Universität Mannheim
Lehrstuhl für Mathematik VI

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Exercises Algebraic Geometry
Sheet 7 - solutions

1. **Exercise: The Cremona transformation:** Let $X \subset \mathbb{P}^2$ be given by $X = V(I)$ with $I = (xy, xz, yz)$, i.e., $X = \{(0:0:1)\} \cup \{(1:0:0)\} \cup \{(0:1:0)\}$. Denote by $\widetilde{\mathbb{P}}^2$ the blowup of $\mathbb{P}^2$ in X. Consider the morphism

$$\begin{array}{rcl} f : \mathbb{P}^2\backslash X & \longrightarrow & \mathbb{P}^2 \\ (x:y:z) & \longmapsto & (xy:yz:xz) \end{array}$$

Show that there is no morphism $F : \mathbb{P}^2 \to \mathbb{P}^2$ with $F_{|\mathbb{P}^2\backslash X} = f$. Show moreover that there exists an isomorphism $\widetilde{F} : \widetilde{\mathbb{P}}^2 \to \widetilde{\mathbb{P}}^2$ which extends f.

(a) Suppose that there would be an extension F of f to the whole projective plane. In particular, on the chart $\{x \neq 0\}$, F would be an extension of the map $k^2\backslash\{(0,0)\} \to \mathbb{P}^2$ given by $(y,z) \mapsto (yz : z : y)$ to the origin in k^2. But then this map would send this origin to $(0:0:0)$ which is not a point of $\mathbb{P}^2$.

(b) It is a general statement on rational maps $f = (f_0, \ldots, f_k) : X \dashrightarrow \mathbb{P}^k$ that they extend to a regular map $\widetilde{F} : \widetilde{X} \to \mathbb{P}^k$ where $\widetilde{X}$ is the blowup of X in the ideal $(f_0, \ldots, f_k)$. In our situation, this shows that there is a morphism $\widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$. But we can say much more: Consider the subvariety Y of $\mathbb{P}^2 \times \mathbb{P}^2$ given by the equations $xc-yb, xc-za$, where $(x:y:z)$ and $(a:b:c)$ are the coordinates on the two factors.

Lemma 1. *The two projections $p_1, p_2 : Y \to \mathbb{P}^2$ induced from the projections $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ both identify Y with the blowup $\widetilde{\mathbb{P}}^2$. The composition $p_2 \circ p_1^{-1}$ is well-defined on $\mathbb{P}^2\backslash X$ and coincides with f, so that the identity on Y gives an extension of f to an automorphism of $\widetilde{\mathbb{P}}^2$.*

Proof. We will show that the preimage $p_1^{-1}(\{z \neq 0\})$ is isomorphic to the blowup of k^2 in the origin. This will be sufficient, as blowing up is a local construction. The affine coordinates on $\{z \neq 0\}$ are (x,y) and the preimage is given by the ideal $(xc-yb, xc-a) \subset k[x,y][a,b,c]$, so that the coordinate ring of the preimage is equal to $k[x,y][c,b]/(xc-yb)$. This obviuosly defines the required blowup. The same is true for the other projection. As the blowup is an isomorphism outside the exceptional locus, we conclude that $p_2 \circ p_1^{-1}$ is well-defined on $\mathbb{P}^2\backslash X$ and an that it is an isomorphism of this open subset of $\mathbb{P}^2$ (i.e., a birational transformation of $\mathbb{P}^2$). Moreover, it is clear that the map $(a:b:c) := (xy:xz:yz)$ defined on $\mathbb{P}^2\backslash X$ (i.e., the map f) *is* just the composition $p_2 \circ p_1^{-1}$ (because Y contains the graph of f as an open subset). $\square$

2. **Exercise:** Let $X \subset k^n$ be affine with $J = I(X)$. Suppose that $0 \in X$, and denote by $\widetilde{I} = ([x_1], \ldots, [x_n]) \subset k[X]$ the ideal of $0 \in X$. Then the coordinate ring $k[C_{X,0}]$ of the tangent cone of X at 0 is the algebra $\oplus_{k\geq 0}\widetilde{I}^k/\widetilde{I}^{k+1}$.

Solution: First note the following obvious ring isomorphism:

$$\bigoplus_{k\geq 0} \frac{\widetilde{I}^k}{\widetilde{I}^{k+1}} \cong \frac{k[\underline{x}]}{J+I} \oplus \frac{I}{J+I^2} \oplus \frac{I^2+J}{I^3+J} \oplus \ldots \oplus \frac{I^k+J}{I^{k+1}+J} \oplus \ldots$$

where this time $I = (x_1, \dots, x_n) \subset k[\underline{x}]$ (note that $0 \in X$ implies $J \subset I$). Define the following isomorphism of graded k-algebras (on the left hand side, the grading is the one induced from the usual grading of $k[\underline{x}]$ on the right hand side, the grading is by the given decomposition)

$$\begin{array}{ccc} \frac{k[\underline{x}]}{LT(J)} & \cong & \frac{k[\underline{x}]}{J+I} \oplus \frac{I}{J+I^2} \oplus \frac{I^2+J}{I^3+J} \oplus \dots \frac{I^k+J}{I^{k+1}+J} \oplus \dots \\ 1 & \longmapsto & ([1], 0, \dots) \\ x_i & \longmapsto & (0, [x_i], \dots) \end{array}$$

Here $LT(g)$ is the leading term of g with respect to the partial order given by the degree (i.e., this is not a monomial ordering as $LT(g)$ is not a monomial). We first show that this is well-defined, i.e., that it sends any $f^{(i)} \in LT(J)$ to zero: obviously $f^{(i)}$ has zero image in $(I^k + J)/(I^{k+1} + J)$ for any $k < i$ or $k > i$, but also for $k = i$: the difference $f - f^{(i)}$ lies in $I^{k+1} \subset I^{k+1} + J$, so that $f^{(i)} \equiv f$ in $(I^k + J)/(I^{k+1} + J)$ and thus $[f^{(i)}] = 0$. Moreover, the above map is obviously surjective, and injectivity follows by the same argument: suppose that for any homogenous $g \in k[\underline{x}]$, the image is zero in $\oplus_{k \geq 0} I^k/(I^{k+1} + J)$, then it is zero on each factor, in particular in $I^k/(I^{k+1} + J)$ with $k = \deg(g)$, so that $g \in I^{k+1} + J$ so that there is $\widetilde{g} \in I^{k+1}$ with $g + \widetilde{g} \in J$, this implies $g \in LT(J)$.

3. **Exercise:** For an affine variety $X \subset k^n$ (with $J = I(X)$) containing the origin, let $\widetilde{X}$ be the blowup of X in the origin (i.e., in the ideal $\widetilde{I} = ([x_1], \dots, [x_n])$). Show that the exceptional divisor $E \subset \widetilde{X}$, seen as an algebraic set in $\mathbb{P}^{n-1} = \{0\} \times \mathbb{P}^{n-1} \subset k^n \times \mathbb{P}^{n-1}$ is contained in the projective zero locus $V_p(LT(J))$ of the initial ideal $LT(J)$ (recall that for a homogenous ideal $J \subset k[y_1, \dots, y_n]$, we denote by $V_p(J) \subset \mathbb{P}^{n-1}$ the projective variety given the vanishing of the elements in J and by $V_a(J) \subset k^n$ its affine cone. Note further that E is actually equal to $V_p(LT(J))$, but we do not prove this here).

   Solution: $\widetilde{X}$ is by definition the closure in $k^n \times \mathbb{P}^{n-1}$ of the graph Γ of $X \backslash \{0\} \to \mathbb{P}^n$ given by $y_i = x_i$ where $(x_1, \dots, x_n), (y_1 : \dots : y_n)$ are coordinates on $k^n \times \mathbb{P}^{n-1}$. Let $J = (f_1, \dots, f_k)$, with $f_i = LT(f_i) + Tail(f_i)$. Then the functions $\widetilde{f_i} := \left(\frac{y_j}{x_j}\right)^{d_i} \cdot f_i$ (with $d_i = \deg(LT(f_i))$) are also zero on Γ and thus on $\widetilde{X}$. But $\widetilde{f_i} \in LT(f_i)(y_1, \dots y_n) + (x_1, \dots, x_n)$, because the relations $\frac{y_k}{x_k} = \frac{y_j}{x_j}$ allows to rewrite the term $\left(\frac{y_j}{x_j}\right)^{d_i}$ such that each monomial of $LT(f_i)$ gets its x-variables replaced by y's. This implies that $(\widetilde{f_i})_{|E} = (\widetilde{f_i})_{|x_i=0} = LT(f_i)(y_1, \dots, y_n)$ is zero. This shows that $I(E) \supset LT(J) \subset k[y_1, \dots, y_n]$. The other direction can actually be shown by the last exercise, it is possible to prove that the coordinate ring of $\widetilde{X}$ is the $k[X]$-algebra $k[\widetilde{X}] = k[X] \oplus \widetilde{I} \oplus \widetilde{I}^2 \oplus \dots$ (where $\widetilde{I} = ([x]_1, \dots, [x]_n) \subset k[X]$) and that the exceptional divisor is $k[\widetilde{X}]/\widetilde{I}k[\widetilde{X}] \cong \oplus_{k \geq 0} \widetilde{I}^k/\widetilde{I}^{k+1}$.