Multivariate sparse FFT based on rank-1 Chebyshev lattice sampling

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***Abstract*—We present a method for the fast reconstruction of high-dimensional sparse algebraic polynomials in Chebyshev form and for the fast approximation of multivariate non-periodic functions from samples, when the frequency locations belonging to the non-zero or largest Chebyshev coefficients are unknown. We only assume that we have given a generally very large index set of possible frequencies, e.g. a d-dimensional full grid. We determine the frequency locations in a dimension-incremental way from samples along reconstructing rank-1 Chebyshev lattices. We demonstrate the high performance of the proposed method in numerical examples in up to 15 dimensions.**

I. Introduction

We consider algebraic polynomials $a_I\colon [-1,1]^d \to \mathbb{R}$ in Chebyshev form,

$$a_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{a}_{\boldsymbol{k}} \prod_{t=1}^{d} T_{k_t}(x_t),\ \hat{a}_{\boldsymbol{k}} \in \mathbb{R}, \quad (1)$$

where $d \in \mathbb{N}$ is the dimension, $I \subset \mathbb{N}_0^d$ is a non-negative index set, $|I| < \infty$, and $T_{\boldsymbol{k}}\colon [-1,1]^d \to [-1,1]$, $T_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{t=1}^{d} T_{k_t}(x_t)$, $\boldsymbol{k} \in \mathbb{N}_0^d$, are multivariate Chebyshev polynomials built from univariate Chebyshev polynomials of the first kind $T_l\colon [-1,1] \to [-1,1]$, $T_l(x) := \cos(l \arccos x)$, for frequencies $l \in \mathbb{N}_0$. For each $l \in \mathbb{N}_0$, T_l is an algebraic polynomial of degree l restricted to the domain $[-1,1]$. We remark that multivariate algebraic polynomials with hyperbolic cross index sets $I = H_n^d := \{\boldsymbol{k} \in \mathbb{N}_0^d\colon \prod_{t=1}^{d} \max(1,k_t) \le n\}$ have already been used for approximations in sparse high-dimensional spectral Galerkin methods, cf. [1, Section 8.5]. Moreover, multivariate algebraic polynomials a_I are used in Petrov-Galerkin discretizations of high-dimensional parametric PDEs, see e.g. [2], [3] for compressed sensing based approaches.

For a given arbitrary index set $I \subset \mathbb{N}_0^d$, $|I| < \infty$, a method for the fast evaluation of an arbitrary polynomial a_I from (1) at the nodes $\boldsymbol{x}_j := \cos(\frac{j}{M}\pi\boldsymbol{z})$, $j = 0,\ldots,M$, of an arbitrary rank-1 Chebyshev lattice

$$\mathrm{CL}(\boldsymbol{z}, M) := \left\{\boldsymbol{x}_j := \cos\left(\tfrac{j}{M}\pi\boldsymbol{z}\right)\colon j = 0,\ldots,M\right\} \subset [-1,1]^d$$

with generating vector $\boldsymbol{z} \in \mathbb{N}_0^d$ and size parameter $M \in \mathbb{N}_0$ was discussed in [4], which only uses easy-to-compute index transforms and a single one-dimensional discrete cosine transform (1d DCT). For a more general definition of d-dimensional rank-k Chebyshev lattices, we refer to [5]. Moreover, reconstruction properties, i.e., conditions on the generating vector $\boldsymbol{z}$ and the size parameter M, such that the fast and exact reconstruction of all Chebyshev coefficients $\hat{a}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, is possible from samples $a_I(\boldsymbol{x}_j)$, $j = 0,\ldots,M$, were discussed in [4] and the term reconstructing rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, M, I)$ was introduced. Methods for obtaining a reconstructing rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, M, I)$ based on a component-by-component (CBC) construction approach were presented in [4]. For general CBC constructions of integration lattices, we refer to the survey [6] and the references therein. The Chebyshev coefficients $\hat{a}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, can be obtained by applying a single 1d DCT to the samples $a_I(\boldsymbol{x}_j)$, $j = 0,\ldots,M$, followed by easy-to-compute index transforms. These computations require $\mathcal{O}(M \log M + d\,|\mathcal{M}(I)|)$ arithmetic operations, where $\mathcal{M}(I) := \{\boldsymbol{h} \in \mathbb{Z}^d\colon (|h_1|,\ldots,|h_d|) \in I\}$ is the extended symmetric index set and $|\mathcal{M}(I)|$ denotes the cardinality of this index set.

Until now, we assumed that we know the frequency index sets I, which contain the locations of the non-zero Chebyshev coefficients $\hat{a}_{\boldsymbol{k}} \neq 0$ of a multivariate algebraic polynomial a_I. However, these frequency locations are unknown in many cases. In general, it is a very challenging task to obtain such frequency index sets I, especially in higher dimensions. We denote a method which determines a set of unknown frequency locations I and corresponding non-zero or approximately largest Chebyshev coefficients in a fast way by sparse FFT. For the periodic case, many approaches exist. For a nice introduction to compressive sensing, we refer to the monograph [7]. Moreover, one-dimensional sparse FFT methods based on efficient filters were introduced in [8], [9], and multivariate methods followed, see e.g. [10] and the references therein. Further methods for the sparse FFT exist, e.g. based on the Chinese Remainder Theorem [11], shifted sampling [12], randomized Kronecker substitution [13] and dimension-incremental projections [14], [15]. Moreover, approaches based Prony's method were proposed in [16]–[18].

In this work, we present a multivariate sparse FFT method for the non-periodic case, which determines the unknown frequency locations I in a dimension-incremental way, one component at a time. Our method is based on dimension-incremental projections parallel to the coordinate axes, cf. [15] and the references therein for a similar idea in the periodic case. As sampling sets, we employ the nodes of reconstructing rank-1 Chebyshev lattices $\mathrm{CL}(\boldsymbol{z}, M, I)$, and we make use of fast reconstruction algorithms. We only require that a very large superset $\Gamma \subset \mathbb{N}_0^d$ of possible frequency locations, the search domain, is known and that we are able to obtain sampling values from the function under consideration. The proposed method is successfully applied in numerical tests for the reconstruction of high-dimensional algebraic polynomials in Chebyshev form a_I in up to 15 dimensions as well as for the approximation of a 9-dimensional non-periodic test function. Our method may be applied for determining sparse approximate solutions of partial differential equations with random coefficients.

II. Method

In this section, we describe the dimension-incremental reconstruction approach. First, we introduce additional notation from [4]. For $M \in \mathbb{N}$ and $l \in \mathbb{Z}$, we define the even-mod relation

$$l \operatorname{emod} M := \begin{cases} l \bmod (2M), & l \bmod (2M) \leq M, \\ 2M - (l \bmod (2M)) & \text{else}, \end{cases}$$

as well as $l \operatorname{emod} 0 := 0$ in the special case $M = 0$. Additionally, we define the index sets $\mathcal{M}_\nu(I) := \{\boldsymbol{h} \in \mathcal{M}(I) \colon h_\nu \geq 0\}$, $\nu \in \{1, \ldots, d\}$, which contain all frequencies $\boldsymbol{k} \in I$ and versions of these frequencies (repeatedly) mirrored at all coordinate axes except the ν-th. Moreover, we denote the projection of a frequency $\boldsymbol{k} \in \mathbb{Z}^d$ to the components $\boldsymbol{i} := (i_1, \ldots, i_m) \in \{1, \ldots, d\}^m$ by $\mathcal{P}_{\boldsymbol{i}}(\boldsymbol{k}) := (k_{i_1}, \ldots, k_{i_m}) \in \mathbb{Z}^m$, and of a frequency index set $I \subset \mathbb{Z}^d$ by $\mathcal{P}_{\boldsymbol{i}}(I) := \{(k_{i_1}, \ldots, k_{i_m}) \colon \boldsymbol{k} \in I\}$.

A. Fast reconstruction for known frequency index sets I

Our method for the dimension-incremental reconstruction is based on the reconstruction method [4] for known frequency index sets I, which we briefly describe in the following. We consider the reconstruction of the Chebyshev coefficients $\hat{a}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, of a multivariate algebraic polynomial in Chebyshev form a_I with frequencies supported on an arbitrary known index set $I \subset \mathbb{N}_0^d$, $|I| < \infty$, from samples $a_I(\boldsymbol{x}_j)$, $j = 0, \ldots, M$, along a suitable rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, M)$. In doing so, we compute the coefficients

$$\tilde{\hat{a}}_l := \sum_{j=0}^{M} (\varepsilon_j^M)^2 \, a_I(\boldsymbol{y}_j) \, \cos\left(\frac{jl}{M}\pi\right) \tag{2}$$

for $l = 0, \ldots, M$ by a 1d DCT of length $M+1$, where $\varepsilon_j^M := 1/\sqrt{2}$ for $j \in \{0, M\}$ and $\varepsilon_j^M := 1$ for $j \in \{1, \ldots, M-1\}$ as well as $\boldsymbol{y}_j := \boldsymbol{x}_j$. Then, we obtain the Chebyshev coefficients $\hat{a}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, of the polynomial a_I by

$$\hat{a}_{\boldsymbol{k}} = \frac{2^d}{M} \frac{\tilde{\hat{a}}_l \; (\varepsilon_l^M)^2}{|\{\boldsymbol{m} \in \mathcal{M}_\nu(\{1\}^d) \colon (\boldsymbol{m} \odot \boldsymbol{k}) \cdot \boldsymbol{z} \operatorname{emod} M = l\}|} \tag{3}$$

with $l := \boldsymbol{k} \cdot \boldsymbol{z} \operatorname{emod} M$ for all frequencies $\boldsymbol{k} \in I$ and any $\nu \in \{1, \ldots, d\}$, if the reconstruction property

$$\begin{gathered} \boldsymbol{k} \cdot \boldsymbol{z} \operatorname{emod} M \neq \boldsymbol{h} \cdot \boldsymbol{z} \operatorname{emod} M \\ \text{for all } \boldsymbol{k} \in I \text{ and } \boldsymbol{h} \in \mathcal{M}_\nu(I),\ \boldsymbol{k} \neq (|h_1|, \ldots, |h_d|), \end{gathered} \tag{4}$$

is fulfilled. A rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, M)$ which fulfills condition (4) will be called *reconstructing rank-1 Chebyshev lattice* for I and is denoted by $\mathrm{CL}(\boldsymbol{z}, M, I)$. The computations in (2) and in (3) can be realized in $\mathcal{O}(M \log M + d\, 2^d |I|)$ arithmetic operations. We can rewrite the computations in (3) as

$$\hat{a}_{\boldsymbol{k}} = \frac{2^{|\boldsymbol{k}|_0 + 1}}{M} \frac{\tilde{\hat{a}}_l \; (\varepsilon_l^M)^2}{|\{\boldsymbol{h} \in \mathcal{M}(\{\boldsymbol{k}\}) \colon \boldsymbol{h} \cdot \boldsymbol{z} \operatorname{emod} M = l\}|}, \tag{5}$$

where $|\boldsymbol{k}|_0 := \sum_{t=1}^d \delta_{k_t, 0}$ denotes the number of non-zero components of a vector $\boldsymbol{k} \in \mathbb{N}_0^d$, and we obtain an arithmetic complexity of $\mathcal{O}(M \log M + d\, |\mathcal{M}(I)|)$, which may be distinctly smaller if only a small amount of components of the frequencies $\boldsymbol{k} \in I$ are non-zero. For a given arbitrary frequency index set $I \subset \mathbb{N}_0^d$, $|I| < \infty$, a reconstructing rank-1 Chebyshev lattice can be obtained by using algorithm [4, Fig. 5] or by using [4, Theorem IV.2] in combination with [19, Algorithm 1 and 2].

B. Dimension-incremental reconstruction

Using the fast reconstruction method from the previous subsection, we describe a method for the dimension-incremental determination of unknown frequency locations belonging to the approximately largest Chebyshev coefficients of a multivariate algebraic polynomial in Chebyshev form a_I or of a multivariate non-periodic function $f \colon [-1,1]^d \to \mathbb{R}$. The method is indicated in Fig. 1 and works similarly to [15, Algorithm 1] from the periodic case.

The proposed algorithm requires several input parameters. The search domain Γ contains all possible frequencies. Moreover, we have to be able to sample the function f under consideration. Threshold parameters θ and θ_{b} are used to distinguish "zero" and "non-zero" Chebyshev coefficients. Additionally, we use the sparsity parameter s to truncate the number of detected frequencies, which may be especially required when the function under consideration has infinitely many non-zero Chebyshev coefficients and we want to determine the approximately largest ones. Finally, the number of detection iterations r controls how many times the sampling is repeated during the determination of the frequency locations in order to obtain higher reliability for the detection, since the frequency detection may fail sometimes due to cancellations within projected and aliased Chebyshev coefficients, see also the discussion in [15, Section 2.2.2] for the periodic case.

Next, we describe the algorithm step-by-step. During the computations, we detect one component of the frequency locations at a time starting with the first component.

In step 1, we start with sampling the function under consideration along the first coordinate direction at the nodes $\boldsymbol{y}_l := (\cos(l\pi/L_1), x'_2, \ldots, x'_d)$, $l = 0, \ldots, L_1$, where $L_1 := \max(\mathcal{P}_1(\Gamma))$ and the higher components $x'_2, \ldots, x'_d$ of the sampling nodes $\boldsymbol{y}_l$ are chosen uniformly at random from $[-1,1]$. Then, we apply a 1d DCT and obtain one-dimensional (projected) Chebyshev coefficients $\tilde{\hat{a}}_{1,k_1}$, $k_1 \in \mathcal{P}_1(\Gamma)$. We determine the ones which are above a certain threshold and store the corresponding frequencies k_1 in the index set $I^{(1)}$.

Input: search domain $\Gamma \subset \mathbb{N}_0^d$, function f (black box), relative thresholds $\theta, \theta_{\mathrm{b}} \in (0,1)$, sparsity parameter $s \in \mathbb{N}$, number of detection iterations $r \in \mathbb{N}$.

(step 1)
$L_1 := \max(\mathcal{P}_1(\Gamma))$, $I^{(1)} := \emptyset$, $z_1 := 1$.
for $i := 1, \ldots, r$ **do**
  Choose $x'_\tau \in [-1,1]$ at random, $\tau = 2, \ldots, d$.
  $\tilde{\hat{a}}_{1,k_1} := \frac{2(\varepsilon_{k_1}^{L_1})^2}{L_1} \sum_{l=0}^{L_1} (\varepsilon_l^{L_1})^2 f(\boldsymbol{y}_l) \cos\left(\frac{lk_1}{L_1}\pi\right)$, $k_1 \in \mathcal{P}_1(\Gamma)$, $\boldsymbol{y}_l := (\cos(\frac{l}{L_1}\pi), x'_2, \ldots, x'_d)$, with 1d DCT.
  $\theta_{\mathrm{abs}} := \theta_{\mathrm{b}} \cdot \max\{|\tilde{\hat{a}}_{1,k_1}| \colon k_1 \in \mathcal{P}_1(\Gamma)\}$.
  $I^{(1)} := I^{(1)} \cup \{k_1 \in \mathcal{P}_1(\Gamma) \colon$ (up to) s-largest $|\tilde{\hat{a}}_{1,k_1}| \geq \theta_{\mathrm{abs}}\}$.
end for i
(step 2)
for $t := 2, \ldots, d$ **do**
(step 2a)
Set $L_t := \max(\mathcal{P}_t(\Gamma))$, $I^{(t)} := \emptyset$.
for $i := 1, \ldots, r$ **do**
  Choose $x'_\tau \in [-1,1]$ at random, $\tau \in \{1, \ldots, d\} \setminus \{t\}$.
  $\tilde{\hat{a}}_{t,k_t} := \frac{2(\varepsilon_{k_t}^{L_t})^2}{L_t} \sum_{l=0}^{L_t} (\varepsilon_l^{L_t})^2 f(\boldsymbol{y}_l) \cos\left(\frac{lk_t}{L_t}\pi\right)$, $k_t \in \mathcal{P}_t(\Gamma)$, $\boldsymbol{y}_l := (x'_1, \ldots, x'_{t-1}, \cos(\frac{l}{L_t}\pi), x'_{t+1}, \ldots, x'_d)$.
  $\theta_{\mathrm{abs}} := \theta_{\mathrm{b}} \cdot \max\{|\tilde{\hat{a}}_{t,k_t}| \colon k_t \in \mathcal{P}_t(\Gamma)\}$.
  $I^{(t)} := I^{(t)} \cup \{k_t \in \mathcal{P}_t(\Gamma) \colon$ (up to) s-largest $|\tilde{\hat{a}}_{t,k_t}| \geq \theta_{\mathrm{abs}}\}$.
end for i
(step 2b)
  Set $\tilde{r} := r$ for $t < d$, $\tilde{r} := 1$ for $t = d$.
  Build reconstructing rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, M_t, \tilde{I})$ for $\tilde{I} := (I^{(1,\ldots,t-1)} \times I^{(t)}) \cap \mathcal{P}_{(1,\ldots,t)}(\Gamma)$.
  for $i := 1, \ldots, \tilde{r}$ **do**
  (step 2c)
    Choose random $x'_\tau \in [-1,1]$, $\tau = t+1, \ldots, d$.
    Set $\boldsymbol{y}_j := (\boldsymbol{x}_j, x'_{t+1}, \ldots, x'_d)$,
    $\boldsymbol{x}_j := \cos(\frac{j}{M_t}\pi\boldsymbol{z})$, $j = 0, \ldots, M_t$.
    Sample f at nodes $\boldsymbol{y}_j$, $j = 0, \ldots, M_t$.
  (step 2d)
    $\tilde{\hat{a}}_l := \sum_{j=0}^{M_t} (\varepsilon_j^{M_t})^2 f(\boldsymbol{y}_j) \cos\left(\frac{jl}{M_t}\pi\right)$, $l = 0, \ldots, M_t$.
    $\tilde{\hat{a}}_{\boldsymbol{k}}^{(1,\ldots,t)} := \frac{2^d (\varepsilon_l^{M_t})^2}{M_t} \frac{\tilde{\hat{a}}_l}{|\{\boldsymbol{m} \in \mathcal{M}_\nu(\{1\}^t) \colon (\boldsymbol{m} \odot \boldsymbol{k}) \cdot \boldsymbol{z} \operatorname{emod} M_t = l\}|}$, for $\boldsymbol{k} \in (I^{(1,\ldots,t-1)} \times I^{(t)}) \cap \mathcal{P}_{(1,\ldots,t)}(\Gamma)$
    with $l := \boldsymbol{k} \cdot \boldsymbol{z} \operatorname{emod} M_t$.
  (step 2e)
    $\theta_{\mathrm{abs}} := \theta \cdot \max_{\boldsymbol{k} \in (I^{(1,\ldots,t-1)} \times I^{(t)}) \cap \mathcal{P}_{(1,\ldots,t)}(\Gamma)} |\tilde{\hat{a}}_{(1,\ldots,t),\boldsymbol{k}}|$.
    $I^{(1,\ldots,t)} := I^{(1,\ldots,t)} \cup \{\boldsymbol{k} \in (I^{(1,\ldots,t-1)} \times I^{(t)}) \cap \mathcal{P}_{(1,\ldots,t)}(\Gamma) \colon$ (up to) s-largest $|\tilde{\hat{a}}_{\boldsymbol{k}}^{(1,\ldots,t)}| \geq \theta_{\mathrm{abs}}\}$.
  end for i
end for t
Output: index set of detected frequencies $I^{(1,\ldots,d)} \subset \Gamma \subset \mathbb{N}_0^d$, corresponding Chebyshev coefficients $\tilde{\hat{a}}_{\boldsymbol{k}}^{(1,\ldots,d)} \in \mathbb{R}$.

Fig. 1. Dimension-incremental reconstruction of Chebyshev coefficients of a function from samples for unknown frequency index set.

Next, we proceed with step 2 for $t := 2, 3, \ldots, d$. In step 2a, we sample the function under consideration along the t-th coordinate direction at the nodes $\boldsymbol{y}_l := (x'_1, \ldots, x'_{t-1}, \cos(l\pi/L_t), x'_{t+1}, \ldots, x'_d)$, $l = 0, \ldots, L_t$, where $L_t := \max(\mathcal{P}_t(\Gamma))$ and the components x'_τ, $\tau \in \{1, \ldots, d\} \setminus \{t\}$, are chosen uniformly at random from $[-1,1]$. Again, we apply a 1d DCT and obtain one-dimensional (projected) Chebyshev coefficients $\tilde{\hat{a}}_{t,k_t}$, $k_t \in \mathcal{P}_t(\Gamma)$. We determine the ones which are above a certain threshold and store the corresponding frequencies k_t in the index set $I^{(t)}$. In step 2b, we build a reconstructing rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, M_t, \tilde{I})$ for $\tilde{I} := (I^{(1,\ldots,t-1)} \times I^{(t)}) \cap \mathcal{P}_{(1,\ldots,t)}(\Gamma)$ using one of the approaches from Subsection II-A. In step 2c, we use sampling nodes $\boldsymbol{y}_j := (\boldsymbol{x}_j, x'_{t+1}, \ldots, x'_d)$, $j = 0, \ldots, M_t$, where the components $x'_{t+1}, \ldots, x'_d$ are chosen uniformly at random from $[-1,1]$, and we sample the function under consideration. In step 2d, we compute (2) using a 1d DCT followed by (3) or (5), which yields (projected) Chebyshev coefficients $\tilde{\hat{a}}_{\boldsymbol{k}}^{(1,\ldots,t)}$, $\boldsymbol{k} \in \tilde{I} \subset \mathcal{P}_{(1,\ldots,t)}(\Gamma) \subset \mathbb{N}_0^t$. In step 2e, we determine the ones which are above a certain threshold and store the corresponding frequencies $\boldsymbol{k}$ in the index set $I^{(1,\ldots,t)}$.

Finally, the algorithm indicated in Fig. 1 returns the frequency index set $I^{(1,\ldots,d)} \subset \Gamma \subset \mathbb{N}_0^d$ and Chebyshev coefficients $\tilde{\hat{a}}_{\boldsymbol{k}}^{(1,\ldots,d)}$, $\boldsymbol{k} \in I^{(1,\ldots,d)}$.

Setting the search domain Γ to the full d-dimensional grid $\hat{G}_n^d := \{\boldsymbol{k} \in \mathbb{N}_0^d \colon \|\boldsymbol{k}\|_\infty \leq n\}$, $n \in \mathbb{N}$, and using [4, Theorem IV.2] in combination with the approach in [19, Algorithm 1 and 2] for building reconstructing rank-1 Chebyshev lattices $\mathrm{CL}(\boldsymbol{z}, M_t, \tilde{I})$, we require $\mathcal{O}(s^2 n)$ samples and $\mathcal{O}(s^3 + s^2 n \log(s\, n))$ arithmetic operations in total for $s \gtrsim \sqrt{n}$, see also [15, Section 2.2.3] from the periodic case. The involved constants in the sampling and arithmetic complexities may be exponential in the dimension d. When we reconstruct a multivariate algebraic polynomial in Chebyshev form $f := a_I$ and set the sparsity parameter s of the algorithm in Fig. 1 to the sparsity $|\operatorname{supp} \hat{a}|$ of the polynomial a_I, $\operatorname{supp} \hat{a} := \{\boldsymbol{k} \in I \colon \hat{a}_{\boldsymbol{k}} \neq 0\}$, we require $\mathcal{O}(|\operatorname{supp} \hat{a}|^2\, n)$ samples and $\mathcal{O}(|\operatorname{supp} \hat{a}|^3 + |\operatorname{supp} \hat{a}|^2\, n \log(|\operatorname{supp} \hat{a}|\, n))$ arithmetic operations in total for $|\operatorname{supp} \hat{a}| \gtrsim \sqrt{n}$, where the constants in the big $\mathcal{O}$ notation may be exponential in the dimension d.

III. Numerical results

The numerical tests were performed in MATLAB using double precision arithmetic. We apply the algorithm in Fig. 1 on random sparse multivariate algebraic polynomials in Chebyshev form a_I and on a 9-dimensional test function, where the latter has infinitely many non-zero Chebyshev coefficients.

A. Sparse multivariate polynomials

We set the refinement $n := 32$ and construct random multivariate algebraic polynomials in Chebyshev form a_I with frequencies supported within the d-dimensional full grid $\hat{G}_{32}^d$. This means, we choose the sparsity as $|\operatorname{supp} \hat{a}|$ many frequencies uniformly at random from $\hat{G}_{32}^d \subset \mathbb{N}_0^d$ and corresponding Chebyshev coefficients $\hat{a}_{\boldsymbol{k}} \in [-1,1]$, $|\hat{a}_{\boldsymbol{k}}| \geq 10^{-6}$, $\boldsymbol{k} \in I = \operatorname{supp} \hat{a}$. For the reconstruction of the multivariate algebraic polynomials in Chebyshev form a_I, we choose the search domain $\Gamma := \hat{G}_{32}^d$ and we build reconstructing rank-1 Chebyshev lattices $\mathrm{CL}(\boldsymbol{z}, M_t, \tilde{I})$ by using algorithm [4, Fig. 5]. We do not truncate the frequency index sets of detected frequencies $I^{(1,\ldots,t)}$, $t \in \{1,\ldots,d\}$, i.e., we set the sparsity parameter $s := |\Gamma|$. We may alternatively set the sparsity parameter $s := |\operatorname{supp}\hat{a}| = |I|$ and obtain the same results. Moreover, we set the number of detection iterations $r := 1$ and the threshold parameters $\theta = \theta_{\mathrm{b}} := 10^{-12}$. All tests are repeated 10 times with newly chosen frequencies $\boldsymbol{k}$ and Chebyshev coefficients $\hat{a}_{\boldsymbol{k}}$. In each test, all frequencies were successfully detected, $I^{(1,\ldots,d)} = \operatorname{supp}\hat{a}$. The used parameters and results are presented in Table I. The column "max. cand." shows the maximal number $\max_{t=2,\ldots,d} |I^{(1,\ldots,t-1)} \times I^{(t)}|$ of frequency candidates of all 10 repetitions and "max. M" the overall maximal size parameter used. Furthermore, the total number of samples for each repetition was computed and the maximum of these numbers for the 10 repetitions can be found in the column "max. #samples". The relative ℓ_2-error $\|(\tilde{\hat{a}}_{\boldsymbol{k}})_{\boldsymbol{k}\in\tilde{I}} - (\hat{a}_{\boldsymbol{k}})_{\boldsymbol{k}\in\tilde{I}}\|_2 / \|(\hat{a}_{\boldsymbol{k}})_{\boldsymbol{k}\in\tilde{I}}\|_2$ of the computed Chebyshev coefficients $(\hat{a}_{\boldsymbol{k}})_{\boldsymbol{k}\in I^{(1,\ldots,d)}}$ was determined for each repetition, where $\tilde{I} := \operatorname{supp}\hat{a} \cup I^{(1,\ldots,d)}$ and $\tilde{\hat{a}}_{\boldsymbol{k}} := 0$ for $\boldsymbol{k} \in \tilde{I} \setminus I^{(1,\ldots,d)}$, and the column "max. rel. ℓ_2-error" contains the maximal value of the 10 repetitions.

In all tests, the relative ℓ_2-error is smaller than $4.2 \cdot 10^{-14}$ and is caused by the utilized IEEE 754 double precision arithmetic. The numbers of used samples "max. #samples" grow for increasing dimensions d. For $|\operatorname{supp}\hat{a}| = 100$, the maximal number of frequency candidates "max. cand." is 3 300 and the maximal size parameter "max. M" is between about 220 000 and 460 000 in Table I for dimensions $d \geq 4$. This is caused by the relatively large numbers of possible frequencies $|\Gamma| = |\hat{G}_{32}^d| = 33^d$ and the small sparsity $|\operatorname{supp}\hat{p}| = 100$, which cause that all 100 non-zero Chebyshev coefficients $\hat{a}_{\boldsymbol{k}} \neq 0$ are already detected in dimension-incremental steps $t \leq 4$ and higher components z_τ, $5 \leq \tau < t$, (except the highest component z_t) of the generating vector $\boldsymbol{z} := (z_1,\ldots,z_t)$ of the reconstructing rank-1 Chebyshev lattices $\Lambda(\boldsymbol{z}, M, \tilde{I})$ to be zero in most cases. Consequently, the numbers of used samples "max. #samples" increase by about 220 000 to 460 000 per additional dimension. We remark that we may have found all non-zero Chebyshev coefficients $\hat{a}_{\boldsymbol{k}} \neq 0$ in a dimension-incremental step $t \leq 4$, but we still need to continue with remaining dimension-incremental steps $t \geq 5$ in order to determine the higher components k_t, $t \in \{5,\ldots,d\}$, of the frequencies $\boldsymbol{k}$.

Additionally, we consider higher sparsity $|\operatorname{supp}\hat{a}| = 1\,000$ in dimensions $d \in \{3,4,\ldots,8\}$. In each test, all frequencies were successfully detected, $I^{(1,\ldots,d)} = \operatorname{supp}\hat{a}$, and we observe a analogous behavior as in the case $|\operatorname{supp}\hat{a}| = 100$. For dimension $d = 6$, we have seven test runs where all 1 000 non-zero Chebyshev coefficients $\hat{a}_{\boldsymbol{k}} \neq 0$ are already found in dimension increment step $t = 5$ yielding size parameters M of about 21 million and numbers of used samples of about 50 million. However, we still have three test runs, where all 1 000 non-zero Chebyshev coefficients $\hat{a}_{\boldsymbol{k}} \neq 0$ are found not until the last dimension increment step $t = 6$ yielding the maximal size parameter "max. M" of about 45 million and the numbers of used samples "max. #samples" of about 74 million in Table I. In dimensions $d \geq 7$, all 1 000 non-zero Chebyshev coefficients $\hat{a}_{\boldsymbol{k}} \neq 0$ are already found in dimension increment steps $t \leq 6$ for all ten test runs. Analogously to the behavior for sparsity $|\operatorname{supp}\hat{a}| = 100$, we expect the numbers of used samples "max. #samples" to increase by about 20 to 45 million per additional dimension for dimensions $d \geq 6$.

TABLE I
RESULTS FOR RECONSTRUCTION OF RANDOM SPARSE MULTIVARIATE ALGEBRAIC POLYNOMIALS IN CHEBYSHEV FORM a_I, $I \subset \hat{G}_{32}^d$, USING ALGORITHM IN FIG. 1 WITH SEARCH DOMAIN $\Gamma := \hat{G}_{32}^d$.

d	$\lvert\operatorname{supp}\hat{a}\rvert$	max. cand.	max. M	max. #samples	max. rel. ℓ_2-error
3	100	3 168	81 642	83 826	4.92e-16
4	100	3 300	221 260	295 118	7.17e-16
5	100	3 300	234 655	537 964	5.45e-16
6	100	3 300	241 391	785 671	1.17e-15
7	100	3 300	456 119	1 614 677	9.37e-16
8	100	3 300	392 251	1 828 842	6.43e-16
9	100	3 300	386 490	2 195 804	7.30e-16
10	100	3 300	414 611	2 710 158	1.78e-15
15	100	3 300	380 502	4 439 451	4.20e-14
3	1 000	15 873	73 856	75 080	5.53e-16
4	1 000	32 604	6 490 663	6 630 162	6.74e-16
5	1 000	33 000	27 021 660	34 116 319	7.44e-16
6	1 000	33 000	44 791 174	74 215 472	1.48e-15
7	1 000	33 000	42 401 071	113 804 504	8.03e-16
8	1 000	33 000	43 799 177	161 481 230	1.49e-15

B. 9-dimensional test function

Next, we apply our method for the non-periodic dimension-incremental reconstruction to a multivariate function $f \in L_{2,w}([-1,1]^d)$, which is not sparse in frequency domain, where $L_{2,w}([-1,1]^d)$ is the weighted Hilbert space of all square integrable functions $f\colon [-1,1]^d \to \mathbb{R}$ with respect to the Chebyshev weight $w(\boldsymbol{x}) := \prod_{t=1}^d 1/\sqrt{1-x_t^2}$ with norm $\|f|L_{2,w}([-1,1]^d)\| := (\int_{[-1,1]^d} |f(\boldsymbol{x})|^2\, w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x})^{1/2}$. The Chebyshev coefficients $\hat{f}_{\boldsymbol{k}}$ of a function $f \in L_{2,w}([-1,1]^d)$ are formally given by $\hat{f}_{\boldsymbol{k}} := 2^{|\boldsymbol{k}|_0}/\pi^d \int_{[-1,1]^d} f(\boldsymbol{x})\, T_{\boldsymbol{k}}(\boldsymbol{x})\, w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, $\boldsymbol{k} \in \mathbb{N}_0^d$, see e.g. [20].

Here, we consider the 9-dimensional test function $f\colon [-1,1]^9 \to \mathbb{R}$,

$$f(\boldsymbol{x}) := \prod_{t\in\{1,3,4,7\}} B_2(x_t) + \prod_{t\in\{2,5,6,8,9\}} B_4(x_t), \qquad (6)$$

where B_2 is a shifted, scaled and dilated B-spline of order 2 and $B_4\colon \mathbb{R} \to \mathbb{R}$ is a shifted, scaled and dilated B-spline of order 4, see Fig. 2 for illustration. We remark that the Chebyshev coefficients of B_2 and B_4 decay like $\sim k^{-3}$ and $\sim k^{-5}$, respectively. We approximate the test function f from (6) by multivariate algebraic polynomials in Chebyshev form a_I. For this, we determine a frequency index set $I = I^{(1,\ldots,9)} \subset \Gamma \subset \mathbb{N}_0^d$ and compute approximated Chebyshev coefficients $\tilde{\hat{a}}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, from sampling values of f using the algorithm in Fig. 1, where the reconstructing rank-1 Chebyshev lattices $\mathrm{CL}(\boldsymbol{z}, M_t, \tilde{I})$ are built using algorithm [4, Fig. 5]. We expect the frequency index set I to "consist of" two

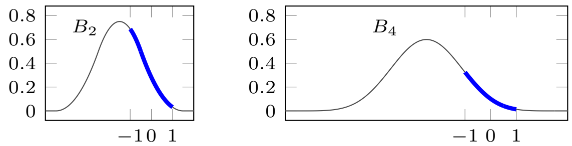

Fig. 2. B-splines B_2 and B_4 considered in interval $[-1,1]$.

TABLE II
RESULTS FOR APPROXIMATION OF TEST FUNCTION $f\colon [-1,1]^9 \to \mathbb{R}$ FROM (6) USING ALGORITHM IN FIG. 1.

Γ	θ	$\lvert I\rvert$	max. cand.	max. total #samples	max. rel. $L_{2,w}$-error
$\hat{G}^9_{32}$	1.0e-02	149	1 116	135 216	1.7e-02
	1.0e-03	485	4 710	898 310	6.7e-03
	1.0e-04	1 431	18 200	5 662 360	4.7e-04
	1.0e-05	3 465	63 800	27 009 528	9.4e-05
H^9_{32}	1.0e-02	147	851	99 181	1.7e-02
	1.0e-03	486	2 979	586 317	5.6e-03
	1.0e-04	1 438	6 038	2 802 539	4.1e-04
	1.0e-05	2 784	9 656	8 340 927	1.3e-04

manifolds, a four-dimensional hyperbolic cross like structure in the dimensions $1,3,4,7$, and a five-dimensional hyperbolic cross like structure in the dimensions $2,5,6,8,9$. All tests were run 10 times and the relative $L_{2,w}([-1,1]^9)$ approximation errors $\|f - \tilde{S}_I f|L_{2,w}([-1,1]^9)\|/\|f|L_{2,w}([-1,1]^9)\|$ are computed, where the approximated Chebyshev partial sum $\tilde{S}_I f := \sum_{\boldsymbol{k}\in I} \tilde{\hat{a}}_{\boldsymbol{k}} T_{\boldsymbol{k}}(\circ)$. We choose the search domain $\Gamma := \hat{G}^9_{32} \subset \mathbb{N}^d_0$ as the 9-dimensional full grid of refinement $n = 32$, which consists of $|\Gamma| = |\hat{G}^9_{32}| \approx 4.641 \cdot 10^{13}$ frequency candidates. The best possible relative $L_{2,w}([-1,1]^9)$ approximation error is about $6.3 \cdot 10^{-5}$ if we use all the corresponding (exactly computed) $|\hat{G}^9_{32}|$ many Chebyshev coefficients as well as about $8.1 \cdot 10^{-5}$ if we use only 3 465 many Chebyshev coefficients. For our method, we set the number of detection iterations $r := 5$ and the threshold parameter $\theta_{\mathrm{b}} := \theta/100$. Moreover, the sparsity parameter $s \in \mathbb{N}$ is set to $|\Gamma|$, i.e., we do not additionally truncate the frequency index sets $I^{(1,\ldots,t)}$. The results for threshold parameter $\theta \in \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$ are shown in Table II. For instance for $\theta = 10^{-4}$, we obtain a maximal relative $L_{2,w}([-1,1]^9)$ approximation error of $4.7 \cdot 10^{-4}$ using 1 431 Chebyshev coefficients $\tilde{\hat{a}}_{\boldsymbol{k}}$ and about 5.7 million samples were taken. We may reduce the numbers of samples by restricting the search domain Γ, e.g. to a hyperbolic cross H^9_{32}, and still obtain comparable maximal relative $L_{2,w}([-1,1]^9)$ approximation errors. For instance for $\theta = 10^{-4}$, we only required about half the number of samples.

IV. CONCLUSION

In this paper, we considered the fast reconstruction of high-dimensional sparse algebraic polynomials in Chebyshev form and the fast approximation of multivariate non-periodic functions from samples. To this end, we used the nodes of reconstructing rank-1 Chebyshev lattices. We presented a dimension-incremental reconstruction method, which determines unknown frequency locations belonging to the nonzero or approximately largest Chebyshev coefficients. We successfully applied the presented method in numerical tests for the reconstruction of high-dimensional sparse polynomials in up to 15 dimensions and for the approximation of a 9-dimensional test function.

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