From the sequence (**) restricted to X' we get: $\operatorname{Ker}\nabla_{t|X'} =$

$$\operatorname{Im}\left(\mathcal{H}^n f_* \Omega^\bullet_{X'} \to \mathcal{H}^n f_* \Omega^\bullet_{X'/S'}\right) =$$

$$\operatorname{Im}\left(\mathbb{R}^n f_* \Omega^\bullet_{X'} \to \mathbb{R}^n f_* \Omega^\bullet_{X'/S'}\right) = \operatorname{Im}\left(R^n f_* \mathbb{C}_{X'} \to R^n f_* f^{-1}\mathcal{O}_{S'}\right)$$

$$= \operatorname{Im}\left(R^n f_* \mathbb{C}_{X'} \to R^n f_* \mathbb{C}_{X'} \otimes_{\mathbb{C}_{S'}} \mathcal{O}_{S'}\right) = \operatorname{Ker}\nabla_t^{top}.$$

d) clear from b)

Brieskorn lattice and Gauß-Manin system

last time: $f: X \to S$ Milnor fibration

$\rightsquigarrow \mathcal{H} := \mathcal{H}^n(f_* \Omega^\bullet_{X/S})$ and $'\mathcal{H} = f_*\Omega^n_{X/S} / d f_* \Omega^{n-1}_{X/S}$

$\mathcal{O}_S$-free, connection $\nabla_t : \mathcal{H} \to {'\mathcal{H}}$, isom. at $0 \in S$

rk: obviously $\mathcal{H} \overset{i}{\hookrightarrow} {'\mathcal{H}}$, i $\mathcal{O}_S$-linear

Malgrange: $(\mathcal{H}, \mathcal{H}', i, \nabla_t)$: (E,F)-connection

Brieskorn: $$''\mathcal{H} := \frac{f_* \Omega^{n+1}_X}{df \wedge d f_* \Omega^{n-1}_X}$$

we have $'\mathcal{H} \xrightarrow{\wedge df} {''\mathcal{H}}$ with coker equal to

$f_* \Omega^{n+1}_X / df \wedge d f_* \Omega^{n-1}_X + df \wedge f_* \Omega^n_X \cong f_* \Omega^{n+1}_X / df \wedge f_* \Omega^n_X$

$\cong \mathcal{O}_{\mathbb{C}^{n+1},0} / J_f \cdot \mathrm{vol}$

we also have $\mathrm{coker}(\mathcal{H} \hookrightarrow {}'\mathcal{H}) \underset{d(-)}{\cong} f_*\Omega_X^{n+1} / df \wedge f_*\Omega_X^n$

hence the inclusions $\mathcal{H} \subset {}'\mathcal{H} \subset {}''\mathcal{H}$ of $\mathcal{O}_S$-mod. are equalities on S', and we have

$$\mathcal{H}(*0) := \mathcal{H} \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_S(*0) = {}'\mathcal{H}(*0) = {}''\mathcal{H}(*0).$$

$\mathcal{H}$, ${}'\mathcal{H}$ and ${}''\mathcal{H}$ are $\mathcal{O}_S$-coherent and even loc. free (later).

$\mathcal{H}(*0)$ is a locally free $\mathcal{O}_S(*0)$-module with connection, which is defined by the Leibniz rule. It is called THE Gauß-Manin connection of f.

<u>Def</u>: Let $\mathcal{N}$ be $\mathcal{O}_S(*0)$-locally free, and $\mathcal{L}$ $\mathcal{O}_S$-coherent with $\mathcal{L}(*0) = \mathcal{N}$, then $\mathcal{L}$ is called a lattice in $\mathcal{N}$. Let ∇ be a connection on $\mathcal{N}$, then $\mathcal{L}$ is called logarithmic, if $(t\nabla_t)(\mathcal{L}) \subset \mathcal{L}$.

extension of the connection to $''\mathcal{H}$:

$$\nabla_t : {}'\mathcal{H} \longrightarrow {}''\mathcal{H} \,,\quad [\alpha] \longmapsto [d\alpha]$$

This implies: let $k \in \mathbb{N}$ s.t. $t^k \, ''\mathcal{H} \subset {}'\mathcal{H}$, i.e. if for $[\omega] \in {}''\mathcal{H}$ we have $f^k \cdot \omega = df \wedge \eta$, then

$$t^k \nabla_t : {}''\mathcal{H} \longrightarrow {}''\mathcal{H}$$
$$[\omega] \longmapsto [d\eta] - k \cdot [f^{k-1}\omega] \qquad (*)$$

example: quasi-homogeneous singularities

Let $f = \sum_{\underline{\alpha} \in \mathbb{N}^n} c_{\underline{\alpha}} \cdot \underline{x}^{\underline{\alpha}} \in \mathbb{C}\{\underline{x}\}$ s.t. $\exists\, w_1, \dots, w_{n+1} \in \mathbb{Q} \cap (0,1]$?

with $\sum_{i=1}^{n+1} \alpha_i \cdot w_i = 1 \quad \forall \underline{\alpha} \in \mathbb{N}^n$ s.t. $c_{\underline{\alpha}} \neq 0$.

Then: $\left(\sum w_i x_i \partial_{x_i}\right)(f) = f \implies f \in J_f$

$$\implies f \cdot \mathrm{vol} = df \wedge \eta \,; \quad \eta = \sum_{i=1}^{n+1} (-1)^{i+1} \cdot w_i \cdot x_i \cdot \frac{\mathrm{vol}}{dx_i} \in f_* \Omega^n_X$$

hence $(t\nabla_t)''\mathcal{H} \subset ''\mathcal{H}$, i.e., $''\mathcal{H}$ is a logarithmic lattice inside $(\mathcal{H}(*0), \nabla)$. Moreover, using (*), we have $(k=1)$: $(t\nabla_t)[\underline{x}^{\beta} \cdot vol] = [d\eta] - [\underline{x}^{\beta} vol]$ where $df \wedge \eta = f \underline{x}^{\beta} vol$. Remember that $f \cdot vol = df \wedge \xi$, then $f \cdot \underline{x}^{\beta} vol = df \wedge \underline{x}^{\beta} \xi \Longrightarrow \eta = \underline{x}^{\beta}\xi$, so that

$$(t\nabla_t)[\underline{x}^{\beta} vol] = [d(\underline{x}^{\beta}\xi) - \underline{x}^{\beta} vol] =$$

$$\left[\left(\sum_{i=1}^{n+1} w_i \partial_{x_i}(x_i \underline{x}^{\beta}) - \underline{x}^{\beta}\right) vol\right] = \left[\left(\left(\overbrace{\sum_{i=1}^{n+1} w_i(\beta_i+1)}^{wt(x^{\beta} vol)}\right) - 1\right) \underline{x}^{\beta} vol\right]$$

<u>On the other hand</u>: from above: $\forall\, \underline{x}^{\beta} vol : \exists\, \tilde{\eta}$ s.t. $d\tilde{\eta} = \underline{x}^{\beta} vol$ and s.t. $df \wedge \tilde{\eta} \in f \cdot \Omega^{n+1}_X$, namely, take: $\tilde{\eta} = \frac{1}{wt(x^{\beta} vol)} \cdot x^{\beta}\xi$. Extending linearly, we conclude that $Im('\mathcal{H} \hookrightarrow ''\mathcal{H}) \subset t \cdot ''\mathcal{H}$.

But $f vol = df \wedge \eta \Rightarrow t \cdot ''\mathcal{H} \subset Im(\wedge df : '\mathcal{H} \hookrightarrow ''\mathcal{H})$, hence

$''\mathcal{H}/t \cdot ''\mathcal{H} \simeq ''\mathcal{H}/'\mathcal{H} \simeq f_* \Omega^{n}_{X/S} \simeq \mathcal{O}_{X,0}/J_f \cdot vol$, hence (Nakayama)

$''\mathcal{H} = \bigoplus_{k=1}^{\mu} \mathcal{O}_S \cdot x^{\beta_k} vol$, where $\mathcal{O}_{X,0}/J_f = \bigoplus_{k=1}^{\mu} \mathbb{C}\, \underline{x}^{\beta_k}$

Corollary: Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ be quasi-hom. i.h.s. mogeneous of degree 1 w.r.t. the weight system $w_1, \dots, w_{n+1}$. Then the monodromy of $R^n f_* \mathbb{C}_{X'}$ is semi-simple with eigenvalues $e^{-2\pi i \cdot wt(\underline{x}^{\beta_k} \cdot vol)}$,

where $\mathcal{O}_{X,0}/J_f = \bigoplus_{k=1}^{\mu} \mathbb{C}\, \underline{x}^{\beta_k}$

regularity: recall: $(\mathcal{N}, \nabla)$ meromorphic connection is regular $\underset{\text{Df}}{\Longleftrightarrow}$ $\exists\, \mathcal{L} \subset \mathcal{N}$ logarithmic lattice.

Theorem: $(\mathcal{H}(*0), \nabla)$ is regular.

sketch of proof (Deligne): - write $X_0 := f^{-1}(0)$, then $\mathcal{H}(*0) = \mathbb{R}^n f_* \Omega^{\bullet}_{X/S}(*X_0)$

- consider a resolution of f

$$\begin{tikzcd} \tilde{X}_0 \subset \tilde{X} \arrow[r, "\pi"] \arrow[d] \arrow[drr, "\tilde{f}"] & X \arrow[d, "f"] \\ \{0\} \arrow[rr] & & S \end{tikzcd}$$

with $\tilde{X}_0$ NCD $\quad \pi_{|\tilde{X}\setminus\tilde{X}_0}$ iso

then $\mathbb{R}^n\tilde{f}_*\Omega^\bullet_{\tilde{X}/S}(*\tilde{X}_0) \simeq \mathbb{R}^n f_*\Omega^\bullet_{X/S}(*X_0) \simeq \mathcal{H}(*0)$

Then $\mathbb{R}^n\tilde{f}_*\Omega^\bullet_{\tilde{X}/S}(\log \tilde{X}_0)/\mathrm{Tors}$ is a logarithmic lattice inside $\mathcal{H}(*0) = \mathbb{R}^n\tilde{f}_*\Omega^\bullet_{\tilde{X}/S}(*\tilde{X}_0)$ $\square$

Rk.: regularity is equivalent to the following statement: let $A_1, \ldots, A_\mu$ be a basis of $R^n f_*\mathbb{C}_{X'}$ consisting of multivalued sections, and $v \in \mathcal{H}_0$ (or in $'\mathcal{H}_0$ or $''\mathcal{H}_0$) with $v = \sum_{i=1}^{\mu} c_i(t)\cdot A_i$, then on any sector $\subset S'$, we have $|c_i(t)| \leq \frac{C}{|t|^k}$ $k \in \mathbb{N}$.

Monodromy theorem (Brieskorn): The eigenvalues of the monodromy of $R^n f_*\mathbb{C}_{X'}$ are roots of unity.

Pf.: $\mathcal{L} \subset \mathcal{H}^{(0)}$ log., show that EV of $\mathrm{Res}(\nabla_{t}, \mathcal{L})$ are algebraic

The Gauß-Manin-system

recall: $f : A \to B$ holomorphic map $\rightsquigarrow$

$f_+ : D^b(\mathcal{D}_A) \to D^b(\mathcal{D}_B)$. However, for N $\mathcal{D}_A$-coherent and holonomic, $\mathcal{H}^i f_+ N$ is not necessarily so (true if $f_{|\mathrm{supp}\,M}$ is proper). Nevertheless

<u>Def.</u>: $f : X \to S$ Milnor fibration of i.h.s, then $M := \mathcal{H}^0 f_+ \mathcal{O}_X \in \mathrm{Mod}(\mathcal{D}_S)$ is called the Gauß-Manin system of f.

<u>Lemma</u>: We have $M := \mathcal{H}^{n+1}(f_* \Omega^\bullet_X[\partial_t], d - df \wedge - \otimes \partial_t)$ with $\mathcal{D}_S$-action $\partial_t(\omega \otimes \partial_t^k) = \omega \otimes \partial_t^{k+1}$ and

$t(\omega \otimes \partial_t^k) = f \cdot \omega \otimes \partial_t^k - k\,\omega \otimes \partial_t^{k-1}$

<u>Proof</u>: Use general rules for calculating the direct image, i.e.: $f = p_2 \circ i_f : X \xrightarrow{i_f} X \times S \xrightarrow{p_2} S$

$\operatorname{Im} i_f =: \Gamma_f = V(\partial_f)$

$$\Longrightarrow \quad f_+ \mathcal{O}_X = p_{2,+}\, i_{f,+} \mathcal{O}_X$$

$$i_{f+} \mathcal{O}_X = \mathcal{O}_X[\partial_t] \simeq \mathcal{O}_{X\times S}(*\Gamma_f)/\mathcal{O}_{X\times S}$$

$$1 \longmapsto \frac{1}{f-t}$$

where $\partial_{x_i}(g \otimes \partial_t^k) = \partial_{x_i} g \otimes \partial_t^k - (\partial_{x_i} f) \cdot g \otimes \partial_t^{k+1}$

$$t \cdot (g \otimes \partial_t^k) = f \cdot g \otimes \partial_t^k - k\, g \otimes \partial_t^{k-1}$$

$$p_{2,+}(\mathcal{O}_X[\partial_t]) = \mathbb{R} p_{2,*} \mathrm{DR}^{\bullet + (n+1)}_{X\times S/S}\left(\mathcal{O}_X[\partial_t]\right)$$

$$\overset{!}{=} \left(p_{2,*} \Omega^{\bullet + (n+1)}[\partial_t],\ d - df \wedge - \otimes \partial_t\right) \quad \text{(M. Saito)} \qquad \square$$

natural filtration: $F^{n-k} M := \operatorname{Im}\left(\bigoplus_{i \le k} \Omega_X^{n+1} \partial_t^i\right) \subset M$

<u>Lemma</u>: $F^n M \simeq$ " $\mathcal{G}$ "

Proof: We have a surjective map $f_* \Omega_X^{n+1} \xrightarrow{\ell} F_n M$

show: $\ker \ell = df \wedge d f_* \Omega_X^{n-1}$: Take $\alpha \in f_* \Omega_X^{n+1}$

with $\alpha \in \ker \ell \Rightarrow \exists\, \eta_0, \dots, \eta_k \in f_* \Omega_X^n$ s.t.

$$\alpha = (d - \partial_t \cdot df \wedge)\left[\sum_{i=0}^{k} \eta_i \partial_t^i\right] = \sum_{i=0}^{k} \left(d\eta_i \partial_t^i - (df \wedge \eta_i)\partial_t^{i+1}\right)$$

$$\Rightarrow df \wedge \eta_k = 0 \underset{\text{Koszul}}{\Longrightarrow} \eta_k = df \wedge \alpha_{k-1},\ \alpha_{k-1} \in f_* \Omega_X^{n-1}$$

$\underline{k=0}$: $\alpha = d(df \wedge \alpha_{k-1}) = -df \wedge d\alpha_{k-1}$ ✓

$\underline{k>0}$: $\underbrace{d\eta_k}_{df \wedge d\alpha_{k-1}} - df \wedge \eta_{k-1} = 0 \Rightarrow df \wedge (d\alpha_{k-1} - \eta_{k-1}) = 0$

$$\Rightarrow \eta_{k-1} = d\alpha_{k-1} + df \wedge \alpha_{k-2} \Rightarrow d\eta_{k-1} = -df \wedge d\alpha_{k-2}$$

$\underline{k=1}$: $\alpha = d\eta_{k-1} = -df \wedge d\alpha_{k-2}$

$\underline{k>1}$: $d\eta_{k-1} - df \wedge \eta_{k-2} = df \wedge (d\alpha_{k-2} - \eta_{k-2}) = 0$

and so on: $d\eta_1 = df \wedge d\alpha_0$ & $d\eta_1 - df \wedge \eta_0 = 0$

$\Rightarrow \eta_0 = d\alpha_0 + df \wedge \beta \quad \beta \in f_* \Omega_X^{n-1}$

and $\alpha = df \wedge d\beta$ $\square$

<u>Coherence, regularity, holonomicity of $\mathcal{M}$:</u>

<u>exercise</u>: $F^{n-k}\mathcal{M} = \partial_t^k F^n \mathcal{M} = \partial_t^k \, {}''\mathcal{H} \quad \forall k \geq 0$

<u>hint</u>: $\omega_1 \partial_t + \omega_0 \in F^{n-1}\mathcal{M} \Rightarrow \omega_1 \partial_t + \omega_0 = \partial_t(\omega_1 + \omega_0')$

<u>Theorem</u>: $\mathcal{M}$ is $\mathcal{D}_S$-coherent and holonomic, with good filtration $F^{n-\bullet}\mathcal{M}$, and $\mathcal{M}_{|S'} = (\mathcal{H}', \nabla)$, i.e. $\mathcal{O}_{S'}$-locally free with connection. Moreover, ∂_t is invertible on $\mathcal{M}_0$, and $\mathcal{M}$ is $\mathcal{D}_S$-regular

<u>Proof</u>: From the $\mathcal{O}_S$-freeness of ${}''\mathcal{H} = F^n\mathcal{M}$ and $F^{n-k}\mathcal{M} = \partial_t^k \, {}''\mathcal{H}$ we deduce that $F^{n-\bullet}$ is good. Notice also that $F^{n-(k-1)}\mathcal{M} = \partial^{k-1} \, {}''\mathcal{H} = \partial^k \partial^{-1} \, {}''\mathcal{H} = \partial^k \, {}'\mathcal{H}$, so that $F^{n-k}\mathcal{M} / F^{n-k+1}\mathcal{M} \simeq {}''\mathcal{H} / {}'\mathcal{H} \simeq f_* \Omega^n_{X|S}$, which is $\mathcal{O}_S$-torsion,

so that $gr^{\bullet}_F M$ is $gr_{\bullet}\mathcal{D}_S$-coherent, hence $\mathcal{M}$ is $\mathcal{D}_S$-coherent.

We have seen: $\nabla_t : {}'\mathcal{H}_0 \xrightarrow{\cong} {}''\mathcal{H}_0$, hence ∂_t is invertible on $\mathcal{M}_0$ (exercise: show this directly from the definition of $\mathcal{M} := \mathcal{H}^0 f_+ \mathcal{O}_S$)

<u>restriction $j^*\mathcal{M}$:</u> as $f_{|X'} : X' \to S'$ is smooth, we have $f_+ \mathcal{O}_{X'} \simeq \mathbb{R} f_* f^{-1}\mathcal{O}_{S'} \simeq R f_* \mathbb{C}_{X'} \otimes_{\mathbb{C}_{S'}} \mathcal{O}_{S'}$

Pf: $f_{|X'}$ smooth $\Longrightarrow f^{-1}\mathcal{O}_{S'} \simeq \Omega^{\bullet}_{X'/S'} \simeq \Omega^{\bullet}_{\Gamma/S'}$, where $\Gamma = \mathrm{Graph}(f) \subset X' \times S' =: Z$. Now consider

$$\begin{array}{ccccccccc}
0 \to & \Omega^{\bullet}_{Z/S'} & \longrightarrow & \Omega^{\bullet}_{Z/S'}(\log \Gamma) & \xrightarrow{res} & i_* \Omega^{\bullet}_{\Gamma/S'} & \to 0 \\
& \| & & \int b & & a \int & \\
0 \to & \Omega^{\bullet}_{Z/S'} & \longrightarrow & \Omega^{\bullet}_{Z/S'}(*\Gamma) & \longrightarrow & DR^{\bullet}(i_{f,+}\mathcal{O}_{X'}) & \to 0
\end{array}$$

where $i : \Gamma \hookrightarrow Z$. We have: b quism $\Rightarrow a$ quism, hence

$$\mathbb{R} f_* \Omega^{\bullet + (n+1)}_{X'/S'} \simeq \mathbb{R} p_* i_* \Omega^{\bullet + (n+1)}_{\Gamma/S'} \simeq \mathbb{R} p_* DR^{\bullet + (n+1)} i_{f,+}\mathcal{O}_{X'} = f_+ \mathcal{O}_{X'}$$

It follows that $\mathrm{char}(\mathcal{M}) \subset T^*_S S \cup T^*_{\{0\}} S$,

hence $\dim \mathrm{char}(\mathcal{M}) = 1 \rightsquigarrow \mathcal{M}$ is holonomic.

<u>regularity</u>: we have that $\mathcal{M} \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_S(*0) \simeq {}^{\text{"}}\mathcal{H} \otimes \mathcal{O}_S(*0) = \mathcal{H}(*0)$

<u>general fact in dim 1</u>: $\mathcal{M} \otimes \mathcal{O}_S(*0)$ regular $\Rightarrow \mathcal{M}$ regular.

V-filtration and mixed Hodge structures

<u>aim:</u> use V-filtration on $\mathcal{H}(*0)$ resp. on M to define Hodge filtration on $H^n(f^{-1}(t),\mathbb{C})$ giving rise to MHS.

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reminder on V-filtration: here only for meromorphic bundles with connection.

<u>starting data:</u> $(\mathcal{H}' \to \mathbb{C}^*, \nabla)$ flat holomorphic bundle (e.g. extension of $R^n f_*(f^{-1}\mathcal{O}_{S'})$ from S' to $\mathbb{C}^*$)

$\mathcal{H}'^{\nabla}$: $\ker \nabla$ is local system

Consider $u: \mathbb{C} \to \mathbb{C}^*$, $\xi \mapsto t := e^{2\pi i \xi}$, then

$u^* \mathcal{H}'^{\nabla}$ is constant, put $H^\infty := H^0(\mathbb{C}, u^* \mathcal{H}'^{\nabla})$

then we have $M \in \mathrm{Aut}(H^\infty)$: $H^\infty \ni A \xmapsto{M} [\xi \mapsto A(\xi+1)]$

more abstract version: $H^\infty := \psi_t(j_! \mathcal{H}'^{\nabla}) \in \mathrm{Mod}(\mathbb{C}[M])$

where $j: \mathbb{C}^* \hookrightarrow \mathbb{C}$

generalized eigenspace decomposition:

$$H^\infty := \bigoplus_{\lambda\in\mathbb{C}} H^\infty_\lambda \ , \quad H^\infty_\lambda := \bigcup_i \ker(M-\lambda)^i$$

and $M = M_{ss}\cdot M_u$, $N := \log M_u$: "nilpotent part of M"

<u>Def.</u>: $\forall \lambda\in\mathbb{C}$, $A\in H^\infty_\lambda$ and $\alpha\in\mathbb{C}$ s.t. $e^{-2\pi i\alpha}=\lambda$,

put $\quad es(A,\alpha) := \left[\xi \longmapsto e^{(2\pi i\alpha\cdot \mathrm{Id}-N)\xi}\cdot A(\xi)\right] \in H^0(\mathbb{C}, u^*\mathcal{H}')$

notice $es(A,\alpha)(\xi+1) = e^{(2\pi i\alpha\cdot \mathrm{Id}-N)\xi}\cdot e^{2\pi i\alpha \mathrm{Id}-N}\cdot A(\xi+1)$

$$= e^{(2\pi i\alpha \mathrm{Id}-N)\xi}\, e^{2\pi i\alpha \mathrm{Id}-N}\cdot M(A)(\xi)$$

$$= e^{(2\pi i\alpha \mathrm{Id}-N)\xi}\left(\lambda^{-1}\cdot \mathrm{Id}\right)\left(M_u^{-1}\right)\cdot M_{ss}\left(M_u\right)(A)(\xi)$$

$$= e^{(2\pi i\alpha \mathrm{Id}-N)\xi}\cdot A(\xi) = es(A,\alpha)(\xi)$$

hence $es(A,\alpha)\in H^0(\mathbb{C}^*, \mathcal{H}')$. It is called an elementary section of $\mathcal{H}'$ of order α.

Using elementary sections, we can give an explicit description of the canonical V-filtration of $(\mathcal{H}', \nabla)$

rules for calculating with elementary sections (exercise)

a) $t \cdot es(A, \alpha) = es(A, \alpha+1)$

b) $\partial_t \cdot es(A, \alpha) = \alpha \cdot es(A, \alpha-1) - \frac{1}{2\pi i} es(N \cdot A, \alpha-1)$

c) Put $C^\alpha := \{ es(A, \alpha) \mid A \in H^\infty_{e^{-2\pi i \alpha}} \}$, then

$H^\infty_{e^{-2\pi i\alpha}} \longrightarrow C^\alpha$, $A \mapsto es(A, \alpha)$ is isom.

d) $t : C^\alpha \to C^{\alpha+1}$ is bijective,

$\partial_t : C^\alpha \to C^{\alpha-1}$ is bijective iff $\alpha \neq 0$

$t\partial_t - \alpha \cdot Id : C^\alpha \to C^\alpha$ is nilpotent

Clear: $C^\alpha = 0$ unless $e^{-2\pi i \alpha}$ is eigenvalue of M

from now on suppose that M is quasi-unipoten $(M^r - Id)^N = 0$

i.e., $C^\alpha = 0 \ \forall \alpha \notin \mathbb{Q}$

Def.: Let $W^w_\bullet H^\infty$ be the monodromy weight filtration with center w, i.e. the unique filtration s.t. $N : W^w_\bullet \subset W^w_{\bullet-2}$ &

$N^e : gr^{W^w}_{w+e} \xrightarrow{\cong} gr^{W^w}_{w-e}$

Def: $V^\alpha := \bigoplus_{\beta \in [\alpha, \alpha+1)} \mathcal{O}_{\mathbb{C}} \cdot C^\beta$, $V^{>\alpha} := \bigoplus_{\beta \in (\alpha, \alpha+1]} \mathcal{O}_{\mathbb{C}} \cdot C^\beta$

then V^α and $V^{>\alpha}$ are $\mathcal{O}_{\mathbb{C}}$-free and

$V^\alpha_{|\mathbb{C}^*} \simeq V^{>\alpha}_{|\mathbb{C}^*} \simeq \mathcal{H}'$ (i.e. $V^\alpha, V^{>\alpha} \subset j_* \mathcal{H}'$)

$gr^\alpha_V := V^\alpha / V^{>\alpha} \simeq C^\alpha$

relation between V-filtration on $\mathcal{H}(*0)$ and M

Th: Let $\mathcal{M}$ be a coherent+holonomic $\mathcal{D}_S$-module with a reg. sing at $0 \in S$ and no others and s.t. ∂_t is invertible on $\mathcal{M}_0$. Then $\mathcal{M} = \bigoplus_{j=1}^{k} \mathcal{M}^{\alpha_j, q_j}$ with $\mathcal{M}^{\alpha_j, q_j} := \mathcal{D}_S / \mathcal{D}_S (t\partial_t - \alpha_j)^{q_j}$

where $\alpha_j \in (-1, 0]$ (suppose that $\alpha \in \mathbb{Q}$).

Canonical V-filtration on $\mathcal{M}$: $C^\alpha := \bigcup_{N \geq 0} \ker (t\partial_t - \alpha)^N$

and $V^\alpha \mathcal{M}$, $V^{>\alpha} \mathcal{M}$ as before. Then $Tors(M) \subset \sum_{\alpha \leq -1} C^\alpha$, hence $V^\alpha \mathcal{M}$ is $\mathcal{O}_S$-free $\forall \alpha > -1$.

<u>notice</u>: $\exists$ exact sequence

$$0 \to \ker \to \mathcal{M} \xrightarrow{\mathrm{loc}} \mathcal{M}(*0) \to \mathrm{coker} \to 0$$

with ker & coker t-torsion. Hence $V^{>-1}\mathcal{M} = V^{>-1}\mathcal{H}(*0)$. Moreover, as ${}''\mathcal{H}$ is $\mathcal{O}_S$-free,

we have ${}''\mathcal{H} \subset V^{>-1}\mathcal{H}(*0) = V^{>-1}\mathcal{M}$.

<u>rk</u>: $V^{\bullet}\mathcal{H}(*0)$ induces $V^{\alpha}\,{}''\mathcal{H} := V^{\alpha}\mathcal{H}(*0) \cap {}''\mathcal{H}$ and

we put $gr_V^{\alpha}\,{}''\mathcal{H} := V^{\alpha}\,{}''\mathcal{H} / V^{>\alpha}\,{}''\mathcal{H}$.

back to isolated hypersurface singularities, $\mathcal{M} = \mathcal{H}^0 f_+ \mathcal{O}_X$

aim: define Hodge filtration on $H^n(f^{-1}(t), \mathbb{C})$

<u>rk</u>: canonical Milnor: consider the cartesian diagram

$$\begin{tikzcd}
X^{\infty} \arrow[r] \arrow[d] & X' \arrow[d, "f_{|X'}"] \\
\tilde{S}' \arrow[r, "u"'] & S'
\end{tikzcd}$$

universal cover

then $X^{\infty} \to \tilde{S}'$ is trivial, and $H^n(X^{\infty}, \mathbb{C}) \simeq H^{\infty}$, where

$H^\infty := \psi_t (R^n f_* \mathbb{C}_{X'})$. We want to put MHS on H^∞.

First notice: $\exists H^\infty_{\mathbb{R}} \subset H^\infty$ real structure: $H^\infty_{\mathbb{R}} = \psi_t R^n f_* \mathbb{R}_{X'}$

similar $H^\infty_{\mathbb{Q}}, H^\infty_{\mathbb{Z}}$ and $M \in \mathrm{Aut}(H^\infty_{\mathbb{R}/\mathbb{Q}/\mathbb{Z}})$.

$$H^\infty \simeq \bigoplus_{\alpha \in (-1,0]} H^\infty_{e^{-2\pi i \alpha}} \xrightarrow{\cong} \bigoplus_{\alpha \in (-1,0]} C^\alpha \simeq \bigoplus_{\alpha \in (-1,0]} gr_V^\alpha \mathcal{H}(*0) \simeq \bigoplus_{\alpha \in (-1,0]} gr_V^\alpha {}''\mathcal{H}$$

<u>Def</u>: - $F^p H^\infty := \bigoplus_{\alpha \in (-1,0]} F^p H^\infty_{e^{-2\pi i \alpha}}$ where $F^p H^\infty_{e^{-2\pi i \alpha}}$

$$:= \partial_t^{n-p} gr_V^{\alpha + n - p} {}''\mathcal{H} \subset gr_V^\alpha \mathcal{H}(*0) \simeq C^\alpha \simeq H^\infty_{e^{-2\pi i \alpha}}$$

<u>Theorem</u>: (Steenbrink, Scherk, Varchenko)

$$(H^\infty, H^\infty_{\mathbb{R}}, F^\bullet, W^n_\bullet H^\infty_{\mathbb{R},\neq 1} \oplus W^{n+1}_\bullet H^\infty_{\mathbb{R},1}) \text{ is a MHS}$$

<u>Sketch of the proof</u>: in 5 steps

<u>Step 1</u>: recall Brieskorn's construction: $\exists F: Y \to S$ projective such that Y smooth, $(F, x) \underset{R}{\sim} (f, 0)$, $F_{|Y' := Y \setminus F^{-1}(0)}: Y' \to S'$ smooth and $F_{|Y \setminus \bar{X}}: Y \setminus \bar{X} \to S$ C^∞-trivial. Hence the primitive part of $R^n F_* \mathbb{C}_{Y'}$ is VPHS of weight n on S'

$\overset{\text{Schmid}}{\Longrightarrow} \exists\ F^{\bullet}_{\lim} H^{\infty}_{Y}$ where $H^{\infty}_{Y} = \psi_t R^n F_* \mathbb{C}_{Y'}$

and $(H^{\infty}_{Y}, H^{\infty}_{Y,\mathbb{R}}, N^n_Y, F^{\bullet}_{\lim})$ is MHS.

Step 2: Define $Y_\infty := Y' \times_{S'} \tilde{S}'$ as above, then

$$0 \to H^n(Y_\infty, X_\infty) \to H^n(Y_\infty) \xrightarrow{i^*} H^\infty \to H^{n+1}(Y_\infty, X_\infty) \to H^{n+1}(Y_\infty) \to 0$$

Fact: $Y_\infty / X_\infty \underset{h.t.}{\simeq} Y_0 = F^{-1}(0)$, because X_0 is cone

over $\partial B_\varepsilon(0) \cap X_0$ and $Y \setminus \overline{X} \to S$ is trivial, hence

$$0 \to H^n(Y_0) \to H^n(Y_\infty) \to H^\infty \to H^{n+1}(Y_0) \to H^{n+1}(Y_\infty) \to 0 \quad \text{M-M}_Y \text{ inv.}$$

Scherk: F can be chosen s.t. $P^n(Y_\infty) \overset{i^*}{\twoheadrightarrow} H^\infty$

where $P^n(Y_\infty)$ = primitive part of $H^n(Y_\infty)$

Steenbrink: $\exists\ F^{\bullet}_{St} H^\infty$ giving rise to MHS on H^∞

& s.t. i^* is MHS-morphism (i.e. $F^{\bullet}_{St} = i^* F^{\bullet} P^n(Y_\infty)$)

Pf uses resolution of F and coh. mixed Hodge complexes

Step 3: Let $\mathcal{M}_Y := \mathcal{H}^0 F_+ \mathcal{O}_Y = \mathbb{R}^{n+1} p_* \, DR^{\bullet}_{Y\times S/S} (i_{F,+} \mathcal{O}_Y)$
$= \mathbb{R}^{n+1} F_* \left(\Omega^{\bullet}_Y [\partial_t], d - \partial_t \cdot dF \wedge \right)$

$\mathcal{M}_Y$ is $\mathcal{D}_S$-coherent and regular holonomic.

The inclusion $i : X \to Y$ induces a $\mathcal{D}_S$-morphism $i^* : \mathcal{M}_Y \to \mathcal{M}$ with $\mathcal{O}_S$-coherent kernel and cokernel (Steenbrink-Schrak)

again put $F^p \mathcal{M}_Y$ = image of $\mathbb{R}^n F_* \left[\bigoplus_{k=0}^{m-p-1} \Omega^m_Y \partial_t^k \right]$ in $\mathcal{M}_Y$

Claim: $F^{\bullet} \mathcal{M}_{Y|S'}$ is the Hodge filtration on $R^n F_* (F^{-1} \mathcal{O}_{S'})$ of the smooth projective family $F_{|Y'} : Y' \to S'$.

<u>Pf</u>: known $F^p R^n F_*(F^{-1}\mathcal{O}_{S'})$ is defined by "filtration bête" $\mathbb{R}^n F_*(\sigma_{\geq p}\Omega^\bullet_{Y'/S'}) \subset \mathbb{R}^n F_* \Omega^\bullet_{Y'/S'}$

We have filtered morphism:

$$dF\wedge : \Omega^\bullet_{Y'/S'} \to \Omega^{\bullet+1}_{Y'}[\partial_t]$$

$$\underset{[0\to\ldots\to 0\to\Omega^p\to\Omega^{p+1}\to\ldots\to\Omega^{n+1}\to 0]}{\underset{\parallel}{\sigma_{\geq p}\Omega^\bullet_{Y'/S'}}} \mapsto F^p\,\Omega^{\bullet+1}_{Y'}[\partial_t]$$

$$\left(\text{recall: } F^p\Omega^k_Y[\partial_t] := \bigoplus_{\ell=0}^{k-p-1} \Omega^k_Y \partial_t^\ell\right)$$

Take gr_p on both sides $\rightsquigarrow \Omega^p_{Y'/S'} \xrightarrow{\wedge dF} \left[\Omega^{p+r}\partial_t^{r-1}\right]_{r>0}$

and on RHS the differential is $-\partial_t dF\wedge$

Hence $dF\wedge$ is a (filtered) quism $\rightsquigarrow$

$$\mathbb{R}^n F_*(\sigma_{\geq p}\Omega^\bullet_{Y'/S'}) = F^p\mathcal{M}_{Y/S'}$$

<u>Step 4</u>: Consider V-filtration of $\mathcal{M}_Y$ and in particular $\bigoplus_{\alpha\in(-1,0]} gr^V_\alpha\left(V^{>1}\mathcal{M}_Y / t\cdot V^{>1}\mathcal{M}_Y\right)$

$\xrightarrow[\cong]{\alpha} H^n(Y_\infty, \mathbb{C})$. Then we have that

$\forall x \in V_{>1}\mathcal{M}_Y \cap F^p\mathcal{M}_Y$: $\alpha([x]) \in F^p_{lim} H^n(Y_\infty, \mathbb{C})$

Claim: $i^* F^p \mathcal{M}_Y = F^p \mathcal{M}$

(can be achieved by chosing basis of $F^n\mathcal{M}$ by forms which extend to $\mathbb{P}^{n+1} \times S$ with pole order along Y which can be controlled)

This implies that $F^p H^\infty \subset F^p_{St} H^\infty$, idea:

let $x \in F^p H^\infty_{e^{-2\pi i \alpha}}$, $\alpha \in (-1, 0]$, then x is represented by $\tilde{x} \in \partial^{n-p}\, {}''\mathcal{H} \cap V^\alpha = F^p\mathcal{M} \cap V^\alpha \Rightarrow \exists\, \tilde{y} \in F^p\mathcal{M}_Y$ with $i^*\tilde{y} = \tilde{x}$ and $\tilde{y} \in V_{>-1}\mathcal{M}_Y$, hence $\tilde{y}$ yields element $y \in F^p H^n(Y_\infty) \simeq \bigoplus gr_V^\alpha (V_{>1}\mathcal{M}_Y / V_{>0}\mathcal{M}_Y)$ and $i^* y \in F^p_{St} H^\infty$.

Step 5: show that $\dim F^p H^\infty = \dim F^p_{St} H^\infty$

$\square$

complement: higher residue pairings and polarizations.

we have: $0 \to \ker(M_Y - id) \to P^n(Y_\infty) \to H^\infty \to 0$

and $S_Y: P^n(Y_\infty) \times P^n(Y_\infty) \to \mathbb{C}$ intersection form (up to $\pm$).

Put $S: H^\infty_1 \times H^\infty_1 \to \mathbb{C}$, $S(a,b) := S_Y(\hat{a}, N_Y \tilde{b})$ and

$S_{|H^\infty_{\neq 1}} := S_Y$, then S is non-deg, $(-1)^n$-symm. on $H^\infty_{\neq 1}$, $(-1)^{n+1}$-symm. on H^∞_1

Th (Hertling): $H^\infty_{\neq 1}$ (resp H^∞_1) underlies PMHS of weight n ($n+1$)

Consider $R := \mathbb{C}\{\{\partial_t^{-1}\}\} := \left\{ \sum_{i\geq 0} c_i \partial_t^{-i} \mid \sum \frac{c_i}{i!} t^i \in \mathbb{C}\{t\} \right\}$

Def.: (K. Saito, M. Saito, Kashiwara, Hertling)

Define a pairing $P: V_{>1}\mathcal{M} \times V_{>1}\mathcal{M} \to \mathbb{C}\{\{\partial_t^{-1}\}\}$ by:

Let $\alpha, \beta \in (-1, 0]$, $a \in C^\alpha$, $b \in C^\beta$ use $C^\gamma \simeq H^\infty_{e^{-2\pi i \gamma}}$

1.) $\alpha + \beta \notin \mathbb{Z} \Rightarrow P(a,b) = 0$

2.) $\alpha + \beta = -1 \Rightarrow P(a,b) = \frac{1}{(2\pi i)^n} S(a,b) \partial_t^{-1}$

$\alpha = \beta = 0 \Rightarrow P(a,b) = \frac{1}{(2\pi i)^{n+1}} S(a,b) \partial_t^{-2}$

3.) $P(\partial_t^{-1} a, b) = P(a, -\partial_t^{-1} b) = \partial_t^{-1} P(a,b)$

rk: P induces Grothendieck's residue pairing on $''\mathcal{H}/\partial_t^{-1}{}''\mathcal{H}$

Spectral numbers and spectral pairs:

aim: encode monodromy eigenvalues and Hodge numbers in an efficient way.

recall: $''\mathcal{H}$ is lattice inside $(\mathcal{H}(*0), \nabla)$ such that $\partial_t^{-1}: {''\mathcal{H}} \to {'\mathcal{H}} \subset {''\mathcal{H}}$. Hence we can put $Sp(\mathcal{L}, \nabla) := \sum\limits_{\alpha \in \mathbb{Q}} \dim_{\mathbb{C}} gr_V^{\alpha}({''\mathcal{H}}/\partial_t^{-1}{''\mathcal{H}}) \cdot \alpha \in \mathbb{Z}[\mathbb{Q}]$

This is a set of μ rational numbers, all are logarithms of eigenvalues of M.

relation to $F^{\bullet}H^{\infty}$: recall $F^p H^{\infty}_{e^{-2\pi i\alpha}} = \partial^{n-p} gr_V^{\alpha+n-p}\, {''\mathcal{H}}$

so that $\partial^{n-p}: gr_V^{\alpha}\, {''\mathcal{H}}/\partial_t^{-1}{''\mathcal{H}} \xrightarrow{\sim} gr_F^p H^{\infty}_{e^{-2\pi i\alpha}}$

$\Longrightarrow \alpha \in Sp(f)$; $\mathrm{mult}(\alpha) = \dim gr_F^p H^{\infty}_{e^{-2\pi i\alpha}}$, $\alpha \in (n-p-1, n-p]$

Corollary (Varch.-St.) The operators $N \in \mathrm{End}(H^{\infty})$ and $f \cdot$ $\mathrm{End}(gr_V^{\bullet}\, {''\mathcal{H}}/\partial_t^{-1}{''\mathcal{H}}) = \mathrm{End}(gr_V^{\bullet}\, \Omega_f^{n+1})$ have the same Jordan form.

<u>example:</u> quasi-homogeneous case

$f = \sum c_\alpha \underline{x}^\alpha$, $\sum \alpha_i w_i = 1$ for weights $w_1, \dots w_{n+1} \in (0,1]$

then "$\mathcal{H} = \bigoplus_{k=1}^{\mu} \mathcal{O}_S \cdot \underline{x}^{\beta_k} \mathrm{vol}$", where $\mathcal{O}_{X,0} / J_f = \bigoplus_{k=1}^{\mu} \mathbb{C}\, \underline{x}^{\beta_k}$

We have seen that $(t\partial_t)(\underline{x}^{\beta_k} \mathrm{vol}) = \left[wt(\underline{x}^{\beta_k} \mathrm{vol}) - 1 \right] \underline{x}^{\beta_k} \mathrm{vol}$

$= \left[\left(\left(\sum_{i=1}^{n+1} w_i (\beta_i + 1) \right) - 1 \right) \underline{x}^{\beta} \mathrm{vol} \right]$. Hence $x^{\beta_k} \cdot \mathrm{vol}$ is an

elementary section of order $wt(\underline{x}^{\beta_k} \mathrm{vol}) - 1$

and these orders are the spectral numbers

<u>exercise:</u> compute the spectrum for the ADE-sing.

$x^2 + y^{k+1}$, $xy^2 + x^{k-1}$, $x^3 + y^4$, $xy^3 + x^4$, $x^3 + y^5$

<u>semi-quasihomogeneous sing.:</u> Let $f = h + g$ where $\exists\, w_1, \dots w_{n+1}$ with $wt(h) = 1$ & $wt(g) > 1$. Then f is a μ-constant deformation of h and $Sp(f) = Sp(h)$, see below.

Properties of the spectrum:

1.) Symmetry: If $\alpha_1, \dots, \alpha_\mu$ are the spectral numbers of $f:(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ s.t. $\alpha_1 \leq \dots \leq \alpha_\mu$, then $\alpha_i + \alpha_{\mu+1-i} = n-1$

Idea of proof: Pairing $P: V^{>-1} \times V^{>-1} \to \mathbb{C}[[\partial_t^{-1}]]$ with the above properties yields non-degenerate pairing $P: gr_V^{\alpha}({}''\mathcal{H}/\partial_t^{-1}{}''\mathcal{H}) \times gr_V^{n-1-\alpha}({}''\mathcal{H}/\partial_t^{-1}{}''\mathcal{H}) \to \mathbb{C}\partial_t^{1-n}$

hence $\alpha_i + \alpha_{\mu+1-i} = n-1$

2.) Intervall: $Sp(f) \subset (-1,n)$ & $V^{n-1} \subset {}''\mathcal{H} \subset V^{>-1}$

Proof: We have seen that ${}''\mathcal{H} \subset V^{>-1}$, hence $gr_V^{\alpha}\, {}''\mathcal{H} = 0 \quad \forall \alpha \leq -1$, symmetry of $Sp(f) \Rightarrow$

$Sp(f) \subset (-1,n) \Longrightarrow V^{n-1} \subset {}''\mathcal{H}$

3.) Semi-continuity: Consider deformation

$F : (\mathbb{C}^{n+1} \times S, 0) \to (\mathbb{C} \times S, 0)$, S smooth. $\forall (t, \underline{y}) \in \mathbb{C} \times S$,

and $B \subset \mathbb{R}$, let $\deg^{B}_{(t,\underline{y})} Sp(F) := \sum_{x \in Crit(F),\, F(x) = t} \sum_{\alpha \in B} n_\alpha^{Sp(F_y, x)}$

where $Sp(F_y, x) = \sum n_\alpha^{Sp(F_y,x)} \cdot \alpha$. Then $\forall a \in \mathbb{R}$ and all

$(t, \underline{y}) \in \mathbb{C} \times S$, we have: $\deg^{(a,a+1]}_{(t,\underline{y})} Sp(F) \leq \deg^{(a,a+1]}_{(0,\underline{0})} Sp(F)$

in particular (Varchenko): $Sp(F_y) = Sp(f)$ if $\mu(F, 0) = \mu(f)$

4.) Thom-Sebastiani property:

Let $f \in \mathbb{C}\{x_1, \dots, x_{n+1}\}$, $g \in \mathbb{C}\{y_1, \dots, y_{m+1}\}$

Thom-Seb.: $\mu(f+g) = \mu(f) \cdot \mu(g)$,

$H^{n+m+1}(X_\infty^{f+g}) = H^n(X_\infty^f) \otimes H^m(X_\infty^g)$

Theorem (Steenbrink-Scherk): $Sp(f) = \{\alpha_i\}_{i \in \{1, \dots, \mu(f)\}}$

$Sp(g) = \{\beta_j\}_{j \in \{1, \dots, \mu(g)\}} \Rightarrow Sp(f+g) = \{\alpha_i + \beta_j + 1\}_{i,j}$

in particular: $\mu(f + y^2) = \{\alpha_i + \frac{1}{2}\}_{\alpha_i \in Sp(f)}$

relation to the Bernstein-Sato polynomial:

recall $f \in \mathbb{C}\{x_1, \dots, x_{n+1}\}$ arbitrary

Björk: $\exists\, P \in \mathcal{D}_{\mathbb{C}^{n+1},0}[s]$ & $B \in \mathbb{C}[s] \setminus \{0\}$ s.t.

$$P f^{s+1} = B(s) \cdot f^s$$

Let $b(s)$ be the unitary generator of the ideal of these $B(s)$'s $\Rightarrow$ roots of $B(s)$ are logarithms of EV of M on H^∞ (Kashiwara-Malgrange). Question: which logarithms.

clear: $b(s) = (s+1)\widetilde{b}(s)$

Malgrange: $\widetilde{b(s)}$ is the minimal polynomial of $\partial_t \cdot t$ on $\widetilde{''\mathcal{H}} / t \cdot \widetilde{''\mathcal{H}} = \widetilde{''\mathcal{H}} / \partial_t^{-1} \widetilde{''\mathcal{H}}$

where $\widetilde{''\mathcal{H}} = \sum_{i \geq 0} (\partial_t \cdot t)^i \, ''\mathcal{H}$ is the saturation of $''\mathcal{H}$ in $\mathcal{H}(*0)$ by $\partial_t t$ and as such logarithmic.

Brieskorn lattices and non-commutative Hodge structures

aim: define and study an axiomatic framework where a Brieskorn lattice "$\mathcal{H}(f)$ can itself be considered as a (generalized or non-commutative) Hodge structure.

Lets start with the slightly restricted definition

Def: (Cecotti-Vafa, Dubrovin, Hertling, Katzarkov-Kontsevich-Pantev)
A regular non-commutative Hodge structure (reg. ncHodge structure) of weight w is a tuple $(H, \mathcal{L}_{\mathbb{R}}, P)$ where $H \to \mathbb{C}_z$ is a holomorphic VB, $\mathcal{L}_{\mathbb{R}}$ is a $\mathbb{R}$-local system on $\mathbb{C}^*$ and $P: \mathcal{L}_{\mathbb{R}} \otimes \iota^* \mathcal{L}_{\mathbb{R}} \to i^w \underline{\mathbb{R}}_{\mathbb{C}^*}$ is $(-1)^w$-symmetric and non-degenerate, $\iota: \mathbb{C}^* \circlearrowleft$, $z \mapsto -z$ s.t.

1.) $\exists$ iso: $H_{|\mathbb{C}^*} \xrightarrow{\simeq} \mathcal{L}_{\mathbb{R}} \otimes \mathbb{C}$, i.e.

2.) The connection ∇ on H corresponding to $\mathcal{L}$ satisfies $(z^2 \nabla_z) H \subset H$

3.) $P: H \otimes i^* H \to z^w \mathcal{O}_{\mathbb{C}}$ & $[z^{-w} P]: H/zH \times H/zH \to \mathbb{C}$ is non-degenerate.

4.) <u>exponential type + compatible Stokes data</u>: replaced by the simpler condition: (H, ∇) is regular at $z=0$

5.) Pureness/oppositeness: Put $\tilde{H} := \overline{\gamma^* H}$, where $\gamma: \mathbb{P}^1_z \to \mathbb{P}^1_z$, $z \mapsto \bar{z}^{-1}$, then $\tilde{H} \to \mathbb{P}^1 \setminus \{0\}$ is <u>holomorphic</u> and the map $\tau: \mathcal{L}_{|S^1} \to \overline{\mathcal{L}}_{|S^1}$, $a \longmapsto z^{-w} a$ defines a flat isomorphism $H_{|\mathbb{C}^*} \xrightarrow{\tau} \tilde{H}_{|\mathbb{C}^*}$ hence a gluing $\hat{H} := H \cup_{\tau} \tilde{H} \longrightarrow \mathbb{P}^1$ to a $\mathbb{P}^1_z$-holomorphic bundle. Then $\hat{H} \to \mathbb{P}^1$ shall be trivial, i.e. $\hat{H} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus \operatorname{rk} H}$

6.) Polarizations: Define $h : H^0(\mathbb{P}^1, \hat{H}) \times H^0(\mathbb{P}^1, \hat{H}) \to \mathbb{C}$ by $h(a,b) := z^{-w} P(a, \tau b)$, which is hermitian. Call $(H, \mathcal{L}_{\mathbb{R}}, P)$ polarized, iff h is positive definit.

rk: An object satisfying 1.) $\to$ 4.) is called a (regular) TERP-structure

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remarks: 1.) a Hodge structure is a nc Hodge structure: given $(V_{\mathbb{R}}, F^{\cdot}V, \overline{F^{w-\cdot}V})$

then: $H' := V \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}] \rightsquigarrow H' \to \mathbb{C}^*$ (algebraic) VB with connection ∇ s.t. $\ker \nabla = V$.

Put: $H := \bigoplus_{p \in \mathbb{Z}} z^{-p} \cdot F^p V \subset H' \supset \tilde{H} := \bigoplus_{p \in \mathbb{Z}} z^{p-w} \overline{F^p V}$

Then H is $\mathbb{C}[z]$-free, $\tilde{H}$ is $\mathbb{C}[z^{-1}]$-free and $H_{|\mathbb{C}^*} = \tilde{H}_{|\mathbb{C}^*} = H' \rightsquigarrow H \cup_{\mathbb{C}^*} \tilde{H} =: \hat{H} \to \mathbb{P}^1$

Exercise: $\hat{H}$ is trivial $\iff V = \bigoplus_{p\in\mathbb{Z}} F^p V \cap \overline{F^{w-p}V}$

i.e. $F^\bullet$ and $\overline{F^{w-\bullet}}$ are opposed, i.e. $(V_{\mathbb{R}}, F^\bullet V)$

is a pure HS of weight w.

Observation: $(H, \mathcal{L}_{\mathbb{R}}, P)$ ncHodge structure defined

by HS $(V_{\mathbb{R}}, F^\bullet V) \rightsquigarrow (z\partial_z)\, \mathcal{H} \subset \mathcal{H}$ logarithmic pole

and monodromy is trivial. However, one can

modify the above construction starting from

a HS + semisimple automorphism or MHS

with $W = W_\bullet(N)$ + ss. autom $\rightsquigarrow$ ncHodge-

structure generated by elementary sections.

similarly: P(M)HS $\rightsquigarrow$ polarised ncHodge-structures

Why ncHodge?: Kontsevich + al: $(H, \mathcal{L}_{\mathbb{Q}}, P)$

should arise as periodic cyclic homology of

certain triangulated categories (smooth + compact).

Proved only partly and only in special cases!

ncHodge structures and Brieskorn lattices:

given $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C}_t,0)$ i.h.s., $f:X\to S$ good represent.

$\rightsquigarrow$ $''\mathcal{H} := f_*\Omega_X^{n+1}/df\wedge d f_*\Omega_X^{n-1} \subset \mathcal{H}(*0) := \mathcal{H}^n(f_*\Omega^n_{X/S})\otimes\mathcal{O}_S(*0)$

$t, \partial_t^{-1}: ''\mathcal{H}\to''\mathcal{H}$ well-defined. As $''\mathcal{H}_{|S^*}$ is flat

$\Rightarrow \exists!\ (''\widetilde{\mathcal{H}}\to\mathbb{C},\nabla)$ s.t. restr. to S^* is $(''\mathcal{H},\nabla)$

<u>Now</u>: Put $''H := \Gamma^{mod}(\mathbb{C},''\widetilde{\mathcal{H}}) = \{s\in\Gamma(\mathbb{C},''\widetilde{\mathcal{H}})\mid s$ has moderate growth at $t=\infty\}$

$=\Gamma(\mathbb{C},''\widetilde{\mathcal{H}}\cap V_{<\infty}''\widetilde{\mathcal{H}})$, this is $\mathbb{C}[t]$-free and $\mathbb{C}[\partial_t^{-1}]$-free of $\mathrm{rk}=\mu$, put $z:=\partial_t^{-1}$ and let H be the analytic VB corresponding to $''H$, seen as a $\mathbb{C}[z]$-module. Then $t\cdot : ''H_z$ is $z^2\nabla_z$, hence H has pole ≤ 2 at $z=0$.

<u>Check</u>: Saito's higher residue pairings $\rightsquigarrow$ Pairing P as in the definition of ncHodge

<u>Regularity</u>: if $(''H,\nabla)$ is smooth outside $t=0$ $\rightsquigarrow (H,\nabla)$ is regular at $z=0$

rk: Kontsevich program fits to this case:
category of max. CM-modules / matrix factorizations

However: A Brieskorn lattice of a local singularity is not necessarily pure.

Theorem (Hertling): $\exists\, r \gg 0$: "$\mathcal{H}(r\cdot f)$ gives rise to a pure polarised ncHodge-structure.

tool to show this: nilpotent orbits of ncHodge-structures.

Def.: Let $\pi : \mathbb{C}^*_r \times \mathbb{C}_z \to \mathbb{C}_z$; $(r,z) := r\cdot z$. Let $(H, \mathcal{L}_{\mathbb{R}}, P)$ be a TERP-structure. Consider the pull-back $\pi^*(H, \mathcal{L}_{\mathbb{R}}, P)$, which is a variation of TERP-structures on $\mathbb{C}^*$. It is called a nilpotent orbit of TERP resp. ncHodge-structures if $\forall\, r \in \mathbb{C}^*,\ |r| \ll 1$, we have that

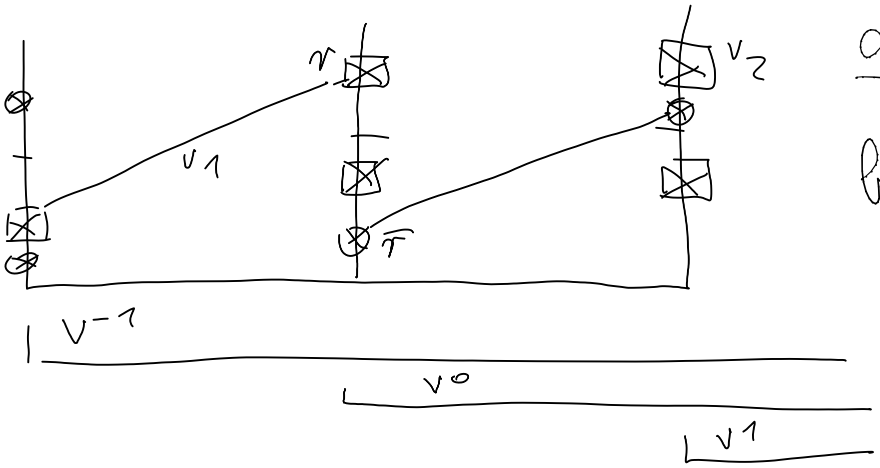
$\left(\pi^*(H, \mathcal{L}_{\mathbb{R}}, P)\right)_{|\mathbb{C}\times\{r\}}$ is pure & polarized.

rk: Let M cplx. manifold, then $(H \to \mathbb{C}\times M, \mathcal{L}_{\mathbb{R}}, P)$ is called a variation of TERP resp. ncHodge -structures, if $(z\nabla_X)H \subset H$, $\forall X \in T_M$ and if the restriction $(H, \mathcal{L}_{\mathbb{R}}, P)_{|\mathbb{C}\times\{x\}}$ is a TERP/ncHodge-str. $\forall x \in M$

example: $H := \mathcal{O}_{\mathbb{C}^*} A_1 \oplus \mathcal{O}_{\mathbb{C}^*} A_2$, $\nabla(A_i)=0$, $\overline{A_1}=A_2$, $S(A_i,A_j)=\delta_{i+j,3}$

$M=\mathbb{C}$: param. space, coord r; $G := \mathcal{O}_{\mathbb{C}\times M}(\underbrace{z^{-1}A_1 + rA_2}_{v_1}) \oplus \mathcal{O}_{\mathbb{C}\times M}\cdot \underbrace{zA_2}_{v_2}$

$Sp = (-1,1)$ constant, $\overline{\gamma^* G} = \mathcal{O}_{(\mathbb{P}^1,0)\times M}(zA_2 + \bar r A_1) \oplus \mathcal{O}_{(\mathbb{P}^1,0)\times M}\, z^{-1}A_1$



check: $G := \pi^* G_{|\mathbb{C}\times\{1\}}$ because $(z\nabla_z - r\nabla_r)(G) \subset G$)

$\hat G$ pure for $|r| \neq 1$, $H^0(\mathbb{P}^1, \hat g) = \mathbb{C}(z^{-1}A_1 + rA_2) \oplus \mathbb{C}(zA_2 + \bar r A_1)$

$|r|=1$, $\hat G_{|\mathbb{P}^1\times\{r\}} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\hat G$ pol. for $|r|>1$

notice: G ncHodge on $M \rightsquigarrow \hat G \to \mathbb{P}^1\times M$ C^∞-bdl. with holomorphic structure in $\mathbb{P}^1$-direction

Hertling, Simpson: $p: \mathbb{P}^1\times M \to M$, $G \to \mathbb{C}\times M$ ncHodge pure pol.

$\rightsquigarrow$ $p_* \hat G \to M$ underlies harmonic bdl.

On the other hand, define filtration as in the singularity case by

$$F^p H^\infty_{e^{-2\pi i\alpha}} := z^{p+1-w} \, gr_V^{\alpha+w-p} \, \mathcal{H}, \text{ where}$$

$H^\infty = \psi_z(\mathcal{L}\otimes\mathbb{C})$. There is a polarizing form S on H^∞ defined by P and we have the correspondence:

<u>Theorem (Hertling-S. '06)</u>: A TERP-structure $(H,\mathcal{L},P)$ induces a nilpotent orbit, i.e., $\pi^*(H,\mathcal{L},P)$ is pure polarized $\forall |r|\ll 1$ iff $(H^\infty, H^\infty_{\mathbb{R}}, F^\bullet, S, N)$ is a PMHS of weight $w-1$ on $H^\infty_{\neq 1}$ resp weight w on H^∞_1. Here $N = \log M_u$ with M monodromy of $\mathcal{L}$.

<u>application</u>: $TERP(r\cdot f) = \pi_{r^{-1}} TERP(f) \Rightarrow TERP(r f)$ p.p $\forall |r| \gg 0$

classifying spaces and period maps: imitate classical construction for (commutative) Hodge structures

Th (Hertling-S. '08): Fix local system $\mathcal{L}_{\mathbb{R}}$ and pairing P, $w \in \mathbb{Z}$, $\alpha_1 \in \mathbb{Q}$, $w - \alpha_1 > \alpha_1$

$$\mathcal{M}_{BL} := \left\{ H \to \mathbb{C} \;\middle|\; \begin{array}{l} (H, \mathcal{L}, P) \text{ is (reg) nc Hodge} \\ Sp(H, \nabla) \subset [\alpha_1, \underbrace{w - \alpha_1}_{\alpha_\mu}] \end{array} \right\}$$

is proj. variety, stratified by locally closed subsets U_{Sp} of points with constant spectrum Sp. Given V nc Hodge on $M \Longrightarrow$ period map $\phi_{BL} : \widetilde{M} \to \mathcal{M}_{BL}$.

idea: H is determined by finite-dim. data

$$H / V^{> \alpha_\mu - 1} \subset V^{\alpha_1} / V^{> \alpha_\mu - 1}$$

Th (Hertling '97): Strata U_{Sp} are affine fibre bundles over classifying spaces $\check{D}_{PMHS}$ of PMHS, in particular smooth.

Consider universal bundle $\mathcal{G} \to \mathbb{C}_z \times \mathcal{M}_{BL}$ s.t. $\mathcal{G}_{|\mathbb{C}_z \times \{H\}} = H$.

$\exists$ real-analytic family $\widehat{\mathcal{G}} \to \mathbb{P}^1_z \times \mathcal{M}_{BL}$ extending $\mathcal{G}$

Put $\mathcal{M}_{BL}^{p.p.} := \{x \in \mathcal{M}_{BL} \mid \hat{g}_{|\mathbb{P}^1 \times \{x\}}$ is pure polarized $\}$.

Then we have a canonical hermitian metric h on $T_{\mathcal{M}_{BL}^{p.p.}}$ and the folling result.

Th: (Hertling-S. '08)

a) $(\mathcal{M}_{BL}^{p.p.}, d_h)$ is metric complete.

b) Fix Sp, consider $\mathcal{U}_{Sp}^{PP} := \mathcal{U}_{Sp} \cap \mathcal{M}_{BL}^{p.p.}$. $\exists$ coherent subsheaf of $T_{\mathcal{U}_{Sp}^{PD}}$ called $T^{hor}_{\mathcal{U}_{Sp}^{PD}}$ s.t.:

i) $G \to \mathbb{C} \times M$ variation of pure pol gen HS with Sp = const

$\phi_{BL} : \tilde{M} \to \mathcal{U}_{Sp}^{PD} \Rightarrow d\phi_{BL} \subset T^{hor}_{\mathcal{U}_{Sp}^{PP}}$

ii) hol. sectional curvature on $T^{hor}_{\mathcal{U}_{Sp}^{PD}} \setminus \{0\}$ $\quad h(R(\xi,\bar{\xi})\xi,\xi) / h^2(\xi,\xi) < c < 0$

c) Supp. that monodromy repr. respects a lattice $\subset G_{\mathbb{R}}$

$\Rightarrow \exists$ good quotient $\mathcal{M}_{BL}^{PD}/\Gamma$ cplx space, s.t. $\phi_{BL} : M \to \mathcal{M}_{BL}^{PD}/\Gamma$

<u>applications</u>: 1.) Vppnc Hodge on $\mathbb{C}^n \rightsquigarrow$ trivial

2.) Vppnc Hodge on $X \setminus Y$ with $\operatorname{codim}_X Y \geq 2 \rightsquigarrow \exists$ extension

<u>example above</u>:

$\mathcal{M}_{BL} = \mathbb{P}^1$, $\mathcal{U}_{(-1,1)} = M$, $\mathcal{U}_{(0,0)} = \{\infty\}$, $\mathcal{M}_{BL}^{PD} = \{r \in \mathbb{P}^1 \mid |r| > 1\}$

$\rho = r^{-1}$, $h(\partial_\rho, \partial_\rho) = \dfrac{1}{\rho^2 \bar{\rho}^2 \cdot (1 - \rho^{-1}\bar{\rho}^{-1})^2} \xrightarrow{\rho \to 0} 1 < \infty \Rightarrow \mathcal{U}_{(1,-1)}$ not complete