Universität Mannheim
Lehrstuhl für Mathematik VI

Christian Sevenheck
28.03.2007

Exercises Algebraic Geometry
Sheet 5

1. (a) Let X be a prevariety.
   i. Show that X is noetherian.
   ii. Let $U \subset X$ be open. Show that U is a prevariety.

   (b) Let $X_1, \dots, X_n$ be prevarieties and suppose that X is obtained by gluing the X_i's along open subsets $U_{ij} \subset X_i$. Show that X is irreducible.

2. (a) Let $\mathcal{F}$ be a sheaf on a topological space X and $s \in \mathcal{F}(U)$, where $U \subset X$ is open. Define the support of s in U by
$$\operatorname{supp}(s) := \{x \in U \mid s(x) \neq 0\}$$
   Show that $\operatorname{supp}(s)$ is closed in U. Moreover, let
$$\operatorname{supp}(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \neq 0\}$$
   (recall that $\mathcal{F}_x$ denotes the germ of $\mathcal{F}$ at $x \in X$). Try to give an example where $\operatorname{supp}(\mathcal{F})$ is not closed.

   (b) Show that for a variety X and $U \subset X$ open, there are no sections $f \in \mathcal{O}_X(U)$ with $\emptyset \subsetneq \operatorname{supp}(f) \subsetneq U$

   (c) Let X be a variety over an algebraically closed field k. Let Y a closed subvariety. For any $U \subset X$ open, consider the ideal
$$\mathcal{I}_Y(U) := \{f \in \mathcal{O}_X(U) \mid f(x) = 0 \ \ \forall x \in Y \cap U\}$$
   Show that $U \mapsto \mathcal{I}_Y(U)$ defines a sheaf, called the ideal sheaf of Y in X.

3. (a) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Define $\mathcal{K}er(\varphi)(U) := ker(\varphi(u) : \mathcal{F}(U) \to \mathcal{G}(U))$, $\widetilde{\mathcal{I}m}(\varphi)(U) := Im(\varphi(u) : \mathcal{F}(U) \to \mathcal{G}(U))$ and $\widetilde{\mathcal{C}oker}(\varphi)(U) := coker(\varphi(u) : \mathcal{F}(U) \to \mathcal{G}(U))$. Show that $\mathcal{K}er(\varphi)$, $\widetilde{\mathcal{I}m}(\varphi)$ and $\widetilde{\mathcal{C}oker}(\varphi)$ are presheaves, but that only $\mathcal{K}er(\varphi)$ is a sheaf. We denote by $\mathcal{C}oker(\varphi)$ resp. $\mathcal{I}m(\varphi)$ the sheafification of $\widetilde{\mathcal{C}oker}(\varphi)$ resp. $\widetilde{\mathcal{I}m}(\varphi)$. We say that φ is injective iff (if and only if) $\mathcal{K}er(\varphi) = 0$.

   (b) Show that the sheafification $\mathcal{F}^+$ of a presheaf $\mathcal{F}$ has the following universal property: for any sheaf $\mathcal{G}$ and any morphism (of presheaves) $\psi : \mathcal{F} \to \mathcal{G}$, there is a unique morphism $\Psi : \mathcal{F}^+ \to \mathcal{G}$ such that $\Psi \circ \phi = \psi$, where $\phi : \mathcal{F} \to \mathcal{F}^+$ is the morphism of presheaves defined in exercise two of the last sheet.

   (c) Show that for any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves, the universal property gives a morphism $\mathcal{I}m(\varphi) \to \mathcal{G}$ which is injective. This means that we can consider $\mathcal{I}m(\varphi)$ as a subsheaf of $\mathcal{G}$ (in general, a subsheaf $\mathcal{F}'$ of $\mathcal{F}$ is a sheaf such that $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$ for any U and the restriction maps ϕ_{UV} of $\mathcal{F}'$ are induced by those of $\mathcal{F}$). We say the φ is surjective iff $\mathcal{I}m(\varphi) = \mathcal{G}$. Show that φ is surjective iff for any $U \subset X$ open and for any $g \in \mathcal{G}(U)$ there is a covering $U = \cup_i U_i$ and $f_i \in \mathcal{F}(U_i)$ such that $\varphi(U_i)(f_i) = g_{|U_i}$. Show that we do not have neccessarily surjectivity of $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$.

4. Let $>$ be a monomial order on $k[x_1, \ldots, x_n]$. Define

$$S_>^{-1}k[\underline{x}] := \left\{ \frac{f}{g} \mid f, g \in k[\underline{x}], \mathrm{LM}(g) = 1 \right\}$$

Show that

(a) $k[\underline{x}] \subset S_>^{-1}k[\underline{x}] \subset k[\underline{x}]_{(\underline{x})}$ (Recall that the latter ring is the localization of $k[\underline{x}]$ at the complement of the maximal ideal $(x_1, \ldots, x_n)$.)

(b) $k[\underline{x}] = S_>^{-1}k[\underline{x}]$ iff $>$ is global.

(c) $k[\underline{x}]_{(x)} = S_>^{-1}k[\underline{x}]$ iff $>$ is local (this means that for any monomial $x^I \neq 1$ we have $1 > x^I$).

5. Compute by hand a standard representation in the following situations.

(a) $>:=$ *lex* (called *lp* in Singular, see below) sowie $>:=$ *lex*(*loc*) (called *ls* in Singular), $f = 1$, $G := \{x - 1\}$.

(b) $>:=$ *Dlex* (called *dp* in Singular), $f = x^4 + y^4 + z^4 + xyz$, $G := J_f := \{\partial_x f, \partial_y f, \partial_z f\}$.

6. Check by hand whether the following functions are contained in the respective ideals.

(a) $f = xy^3 - z^2 + y^5 - z^3, I_1 = (-x^3 + y, x^2y - z) \subset \mathbb{Q}[x, y, z]$.

(b) $g = x^3z - 2y^2, I_2 = (yz - y, xy + 2x^2, y - z) \subset \mathbb{Q}[x, y, z]$.

7. Determine the closure of the image of the following map

$$\begin{array}{rcl} \varphi : \mathbb{C}^2 & \longrightarrow & \mathbb{C}^4 \\ (s, t) & \longmapsto & (s^3, s^2t, st^2, t^3) \end{array}$$

Is $\mathrm{Im}(\varphi) = \overline{\mathrm{Im}(\varphi)}$?

(Hint: For this and the following exercises, you might use **Singular**. Computations in Singular are done by declaring a ring like

```
ring R=0,(x,y,z),lp;
```

here 0 is the characteristic, x, y, z are the variables and *lp* is the monomial order (lex in our case). Then ideals are defined like

```
ideal i=(f,g,h);
```

where f, g, h are polynomials in x, y, z. The most important procedure is

```
ideal j=std(i);
```

which computes a standard basis (i.e., Gröbner basis in our case) of the ideal i. See

`http://www.singular.uni-kl.de`

for further information).

8. Do exercise 6. in the ring $\mathbb{Q}[x, y, z]_{(x,y,z)}$.

9. Compute the kernel of the homomorphism of rings

$$\begin{array}{rcl} \varphi : \mathbb{Q}[x, y, z] & \longrightarrow & \mathbb{Q}[t]/(t^{12}) \\ (x, y, z) & \longmapsto & (t^5, t^7 + t^8, t^{11}) \end{array}$$