Fourier extension and sampling on the sphere

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***Abstract*—We present different sampling methods for the approximation of functions on the sphere. In this note we focus on Fourier methods on the sphere based on spherical harmonics and on the double Fourier sphere method. Further longitude-latitude transformation is combined with Fourier extension to allow the use of bi-periodic Fourier series on the sphere. Fourier extension with hermite interpolation is introduced and double Fourier sphere method is discussed shortly.**

I. Introduction

We discuss different methods for the approximation of functions on the sphere and and start with the representation of functions on the sphere with respect to the spherical harmonics. Finite expansions in spherical harmonics are known as spherical polynomials. The fast evaluation of spherical polynomials on special grids or on arbitrary nodes are know as spherical Fourier transform or as nonequispaced spherical Fourier transform, respectively. We summarize these methods in Section II. The evaluation of the coefficients of a finite spherical harmonic expansion leads to quadrature rules on the sphere. We discuss special quadrature rules, i.e., we are interested in special point distributions, the spherical t-designs, which allow the efficient evaluation of the spherical Fourier coefficients, see Section III. We refer to methods for the fast and efficient evaluation of numerical t-designs, based on the nonequispaced spherical Fourier transform. A method, known as double Fourier sphere avoids the transform from spherical harmonic expansion (2) to the bi-periodic Fourier series (3) and starts with the representation (3) directly. In order to overcome the pole problem, one can use a method known as Fourier extension. In Section IV we introduce a simple Fourier extension method and apply this approach to the sphere in Section V. Finally we refer to spectral methods in Section VI.

II. Fast spherical Fourier transform

Functions on the unit sphere $f(x,y,z)$ are restricted to points on this sphere, this means the variables must satisfy $x^2+y^2+z^2=1$. This restriction can be taken into account implicitly by the longitude-latitude coordinate transformation

$$x=\cos\varphi\sin\theta,\quad y=\sin\varphi\sin\theta,\quad z=\cos\theta, \qquad (1)$$
$$(\theta,\varphi)\in[0,\pi]\times[-\pi,\pi).$$

After this transformation all computations can be done using $f(\theta,\varphi)$ without a restriction on the variables φ and θ. It is well known that the eigenfunctions of the spherical Laplace-Beltrami operator $\Delta_{\mathbb{S}^2}$ are the spherical harmonics Y_n^k of degree n and order k, cf. [1]–[3],

$$Y_n^k(\mathbf{x})=Y_n^k(\theta,\varphi):=\sqrt{\frac{2n+1}{4\pi}}P_n^{|k|}(\cos\theta)\mathrm{e}^{\mathrm{i}k\varphi}$$

with the notation $\mathbf{x}=\mathbf{x}(\theta,\varphi)\in\mathbb{S}^2$, where the associated Legendre functions $P_n^k:[-1,1]\to\mathbb{R}$ and the Legendre polynomials $P_n:[-1,1]\to\mathbb{R}$ are given by

$$P_n^k(x) := \left(\frac{(n-k)!}{(n+k)!}\right)^{1/2}\left(1-x^2\right)^{k/2}\frac{\mathrm{d}^k}{\mathrm{d}x^k}P_n^k(x),$$
$$P_n(x) := \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n$$

for $n\in\mathbb{N}_0,\ k=0,\dots,n$. In spherical coordinates the surface element reads as $\mathrm{d}\mu_{\mathbb{S}^2}(\mathbf{x})=\sin\theta\mathrm{d}\theta\mathrm{d}\varphi$ and the spherical harmonics obey the orthogonality relation

$$\int_0^{2\pi}\int_0^{\pi}Y_n^k(\theta,\phi)\overline{Y_m^l(\theta,\phi)}\sin\theta\mathrm{d}\theta\mathrm{d}\varphi=\delta_{k,l}\delta_{n,m}.$$

Moreover, the spherical harmonics form an orthonormal basis of the space of all square integrable functions $L_2(\mathbb{S}^2):=\{f:\mathbb{S}^2\to\mathbb{C}\ :\ \int_{\mathbb{S}^2}|f(\mathbf{x})|^2\mathrm{d}\mu_{\mathbb{S}^2}(\mathbf{x})<\infty\}$. Hence, every $f\in L_2(\mathbb{S}^2)$ has an unique expansion in spherical harmonics

$$f=\sum_{n=0}^{\infty}\sum_{k=-n}^{n}\hat{f}_n^kY_n^k.$$

We say that f is a spherical polynomial of degree at most N if $\hat{f}_n^k=0,\ n>N$, and we denote by $\Pi_N(\mathbb{S}^2)$ the space of all spherical polynomials of degree at most N. We remark that the dimension of $\Pi_N(\mathbb{S}^2)$ is $d_N:=(N+1)^2$.

The evaluation of a spherical polynomial

$$p=\sum_{n=0}^{N}\sum_{k=-n}^{n}\hat{p}_n^kY_n^k\in\Pi_N(\mathbb{S}^2) \qquad (2)$$

on a sampling set $X_M=\{\mathbf{x}_1,\dots,\mathbf{x}_M\}\subset\mathbb{S}^2$ can be expressed by a matrix-vector multiplication

$$\mathbf{p}=\mathbf{Y}^N\hat{\mathbf{p}},$$

where $\mathbf{Y}^N$ is the nonequispaced spherical Fourier matrix

$$\mathbf{Y}^N:=\left(Y_k^n(\mathbf{x}_i)\right)_{i=1,\dots,M;\ n=0,\dots,N,|k|\le n}\in\mathbb{C}^{M\times d_N},$$

$\mathbf{p}$ is the vector of the sampling values

$$\mathbf{p} = (p(\mathbf{x}_1), \ldots, p(\mathbf{x}_M))^\top \in \mathbb{C}^M$$

and $\hat{\mathbf{p}}$ is the vector of spherical Fourier coefficients

$$\hat{\mathbf{p}} := (\hat{p}_n^k)_{n=0,\ldots,N,|k|\le n} \in \mathbb{C}^{d_N}.$$

Recently, fast approximate algorithms for the matrix times vector multiplication with the nonequispaced spherical Fourier matrix $\mathbf{Y}^N$ and its adjoint $\overline{\mathbf{Y}^N}^\top$ have been proposed in [4], [5]. The arithmetic complexity for the so called fast spherical Fourier transform and its adjoint is $\mathcal{O}(N^2 \log^2 N + M \log^2(1/\epsilon))$, where $\epsilon > 0$ is a prescribed accuracy of the approximate algorithms. An implementation of these algorithms in C, Matlab or Octave can be found on the Internet [6]. The key idea is to first perform a change of basis such that the polynomial p in (2) takes the form

$$p(\vartheta, \varphi) = \sum_{n=-N}^{N} \sum_{k=-N}^{N} c_k^n \mathrm{e}^{\mathrm{i}k\vartheta} \mathrm{e}^{\mathrm{i}n\varphi} \tag{3}$$

of an ordinary two-dimensional Fourier sum with new complex coefficients c_k^n. This basis exchange was first suggested in [7], [8].

Then, the evaluation of the function p can be performed using the fast Fourier transform for nonequispaced nodes (NFFT; see for example [9], [10]). The efficient evaluation of gradients and Hessians of spherical polynomials can be done as well [11].

The coefficients $\hat{p}_k^n$ in (2) can be obtained from values of the function p on a set of arbitrary nodes (ϑ_i, φ_i) provided that a quadrature rule with weights w_i and sufficient high degree of exactness is available (see also [12], [13]). Then the sum

$$\hat{p}_k^n = \sum_{i=1}^{M} w_i p\left(\vartheta_i, \varphi_i\right) \overline{Y_k^n\left(\vartheta_i, \varphi_i\right)} \tag{4}$$

can be efficiently realized by the adjoint nonequispaced spherical Fourier transform.

The corresponding sampling problem is the computation of Fourier coefficients $\hat{p}_n^k$ of a function from sampled values at scattered nodes. A least squares approximation and an interpolation of the given data is considered in [14] and based on the fast spherical Fourier transforms.

III. Spherical Designs

Distributing points on the unit sphere $\mathbb{S}^2$ in the Euclidean space $\mathbb{R}^3$ in some optimal sense is a challenging problem, cf. [15]. The concept of spherical t-designs, which was introduced in [16]. There a spherical t-design on $\mathbb{S}^2$ is defined as a finite set $X_M = \{\mathbf{x}_1, \ldots, \mathbf{x}_M\} \subset \mathbb{S}^2$ satisfying

$$\int_{\mathbb{S}^2} p(\mathbf{x}) \mathrm{d}\mu_{\mathbb{S}^2}(\mathbf{x}) = \frac{4\pi}{M} \sum_{i=1}^{M} p(\mathbf{x}_i), \qquad \text{for all } p \in \Pi_t(\mathbb{S}^2), \tag{5}$$

where $\mu_{\mathbb{S}^2}$ is the surface measure on $\mathbb{S}^2$ and $\Pi_t(\mathbb{S}^2)$ is the space of all spherical polynomials with degree at most t. Such point sets provide equal weights quadrature formulae on the sphere $\mathbb{S}^2$, which have many applications. In the Hilbert space $\Pi_t(\mathbb{S}^2)$ with standard inner product the worst case quadrature error for the point set X_M is defined by

$$E_t(X_M) := \sup_{p \in \Pi_t(\mathbb{S}^2), \|p\|_2 \le 1} \left| \int_{\mathbb{S}^2} p(\mathbf{x}) \mathrm{d}\mu_{\mathbb{S}^2}(\mathbf{x}) - \frac{4\pi}{M} \sum_{i=1}^{M} p(\mathbf{x}_i) \right|.$$

For the general setting of quadrature errors in reproducing kernel Hilbert spaces we refer to [17]. Of course, a spherical t-design X_M is a global minimum of the worst case quadrature error with $E_t(X_M) = 0$, cf. (5). In [18] the authors present a variational characterization of spherical t-designs which involves a squared quadrature error

$$A_t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M) := \frac{1}{M^2} \sum_{n=1}^{t} \sum_{k=-n}^{n} \left| \sum_{i=1}^{M} Y_n^k(\mathbf{x}_i) \right|^2 = \left(\frac{1}{4\pi} E_t(X_M) \right)^2.$$

In [11] the authors developed efficient algorithms for numerical spherical t-designs, i.e., we compute point sets X_M, such that $A_t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M) \le \varepsilon$, where ε is a given accuracy, say $\varepsilon = 1\mathrm{e}^{-10}$. There optimization algorithms on Riemannian manifolds for attacking this highly nonlinear and nonconvex minimization problem are considered. The proposed methods make use of fast spherical Fourier transforms, which were already successfully applied in [14], [19] for solving high dimensional linear equation systems on the sphere. The computed spherical t-designs are available from [20]. There we also present results for Gauss-type quadrature results on $\mathbb{S}^2$, which can be obtained by similar methods.

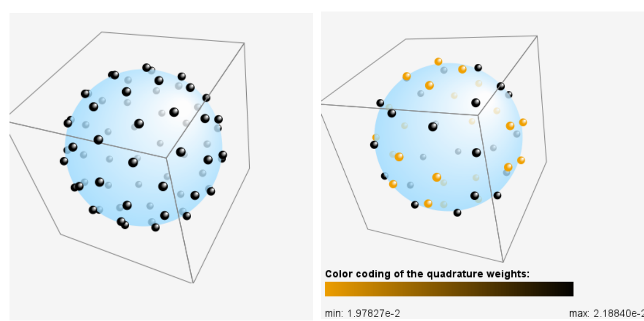

Fig. 1. Spherical 11-design with 70 nodes (left) and 48 nodes (right) on the sphere which are exact for spherical polynomials up to degree 11. The color represents the weights of the quadrature rule. Different point sets available from [20], see also [21].

IV. Fast Fourier Extension

Approximating a function $f(x)$ on the interval $x \in [-1, 1]$ by Fourier series

$$t_N(x) = \sum_{k=-N/2}^{N/2-1} f_k \mathrm{e}^{\pi \mathrm{i} k x} \tag{6}$$

can be done using the fast Fourier transform (FFT) to compute the coefficients

$$f_k = \frac{1}{N} \sum_{l=0}^{N-1} f(y_l) \mathrm{e}^{-2\pi \mathrm{i} k l / N}, \tag{7}$$

with equispaced points $y_l = -1 + 2\frac{l}{N}$, $l = 0, \dots, N-1$. The convergence rate of $t_N(x)$ to $f(x)$ is well known:

Theorem IV.1. *([22, Theorem 7.2]) If a periodic function f is $\nu \geq 1$ times differentiable and $f^{(\nu)}$ is of bounded variation V on $[0, 2\pi]$, then its degree N trigonometric interpolant (Fourier series approximation)* (6) *satisfies*

$$||f - t_N||_\infty \leq \frac{2V}{\pi \nu N^\nu}. \tag{8}$$

For functions f that are non-periodic a loss of smoothness occurs in the implicit periodic extension intrinsic to Fourier series and the Gibbs phenomenon occurs in the Fourier series approximation. To recover this smoothness up to C^r-continuity and thereby recovering the convergence rate, see Theorem (IV.1), we extend the function $f(x)$, $x \in [-1, 1]$ to $f_{\text{ext}}(x)$, on a larger interval $x \in [-1, 2T-1]$ for $T > 0$. This is known as a Fourier extension, see [23] and the references therein. The extended function is defined as

$$f_{\text{ext}}(x) = \begin{cases} f(x), & x \in [-1, 1] \\ P(x), & x \in [1, 2T-1] \end{cases} . \tag{9}$$

In (9) $P(x)$ is a polynomial of degree $2r+1$ computed via two-point Taylor interpolation such that $f_{\text{ext}}(x) \in C^r$ and periodic on the larger interval. The formulas for Fourier series and its coefficients change because now the extended function $f_{\text{ext}}(x)$ is used:

$$t_N(x) = \sum_{k=-N/2}^{N/2-1} f_k \mathrm{e}^{\pi \mathrm{i} k x / T}, \tag{10}$$

$$f_k = \frac{1}{N} \sum_{l=0}^{N-1} f_{\text{ext}}(y_l) \mathrm{e}^{-2\pi \mathrm{i} k l / N} \tag{11}$$

with $y_l = -1 + 2T\frac{l}{N}$, $l = 0, \dots, N-1$.

Lemma IV.2. *The two-point Taylor interpolation can be supplied with conditions on the endpoints of an interval $[m-q, m+q] = [1, 2T-1]$ in the form of function values or derivatives: $P^{(j)}(m-q) = a_j$, $P^{(j)}(m+q) = b_j$, $j = 0, \dots, r$. The resulting interpolating polynomial $P(x)$ of degree $2r+1$ will then satisfy the given conditions in $x = m-q$ and $x = m+q$. With $y = \frac{x-m}{q} = \frac{x-T}{T-1}$,*

$$P(x) = \sum_{j=0}^{r} B(r, j, y) q^j a_j + \sum_{j=0}^{r} B(r, j, -y)(-q)^j b_j \tag{12}$$

$$B(r, j, y) = \sum_{k=0}^{r-j} \binom{r+k}{k} \frac{1}{j! 2^{r+1} 2^k} (1-y)^{r+1} (1+y)^{k+j}.$$

Proof. See ([24, Proposition 3.2]). □

If the function f is known analytically, then in general also the values $f^{(r)}(-1)$ and $f^{(r)}(1)$ are known and the extended function f_{ext} can be computed straightforward using the condition $P^{(j)}(-1) = f^{(j)}(-1)$ and $P^{(j)}(1) = f^{(j)}(1)$ for $j = 0, \dots, r$ in Lemma IV.2. This method is e.g. applied in [24], [25]. In the following we suggest to compute the derivatives $f^{(r)}(-1)$ and $f^{(r)}(1)$ using the following Chebyshev approximation. We denote the Chebyshev polynomials $T_j(x) := \cos(j \text{ arccos(x)})$ and the approximating polynomial is of the form

$$p_{N_{\text{Cheb}}}(x) = {\sum_{k=0}^{N_{\text{Cheb}}}}'' c_k T_k(x), \tag{13}$$

where the double prime indicates that the first and last element of the sum ($k = 0$ and $k = N_{\text{Cheb}}$) should be divided by 2. The coefficients c_k used in sum (13) can be computed using the discrete cosine transform (DCT)

$$c_k = \frac{2}{N_{\text{Cheb}}} {\sum_{j=0}^{N_{\text{Cheb}}}}'' p_{N_{\text{Cheb}}} \left(\cos \left(\frac{j\pi}{N_{\text{Cheb}}} \right) \right) \cos \left(\frac{jk\pi}{N_{\text{Cheb}}} \right), \quad k = 0, \dots, N_{\text{Cheb}}. \tag{14}$$

Lemma IV.3. *Once the coefficients c_k* (14) *for the Chebyshev series* (13) *are computed, the coefficients d_k of the derivative*

$$p'_{N_{\text{Cheb}}}(x) = {\sum_{k=0}^{N_{\text{Cheb}}-1}}'' d_k T_k(x) \tag{15}$$

are found by the recurrence relation:

$$d_k = d_{k+2} + 2(k+1) c_{k+1}, \; k = N_{\text{Cheb}} - 1, \dots, 0 \tag{16}$$

with $d_{N+1} = d_N = 0$.

Proof. See ([26], Satz 6.2.9) □

To get the rth order derivative, the coefficients d_k of the $(r-1)$th derivative are used as c_k in the recurrence relation (16).

Algorithm 1 Fast Fourier Extension using 2-point Taylor interpolation

Input: N, $f(x)$, $x \in [-1, 1]$, N_{Cheb}, r, T
1: Evaluate $f(x)$ in N_{Cheb} Chebyshev points.
2: Compute Chebyshev coefficients c_k (14) via DCT.
3: Compute Chebyshev coefficients in (15) for derivatives d_k via recurrence relation (16).
4: Evaluate Chebyshev series (13) and (15) in $x = -1$ and $x = 1$ to obtain interpolation values for the Taylor polynomial (12).
5: Evaluate $f_{\text{ext}}(x)$ in N equidistant points $y_l = -1 + 2T\frac{l}{N}$, $l = 0, \dots, N-1$.
6: Compute the coefficients of the Fourier series f_k (11) via the FFT.
Output: f_k

Figure 2 shows on the left the extension of $f(x) = x$ that is C^4-smooth and on the right the convergence rate of $t_N(x)$ to $f(x)$, which follows the theoretical convergence rate.

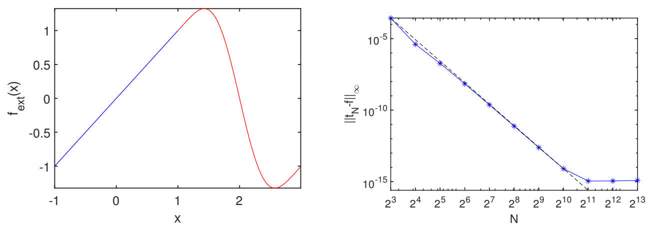

Fig. 2. Example: Left: $f(x) = x$ for $x \in [-1,1]$ in blue and $P(x)$ for $x \in [1, 2T-1]$ in red. Right: Maximum error $||f(x) - t_N(x)||_\infty$ on $x \in [-1,1]$. In blue the observed maximum error (with oversampling of 10). In black, dotted line the theoretical convergence rate $\mathcal{O}(\frac{1}{N^5})$. Parameters: $r=4$, $N_{\text{Cheb}} = 2$, $T = 2$.

V. Fourier extension on the sphere

There are two major drawbacks, to complicate for the direct computation with (3).

1) Loss of periodicity in the θ-direction. In the φ-direction the function is 2π-periodic and sufficiently smooth.
2) For naive grids in (θ, φ): oversampling near the singularities located at the north and south pole.
   This is due to the mapping of $(0,\varphi)$ and (π, φ), $\varphi \in [-\pi, \pi)$ to $(0,0,1)$ and $(0,0,-1)$ respectively.

The first drawback, loss of periodicity in θ-direction can be avoided with a Fourier extension along this direction.
A simple way of extending would be via the double Fourier sphere method (DFS). The DFS extends $f(\theta,\varphi)$ as follows:

$$\tilde{f}(\theta,\varphi) = \begin{cases} f(\theta,\varphi), & (\theta,\varphi) \in [0,\pi] \times [-\pi, 0] \\ f(\theta,\varphi), & (\theta,\varphi) \in [0,\pi] \times [0,\pi] \\ f(-\theta, \varphi - \pi), & (\theta,\varphi) \in [-\pi,0] \times [0,\pi] \\ f(-\theta, \varphi + \pi), & (\theta,\varphi) \in [-\pi,0] \times [-\pi,0] \end{cases} . \tag{17}$$

This succeeds in recovering C^1-smoothness using an interval twice the width of the original interval. Using this extension to compute Fourier coefficients directly will not converge fast. In [27] the DFS extension and structure-preserving Gaussian elimination are combined, which achieves faster convergence. With the fast Fourier extension (FFE) it should be possible to recover C^r-smoothness, for a chosen value for r. And therefore also increase the convergence rate.
Using the idea of FFE a function $f(\theta,\varphi)$, $(\theta,\varphi) \in [0,\pi] \times [-\pi,\pi]$, can be extended as $f_{\text{ext}}(\theta,\varphi)$, on an interval $(\theta,\varphi) \in [-b,\pi] \times [-\pi,\pi]$, where $b > 0$ can be chosen. Note that $f_{\text{ext}}(\theta,\varphi)$ is bi-periodic, it is again interesting to use the Fourier series approximation, see also (3),

$$p_N(\theta,\varphi) = \sum_{k=0}^{M-1} \sum_{n=0}^{N-1} f_k^n \mathrm{e}^{\mathrm{i}n\theta} \mathrm{e}^{\mathrm{i}k\varphi} \tag{18}$$

as an approximation to $f_{\text{ext}}(\theta,\varphi)$ on $[-b,\pi] \times [-\pi,\pi]$ and thus to $f(\theta,\varphi)$ on $[0,\pi] \times [-\pi,\pi]$. For equispaced nodes in both directions, $\theta_h = -b + (\pi + b)h/N$, $h = 0, \ldots, N-1$ and $\varphi_l = -\pi + 2\pi l/M$, $l = 0, \ldots, M-1$ the Fourier coefficients f_k^n can be computed by applying the FFT to the matrix composed of elements $f_{\text{ext}}(\theta_h, \varphi_l)$

$$f_k^n = \sum_{l=-M/2}^{M/2-1} \underbrace{\left(\sum_{h=-N/2}^{N/2-1} f_{\text{ext}}(\theta_h, \varphi_l) \mathrm{e}^{\mathrm{i}n\theta_h} \right)}_{F_{\text{ext}}^l} \mathrm{e}^{\mathrm{i}k\varphi_l} \tag{19}$$

$$k = -M/2, \ldots, M/2-1, \quad n = -N/2, \ldots, N/2-1.$$

The FFT is done M times, for each value of φ_l on a vector of size N (all values of θ_h), this results in M vectors F_{ext}^l of size $(N \times 1)$. Combining these columnwise in a matrix

$$F_{\text{ext}} = \begin{bmatrix} F_{\text{ext}}^0 & F_{\text{ext}}^1, & \ldots, & F_{\text{ext}}^{M-1} \end{bmatrix} \tag{20}$$

and then applying FFT to the rows of F_{ext} results in coefficients f_k^n.

Algorithm 2 shows the implementation of this. The input $\boldsymbol{N}_{\text{Cheb}}$ is a vector of size M. Because for different values of φ, Chebyshev approximation might require more modes to approximate $f(\theta,\varphi)$ accurately.

Algorithm 2 Fast Fourier Extension using 2-point Taylor interpolation applied on a sphere.

Input: N, M, $f(\theta,\varphi)$, $(\theta,\varphi) \in [0,\pi] \times [-\pi,\pi]$, $\boldsymbol{N}_{\text{Cheb}}$, r
for $l = [0 : M-1]$ **do**
  Evaluate $f(\theta, \varphi_l)$ in $\boldsymbol{N}_{\text{Cheb}}(l+1)$ Chebyshev points.
  Compute Chebyshev coefficients via DCT.
  Compute Chebyshev coefficients for derivatives of Chebyshev series via recurrence relation. Then we obtain an approximation for the derivatives of f.
  Evaluate Chebyshev series in $x = 0$ and $x = \pi$ to obtain interpolation values for the two-point Taylor interpolation.
  Evaluate $f_{\text{ext}}(\theta, \varphi_l)$ in N equidistant points $\theta_h = -b + (b+\pi)h/N$, $h = 0, \ldots, N-1$.
  Apply the FFT to $\begin{bmatrix} f_{\text{ext}}(\theta_0, \varphi_l) \\ f_{\text{ext}}(\theta_1, \varphi_l) \\ \vdots \\ f_{\text{ext}}(\theta_{N-1}, \varphi_l) \end{bmatrix}$ to obtain F_{ext}^l.
end for
Combine all F_{ext}^l in a matrix: $F_{\text{ext}} = \begin{bmatrix} F_{\text{ext}}^0 & F_{\text{ext}}^1 & \ldots & F_{\text{ext}}^{M-1} \end{bmatrix}$.
Apply the FFT rowwise to F_{ext} to obtain the 2D-Fourier coefficients f_k^n.
Output: f_k^n $(N \times M)$-matrix

Consider the function $f(\theta,\varphi) = \theta \cos\varphi \sin\theta \cos\theta$. This function can be written as $f(\theta,\varphi) = g(\theta)h(\varphi)$ and therefore it suffices to extend $g(\theta)$ independent of φ.
Figure 3 shows the convergence rate of $p_N(\theta,\varphi)$ for both methods of extending $f(\theta,\varphi)$ measured in the maximum

norm over the relevant interval $(\theta, \varphi) \in [0, \pi] \times [-\pi, \pi]$. Red shows the DFS (without low-rank approximation), blue shows FFE with $r = 3$ and green shows FFE with $r = 5$. All observed convergence rates follow the theoretical convergence rate based on the smoothness of the extended function as stated in (IV.1).

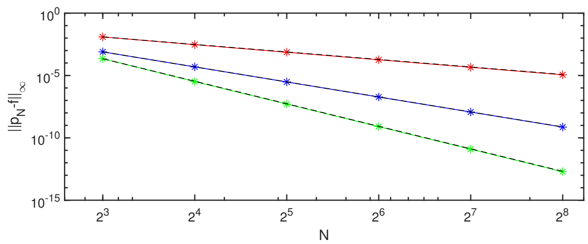

Fig. 3. Theoretical convergence rate $\frac{1}{N^{(r+1)}}$ for $r = 1, 3$ and 5. Red: observed convergence rate of DFS. Blue: observed convergence rate of FFE using interpolation values up to the third derivative. Green: FFE using up to fifth derivative. For Chebyshev approximation in FFE: $N_{\text{cheb}} = 21$.

Figure 4 shows the extended functions $\tilde{f}(\theta, 0)$ and $g_{\text{ext}}(\theta)$ for $g(\theta) = \theta \sin\theta \cos\theta$. For the FFE $r = 3$ is used, the higher smoothness is apparent from the figure.

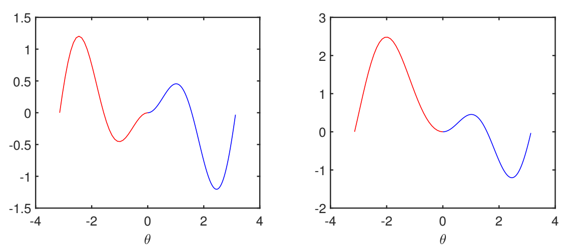

Fig. 4. The extended function $\tilde{f}(\theta, \varphi)$ for the DFS method (with $\varphi = 0$) (left) and $g_{\text{ext}}(\theta)$ for FFE with $r = 3$ (right).

VI. Fast spectral method on the sphere

In many applications, where nonlinear partial differential equations have to be solved on the sphere, so-called Fourier spectral techniques have proven to be very efficient and accurate. These methods benefit from exponential convergence rates for smooth solutions and from fast Fourier transforms. Methods based on the spherical Fourier transform where presented in [28, Section 3.3] and [29]. Methods based on DFS were presented in [27], [30].

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