Universität Mannheim
Lehrstuhl für Mathematik VI

Christian Sevenheck
13.06.2007

Exercises Algebraic Geometry
Sheet 8 - solutions

1. **Exercise:** Calculate the divisor of x/y on the Segre quadric $X = V(xy - zw) \subset \mathbb{P}^3$.

   It is easy to see that $(x, xy-zw) = (x, zw) = (x, z) \cap (x, w)$ and similarly $(y, xy-zw) = (y, zw) = (y, z) \cap (y, w)$. If we denote by $X_1 = V(x, z) \subset X$, $X_2 = V(x, w) \subset X$ and by $Y_1 = V(y, z) \subset X$, $Y_2 = V(y, w) \subset X$, then the divisor of the rational function x/y is given as

   $$\left(\frac{x}{y}\right) = X_1 + X_2 - Y_1 - Y_2 \in \mathrm{Div}(X)$$

2. **Exercise:** Determine the divisor of $x_1/x_0 - 1$ on the circle $X = V(x_1^2 + x_2^2 - x_0^2) \subset \mathbb{C}^3$.

   The function $x_1/x_0 - 1$, written as $\frac{x_1 - x_0}{x_0}$ gives the divisor $2X_2 - X_{12} - X_{21}$ where we write $X_2 = V(x_2, x_1 - x_0) \subset X$, $X_{12} = V(x_1 + ix_0, x_2) \subset X$ and $X_{21} = V(x_1 + ix_0, x_2)$. To see the multiplicities, we argue as above, e.g., we have that $x_0 - x_1 = x_2^2(x_0 + x_1)^{-1} \in k[X]_{(x_1 - x_0, x_2)}$ where x_2 is a generator of the maximal ideal of the local ring $k[X]_{(x_1 - x_0, x_2)}$.

3. **Exercise:** Calculate the divisor of y on the cone $X = V(xy - z^2) \subset k^3$.

   Let us first describe the set-theoretic vanishing locus of the function y on X: Obviously, we have $(xy - z^2, y) = (y, z^2)$. This means that the divisor of y is supported by the irreducible variety $Y = V(y, z) \subset X$. In order to determine the multiplicity of y, let us consider the local ring $\mathcal{O}_{X,Y}$, i.e., the localization of the coordinate ring $k[X] = k[x, y, z]/(xy - z^2)$ at the prime ideal $(y, z) \subset k[X]$. This localization has a maximal ideal generated by z, because in this localization, we have that $y = x^{-1}z^2 \subset (z)$. The very same equation tells us that the multiplicity $\nu_Y(y)$ we are looking for is two: y is a z^2 times a unity $(x{-}1)$. We conclude that

   $$(y) = 2Y \in \mathrm{Div}(X)$$

4. **Exercise:** Prove that for any smooth variety X, $Cl(X \times k) \cong Cl(X)$.

   We define the map

   $$\begin{array}{rcl} \pi^* : \mathrm{Div}(X) & \longrightarrow & \mathrm{Div}(X \times k) \\ D = \sum_i a_i X_i & \longmapsto & \pi^* D = \sum_i a_i \pi^{-1}(X_i) \end{array}$$

   where $\pi : X \times k \to X$ denotes the projection. If $D = (f/g)$ is the divisor of a rational function $f/g \in k(X)$, then $\pi^* D$ is just the divisor of the same f/g, this time seen as an element in $k(X)(t)$ (t being the coordinate on k in $X \times k$). Therefore, the map π^* sends divisors of rational functions on X to divisors of rational functions on $X \times k$ and thus descends to a map $\pi^* : Cl(X) \to Cl(X \times k)$.

   Next we would like to show that π^* is both injective and surjective. In order to do that, we need to discuss the possible irreducible codimension one subvarieties of $X \times k$: Two types of such prime divisors C can occur: Either C is dominant over X, i.e., i.e., $\pi(C)$ is a dense subset of X, or the closure of $\pi(C)$ is a prime divisor of X. There cannot be any other type of prime divisors on $X \times k$, because if $\pi(C)$ would be of dimension strictly smaller then $\dim(X) - 1$, then one could find a chain $\pi(C) \subsetneq \widetilde{C} \subsetneq X$, and $\pi^{-1}(\widetilde{C})$ would lie between C and $X \times k$ so that C would not be a divisor on $X \times k$.

Let us show that π^* is injective: Suppose that $D \in \mathrm{Div}(X)$ and that $\pi^*(D) = (f/g)$ for some $f, g \in k[X][t]$ with f, g relatively prime. Then f and g are necessarily elements in $k[X]$, as otherwise $\pi^*(D)$ would have components which are dominant over X. These cannot be of the form $\pi^{-1}(C_i)$ for some prime divisor $C_i \in \mathrm{Div}(X)$, so that (f/g) would not be of the form $\pi^*(D)$.

It follows from this discussion that any prime divisor $\widetilde{C}$ on $X \times k$ which projects to (a dense subset of) a divisor C on X is of the form $\widetilde{C} = \pi^*(C)$, in particular, $\widetilde{C}$ is in the image of π^*. In order to prove surjectivity of π^*, we need to show that any prime divisor C on $X \times k$ dominant over X is linearly equivalent to a divisor which projects to a divisor on X. Let $I \subset k[X][t]$ be the defining ideal of C. Consider the map $k[X][t] \to k(X)[t]$ and let $\widetilde{I} \subset k(X)[t]$ be the image of I. The ring $k(X)[t]$ is a principal ideal domain (because $k(X)$ is a field) so that $\widetilde{I} = (f)$ for some $f \in k(X)[t] \subset k(X)(t)$. This means that we can consider the divisor $(f) \in \mathrm{Div}(X \times k)$ of f, and the fact that f is an element in $k(X)[t]$ shows that this divisor contains C and perhaps some other divisor of type $\pi^{-1}(D)$ with $D \in \mathrm{Div}(X)$, but no other divisor dominant over X. This shows that C is linearly equivalent to a divisor in the image of $\mathrm{Div}(X) \to \mathrm{Div}(X \times k)$, so that π^* on the class groups is surjective.

Remark: The statement just proved is valid in a more general context, namely, it is sufficient to suppose that X is regular in codimension one, that is, that the (closed) subset of points x such that X is singular at x is of codimension at least two in X.