# Fast and exact reconstruction of arbitrary multivariate algebraic polynomials in Chebyshev form


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***Abstract*—We describe a fast method for the evaluation of an arbitrary high-dimensional multivariate algebraic polynomial in Chebyshev form at the nodes of an arbitrary rank-1 Chebyshev lattice. Our main focus is on conditions on rank-1 Chebyshev lattices allowing for the exact reconstruction of such polynomials from samples along such lattices. We present an algorithm for constructing suitable rank-1 Chebyshev lattices based on a component-by-component approach. Moreover, we give a method for the fast and exact reconstruction.**


## I. Introduction

We denote the Chebyshev polynomials of the first kind by $T_k : [-1,1] \to [-1,1]$, $T_k(x) := \cos(k \arccos x)$, $k \in \mathbb{N}_0$. Note that for each $k \in \mathbb{N}_0$, $T_k$ is an algebraic polynomial of degree $\deg(T_k) = k$ restricted to the domain $[-1,1]$. Moreover, we define the multivariate Chebyshev polynomials $T_{\boldsymbol{k}} : [-1,1]^d \to [-1,1]$, $T_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{t=1}^d T_{k_t}(x_t)$ for $d \in \mathbb{N}$, $\boldsymbol{x} := (x_1,\ldots,x_d)^\top \in [-1,1]^d$ and $\boldsymbol{k} := (k_1,\ldots,k_d)^\top \in \mathbb{N}_0^d$.

Let $\Pi_I := \operatorname{span}\{T_{\boldsymbol{k}}(\circ) \colon \boldsymbol{k} \in I\}$, where $I \subset \mathbb{N}_0^d$, $d \in \mathbb{N}$, is a non-negative index set of finite cardinality, $|I| < \infty$. Then, each multivariate polynomial $p \in \Pi_I$ can be written as

$$p(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \prod_{t=1}^d T_{k_t}(x_t), \quad \hat{p}_{\boldsymbol{k}} \in \mathbb{R}, \quad (1)$$

where $\boldsymbol{x} \in [-1,1]^d$. We remark that if the index set $I = I_n^d := \{\boldsymbol{k} \in \mathbb{N}_0^d \colon \|\boldsymbol{k}\|_1 \le n\}$, $n \in \mathbb{N}_0$, is the $\ell_1$-ball, then $\Pi_I$ is the space of all algebraic polynomials of (total) degree $\le n$ in $d$ variables restricted to the domain $[-1,1]^d$. Moreover, polynomials with hyperbolic cross index sets $I = H_n^d := \left\{\boldsymbol{k} \in \mathbb{N}_0^d \colon \prod_{t=1}^d \max(1,|k_t|) \le n\right\}$, where $n, d \in \mathbb{N}$, have already been used for approximations in sparse high-dimensional spectral Galerkin methods, cf. [1, Section 8.5].

In this paper, for a given arbitrary index set $I \subset \mathbb{N}_0^d$ of finite cardinality, we present a method for the fast evaluation of a polynomial $p$ from (1) at the nodes $\boldsymbol{x}_j := \cos(\frac{j}{M}\pi \boldsymbol{z})$, $j = 0,\ldots,M$, of a $d$-dimensional rank-1 Chebyshev lattice

$$\mathrm{CL}(\boldsymbol{z}, M) := \left\{\boldsymbol{x}_j := \cos\left(\tfrac{j}{M}\pi \boldsymbol{z}\right) : j = 0,\ldots,M\right\}$$

with the generating vector $\boldsymbol{z} \in \mathbb{N}_0^d$ and the size parameter $M \in \mathbb{N}_0$, where the cosine is applied component-wise. For a more general definition of $d$-dimensional rank-$k$ Chebyshev lattices, we refer to [2]. Moreover, we discuss conditions on a rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, M)$ such that the fast and exact reconstruction of all coefficients $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, from sampling values $p(\boldsymbol{x}_j)$ taken at the corresponding nodes $\boldsymbol{x}_j$, $j = 0,\ldots,M$, is possible. Both, for the fast evaluation and reconstruction, we only apply a single one-dimensional discrete cosine transform of type I (DCT-I) and additionally compute simple index transforms, see also [3]. Note that for the special case $I = I_n^d$, constructions of rank-1 Chebyshev lattices suitable for the exact reconstruction were already discussed in [2], [4] and the references therein. Here, we present an algorithm based on component-by-component (CBC) construction for arbitrary index sets $I \subset \mathbb{N}_0^d$ using ideas from [5]–[7].

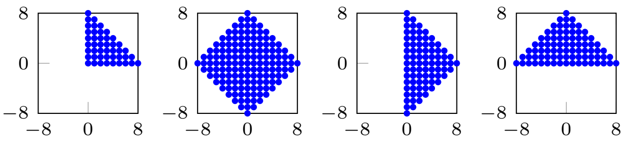
We remark that our considerations for the reconstruction of the coefficients $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, of a polynomial $p$ from (1) with known index set $I \subset \mathbb{N}_0^d$ in this paper establish a basis for the reconstruction of a polynomial $p$ with unknown index set $I$ using a method similar to the one presented in [8].

The remaining parts of this paper are organized as follows: In Secion II, we give prerequisites for the subsequent sections. We discuss the fast evaluation and reconstruction in Section III. In Section IV, we point out relations of our results to existing work. Afterwards, in Section V, we present computed rank-1 Chebyshev lattices suitable for reconstruction. Finally, in Section VI, we summarize the results of this paper.

## II. Prerequisites

### A. One-dimensional DCT-I

First, we recall results for the fast reconstruction of a one-dimensional (algebraic) polynomial $p$. We are able to reconstruct the coefficients $\hat{p}_0,\ldots,\hat{p}_n \in \mathbb{R}$ of a polynomial $p$ from (1) with $I := I_n^1$ from sampling values $p(x_j)$ at the Chebyshev nodes $x_j := \cos(j\pi/n)$, $j = 0,\ldots,n$. For this, we apply a one-dimensional DCT-I to the sampling values $p(x_j)$ and we obtain $\hat{a}_k := \sum_{j=0}^n (\varepsilon_j^n)^2\, p(x_j)\, \cos(jk\pi/n) =$



$|I_8^2| = 45$ $\quad|\mathcal{M}(I_8^2)| = 145$ $\quad|\mathcal{M}_1(I_8^2)| = 81$ $\quad|\mathcal{M}_2(I_8^2)| = 81$

Fig. 1. Index sets $I_8^2$, $\mathcal{M}(I_8^2)$, $\mathcal{M}_1(I_8^2)$, $\mathcal{M}_2(I_8^2)$ (from left to right).

$\sum_{k'\in I_n^1} \hat{p}_{k'} \sum_{j=0}^{n} (\varepsilon_j^n)^2 \cos(jk'\pi/n) \cos(jk\pi/n)$ for $k \in I_n^1$, $\varepsilon_l^n := 1/\sqrt{2}$ for $l \in \{0, n\}$ and $\varepsilon_l^n := 1$ for $l \in \{1, \ldots, n-1\}$, since $T_k(x_j) = T_k(\cos(j\pi/n)) = \cos(jk\pi/n)$. The following orthogonality relation follows straightforward

$$\frac{2}{n}\varepsilon_k^n \varepsilon_{k'}^n \sum_{j=0}^{n} (\varepsilon_j^n)^2 \cos\left(\frac{jk\pi}{n}\right) \cos\left(\frac{jk'\pi}{n}\right) = \delta_{k,k'},\ k, k' \in I_n^1, \tag{2}$$

where $\delta_{k,k'}$ is Kronecker's delta, see e.g. [9, Section 2.4]. This yields the coefficients $\hat{p}_k = \frac{2(\varepsilon_k^n)^2}{n}\,\hat{a}_k$ for $k \in I_n^1$. Note that the DCT-I can be computed by means of a fast algorithm in $\mathcal{O}(n \log n)$ arithmetic operations.

### B. *Index sets and tensor-products of cosines*

Let $I \subset N_0^d$ be an arbitrary index set of finite cardinality. For the description of the approach for the fast evaluation and reconstruction, we define the extended symmetric index set

$$\mathcal{M}(I) := \{\boldsymbol{h} \in \mathbb{Z}^d \colon (|h_1|, \ldots, |h_d|)^\top \in I\},$$

which contains all frequencies $\boldsymbol{k} \in I$ and versions of these frequencies $\boldsymbol{k}$ mirrored at all coordinate axes. Moreover, we define the index sets

$$\mathcal{M}_s(I) := \{\boldsymbol{h} \in \mathcal{M}(I) \colon h_s \geq 0\},\ s \in \{1, \ldots, d\},$$

which contain all frequencies $\boldsymbol{k} \in I$ and versions of these frequencies mirrored at all coordinate axes except the $s$-th. For instance, in the case $d = 2$ and $n = 8$, the index set $I_8^2$ as well as the corresponding extended symmetric index set $\mathcal{M}(I_8^2)$ and mirrored index sets $\mathcal{M}_1(I_8^2)$, $\mathcal{M}_2(I_8^2)$ are depicted in Fig. 1.

Next, we remark that for $y_1, y_2 \in \mathbb{R}$, we have $\cos(y_1)\cos(y_2) = \frac{1}{2}(\cos(y_1 + y_2) + \cos(y_1 - y_2))$. Using induction on the dimension $d \in \mathbb{N}$ and due to $\cos(x) = \cos(-x)$ for all $x \in \mathbb{R}$, we obtain for $\boldsymbol{y} := (y_1, \ldots, y_d)^\top \in \mathbb{R}$

$$\prod_{t=1}^{d} \cos(y_t) = \sum_{\boldsymbol{m}\in\mathcal{M}_s(\{\boldsymbol{1}\})} \frac{1}{2^{d-1}} \cos(\boldsymbol{m}\cdot\boldsymbol{y}) \tag{3}$$

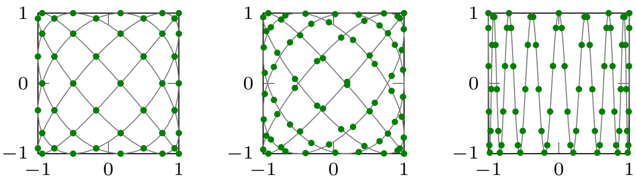
$$= \sum_{\boldsymbol{m}\in\mathcal{M}(\{\boldsymbol{1}\})} \frac{1}{2^{d}} \cos(\boldsymbol{m}\cdot\boldsymbol{y}), \tag{4}$$

where $\boldsymbol{1} := (1, \ldots, 1)^\top \in \mathbb{N}^d$ and $\boldsymbol{m}\cdot\boldsymbol{y} := \sum_{t=1}^{d} m_t y_t$.

## III. Fast evaluation and reconstruction of multivariate polynomials from $\Pi_I$ along rank-1 Chebyshev lattices using DCT-I

### A. *Fast evaluation at the nodes of rank-1 Chebyshev lattices*

Briefly, we describe a simple method for the fast evaluation of a polynomial $p$ from (1) with arbitrary index set $I \subset \mathbb{N}_0^d$



(a) $\boldsymbol{z} := (8, 9)^\top$, $M := 72$, $|\mathrm{CL}(\boldsymbol{z}, M)| = 45$. (b) $\boldsymbol{z} := (8, 9)^\top$, $M := 73$, $|\mathrm{CL}(\boldsymbol{z}, M)| = 74$. (c) $\boldsymbol{z} := (1, 16)^\top$, $M := 76$, $|\mathrm{CL}(\boldsymbol{z}, M)| = 77$.

Fig. 2. Rank-1 Chebyshev lattices $\mathrm{CL}(\boldsymbol{z}, M)$.

at the nodes $\boldsymbol{x}_j := \cos(\frac{j}{M}\pi\boldsymbol{z})$, $j = 0, \ldots, M$, of an arbitrary $d$-dimensional rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, M)$. Examples for two-dimensional rank-1 Chebyshev lattices are shown in Fig. 2. We remark that not all $(M + 1)$ nodes $\boldsymbol{x}_j$, $j = 0, \ldots, M$, have to be distinct, i.e., $|\mathrm{CL}(\boldsymbol{z}, M)| \in \{1, \ldots, M + 1\}$, see e.g. Fig. 2a. Due to (3), we have

$$p(\boldsymbol{x}_j) = \sum_{\boldsymbol{k}\in I} \frac{\hat{p}_{\boldsymbol{k}}}{2^{d-1}} \sum_{\boldsymbol{m}\in\mathcal{M}_s(\{\boldsymbol{1}\})} \cos\left(\frac{j}{M}\pi\,(\boldsymbol{m}\odot\boldsymbol{k})\cdot\boldsymbol{z}\right),$$

$j = 0, \ldots, M$, for any $s \in \{1, \ldots, d\}$ and for each polynomial $p$ from (1), where $\boldsymbol{m}\odot\boldsymbol{k} := (m_1k_1, \ldots, m_dk_d)^\top$. For $M \in \mathbb{N}$ and $l \in \mathbb{Z}$, we define the even-mod relation

$$l \,\mathrm{emod}\, M := \begin{cases} l \bmod (2M), & l \bmod (2M) \leq M, \\ 2M - (l \bmod (2M)) & \text{else}. \end{cases}$$

For each $l \in I_M^1$, we consider the frequencies $\boldsymbol{k} \in I$ and $\boldsymbol{m} \in \mathcal{M}_s(\{\boldsymbol{1}\})$, such that $l = (\boldsymbol{m}\odot\boldsymbol{k})\cdot\boldsymbol{z} \,\mathrm{emod}\, M$. Since we have $\cos(jl\pi/M) = \cos\left(\frac{j}{M}\pi\,(\boldsymbol{m}\odot\boldsymbol{k})\cdot\boldsymbol{z}\right)$ for $j = 0, \ldots, M$ in the case $l = (\boldsymbol{m}\odot\boldsymbol{k})\cdot\boldsymbol{z} \,\mathrm{emod}\, M$, we obtain $p(\boldsymbol{x}_j) = \sum_{l=0}^{M} (\varepsilon_l^M)^2\, \hat{b}_l \cos(jl\pi/M)$ with the coefficients

$$\hat{b}_l := \sum_{\boldsymbol{k}\in I} \sum_{\substack{\boldsymbol{m}\in\mathcal{M}_s(\{\boldsymbol{1}\}) \\ (\boldsymbol{m}\odot\boldsymbol{k})\cdot\boldsymbol{z} \,\mathrm{emod}\, M = l}} \frac{\hat{p}_{\boldsymbol{k}}}{2^{d-1}\,(\varepsilon_l^M)^2} \quad \text{for } l \in I_M^1. \tag{5}$$

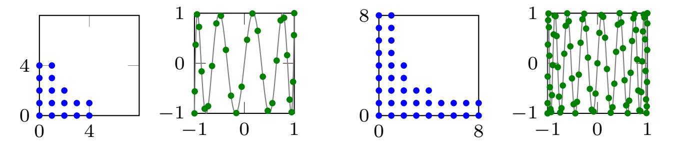
Therefore, for any $s \in \{1, \ldots, d\}$, we build the index set $\mathcal{M}_s(I)$ and we compute the coefficients $\hat{b}_l$ by (5) for $l \in I_M^1$. Then, we apply a one-dimensional DCT-I to these coefficients $\hat{b}_l$ and this yields the function values $p(\boldsymbol{x}_j)$ for $j = 0, \ldots, M$. In total, we require $\mathcal{O}(M \log M + d\,2^d|I|)$ arithmetic operations.

### B. *Fast and exact reconstruction*

In this section, we consider the fast reconstruction of a polynomial $p$ from (1) with arbitrary index set $I \subset \mathbb{N}_0^d$, $|I| < \infty$. Our approach is based on applying a one-dimensional DCT-I to the sampling values $p(\boldsymbol{x}_j)$ at the nodes $\boldsymbol{x}_j := \cos(j\pi\boldsymbol{z}/M)$, $j = 0, \ldots, M$, of a rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, M)$ fulfilling a certain property. Concretely, we compute the coefficients

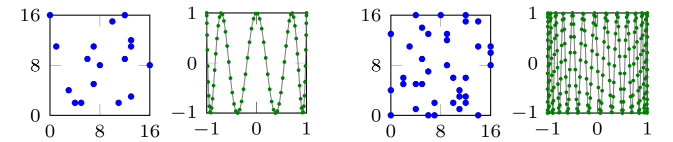
$$\hat{a}_l := \sum_{j=0}^{M} (\varepsilon_j^M)^2\, p(\boldsymbol{x}_j) \cos\left(\frac{jl}{M}\pi\right) \tag{6}$$

$$= \sum_{j=0}^{M} (\varepsilon_j^M)^2 \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \left(\prod_{t=1}^{d} \cos\left(\frac{j}{M}\pi\,k_t\,z_t\right)\right) \cos\left(\frac{jl}{M}\pi\right)$$



(a) $H_4^2$ and CL $\left((1,9)^\top, 29\right)$. (b) $H_8^2$ and CL $\left((1,17)^\top, 82\right)$.

Fig. 3. Examples for hyperbolic cross index sets $I = H_n^2$ and corresponding rank-1 Chebyshev lattices $\mathrm{CL}(\boldsymbol{z}, M)$ fulfilling condition (7).



(a) $I$ ($|I|=17$), CL $\left((1,8)^\top, 77\right)$. (b) $I$ ($|I|=37$), CL $\left((1,27)^\top, 249\right)$.

Fig. 4. Examples for arbitrarily chosen index sets $I \subset \mathbb{N}_0^2$ and corresponding rank-1 Chebyshev lattices $\mathrm{CL}(\boldsymbol{z}, M)$ fulfilling condition (7).

for $l \in I_M^1$. Due to (4), this means $\hat{a}_l = \sum_{\boldsymbol{k}\in I} \frac{\hat{p}_{\boldsymbol{k}}}{2^d} \sum_{\boldsymbol{m}\in\mathcal{M}(\{\boldsymbol{1}\})} \sum_{j=0}^{M} (\varepsilon_j^M)^2 \cos\left(\frac{j}{M}\pi\,(\boldsymbol{m}\odot\boldsymbol{k})\cdot\boldsymbol{z}\right) \cos\left(\frac{jl}{M}\pi\right)$ for $l \in I_M^1$. We consider the indices $l := \boldsymbol{k}\cdot\boldsymbol{z} \operatorname{emod} M$ for $\boldsymbol{k} \in I$. Since we have $\{\boldsymbol{m}\odot\boldsymbol{k} \colon \boldsymbol{m} \in \mathcal{M}(\{\boldsymbol{1}\})\} = \mathcal{M}(\{\boldsymbol{k}\})$ for $\boldsymbol{k} \in I$ and due to the orthogonality condition (2), we are able to exactly reconstruct all the coefficients $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, of the polynomial $p$ from (1) using the computed coefficients $\hat{a}_l$, $l := \boldsymbol{k}\cdot\boldsymbol{z} \operatorname{emod} M$ for $\boldsymbol{k} \in I$, from (6) if and only if

$$\begin{aligned}&\boldsymbol{k}\cdot\boldsymbol{z} \operatorname{emod} M \neq \boldsymbol{h}\cdot\boldsymbol{z} \operatorname{emod} M \\ &\text{for all } \boldsymbol{k} \in I \text{ and } \boldsymbol{h} \in \mathcal{M}(I),\ \boldsymbol{k} \neq (|h_1|,\ldots,|h_d|)^\top. \end{aligned} \tag{7}$$

Examples for two-dimensional hyperbolic cross index sets $I = H_4^2$ and $I = H_8^2$ with corresponding rank-1 Chebyshev lattices $\mathrm{CL}(\boldsymbol{z}, M)$ fulfilling condition (7) are depicted in Fig. 3 as well as two-dimensional examples for index sets $I$ with less structure with corresponding $\mathrm{CL}(\boldsymbol{z}, M)$ in Fig. 4. Moreover, the rank-1 Chebyshev lattices $\mathrm{CL}(\boldsymbol{z}, M)$ in Fig. 2 fulfill condition (7) for the $\ell_1$-ball index set $I = I_8^2$ in Fig. 1.

Due to the symmetry of the emod operator, we can reduce the number of tests in condition (7) by a factor of (about) two.

**Lemma III.1.** *For $M \in \mathbb{N}_0$ and $l \in \mathbb{Z}$, we have $l \operatorname{emod} M = (-l) \operatorname{emod} M$.*

*Proof.* Considering the two different cases in the definition of the emod operator, the assertion follows straight forward. □

**Lemma III.2.** *For a given arbitrary index set $I \subset \mathbb{N}_0^d$ of finite cardinality, $|I| < \infty$, let $\tilde{I} \subset \mathbb{Z}^d$ be an arbitrary index set with the property $\mathcal{M}(I) = \tilde{I} \cup \{-\boldsymbol{h} \colon \boldsymbol{h} \in \tilde{I}\}$. Then, condition (7) is equivalent to*

$$\begin{aligned}&\boldsymbol{k}\cdot\boldsymbol{z} \operatorname{emod} M \neq \boldsymbol{h}\cdot\boldsymbol{z} \operatorname{emod} M \\ &\textit{for all } \boldsymbol{k} \in I \textit{ and } \boldsymbol{h} \in \tilde{I},\ \boldsymbol{k} \neq (|h_1|,\ldots,|h_d|)^\top. \end{aligned}$$

*Proof.* Due to $(-\boldsymbol{h})\cdot\boldsymbol{z} = -(\boldsymbol{h}\cdot\boldsymbol{z})$ for $\boldsymbol{h} \in \mathbb{Z}^d$, we obtain

$$(-\boldsymbol{h})\cdot\boldsymbol{z} \operatorname{emod} M = \boldsymbol{h}\cdot\boldsymbol{z} \operatorname{emod} M \text{ for } \boldsymbol{h} \in \mathbb{Z}^d \tag{8}$$

from Lemma III.1 and the assertion follows. □

**Input:** index set $I_{\text{input}} \subset \mathbb{N}_0^d$, parameter $s \in \{1,\ldots,d\}$.
Determine suitable initial size parameter $M_{\text{start}}$, see e.g. Remark IV.4.
**for** $t := 1,\ldots,d$ **do**
  **for** $z_t := 0,\ldots,M_{\text{start}}$ **do**
    **if** condition (9) is valid for $I := \{(k_1,\ldots,k_t)^\top \colon \boldsymbol{k} \in I_{\text{input}}\}$, $\boldsymbol{z} := (z_1,\ldots,z_t)^\top$, $M := M_{\text{start}}$ **then**
      **break**
    **end if**
  **end for**
**end for**
**for** $M := |I_{\text{input}}| - 1,\ldots,M_{\text{start}}$ **do**
  **if** condition (9) is valid for $I := I_{\text{input}}$, $\boldsymbol{z} := (z_1,\ldots,z_d)^\top$, $M$ **then**
    **break**
  **end if**
**end for**
**Output:** generating vector $\boldsymbol{z} \in \mathbb{N}_0^d$ and size parameter $M \in \mathbb{N}_0$ fulfilling condition (7) for index set $I := I_{\text{input}}$.

Fig. 5. Algorithm for construction of rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, M)$ suitable for reconstruction of multivariate polynomials (1) supported on the index set $I := I_{\text{input}}$.

**Corollary III.3.** *For any $s \in \{1,\ldots,d\}$, condition (7) is equivalent to*

$$\begin{aligned}&\boldsymbol{k}\cdot\boldsymbol{z} \operatorname{emod} M \neq \boldsymbol{h}\cdot\boldsymbol{z} \operatorname{emod} M \\ &\textit{for all } \boldsymbol{k} \in I \textit{ and } \boldsymbol{h} \in \mathcal{M}_s(I),\ \boldsymbol{k} \neq (|h_1|,\ldots,|h_d|)^\top. \end{aligned} \tag{9}$$

If condition (7) or (9) is fulfilled, we can reconstruct the coefficients $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, in the following way. We apply a DCT-I to the sampling values $p(\boldsymbol{x}_j) = p(\cos(j\pi\boldsymbol{z}/M))$, $j = 0,\ldots,M$, which yields the coefficients $\hat{a}_l$, $l \in I_M^1$, in (6). Then, we obtain the coefficients of the polynomial $p$ by

$$\hat{p}_{\boldsymbol{k}} = \frac{2^d(\varepsilon_l^M)^2}{M} \frac{\hat{a}_l}{|\{\boldsymbol{m} \in \mathcal{M}_s(\{\boldsymbol{1}\}) \colon (\boldsymbol{m}\odot\boldsymbol{k})\cdot\boldsymbol{z} \operatorname{emod} M = l\}|}$$

with $l := \boldsymbol{k}\cdot\boldsymbol{z} \operatorname{emod} M$ for all $\boldsymbol{k} \in I$ and any $s \in \{1,\ldots,d\}$. Using a fast algorithm for the DCT-I, this computation can be performed in $\mathcal{O}(M\log M + d\,2^d|I|)$ arithmetic operations.

Again, we stress the fact that the index set $I \subset \mathbb{N}_0^d$, $|I| < \infty$, may be arbitrarily chosen. Upper bounds on the size parameter $M$ for the existence of a rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, M)$ fulfilling condition (7) are discussed in Section IV-B. A method for the construction of a suitable generating vector $\boldsymbol{z} \in \mathbb{N}_0^d$ is described in the following subsection.

### C. Construction of suitable rank-1 Chebyshev lattices

In Fig. 5, we present an algorithm for the construction of a rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, M)$ which allows for the exact reconstruction of the coefficients $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, of a polynomial $p$ from (1) based on samples taken at the nodes of

$\mathrm{CL}(\boldsymbol{z}, M)$, where $I \subset \mathbb{N}_0^d$, $|I| < \infty$, is an arbitrary index set. Our algorithm is based on [7, Algorithm 1 and 2] and uses a CBC search for the generating vector $\boldsymbol{z} \in \mathbb{N}_0^d$.

## IV. Relations to existing work

### A. Padua points and higher-dimensional rank-$s$ Chebyshev lattices

In [10], special sampling points were discussed in the two-dimensional case, so-called Padua points. For a parameter $n \in \mathbb{N}$, these are the nodes $\boldsymbol{x}_j := (\cos(j\pi/(n+1)), \cos(j\pi/n))^\top = \cos(j\pi\boldsymbol{z}/M)$, $j = 0, \ldots, M$, of the rank-1 Chebyshev lattice $\mathcal{A}_n := \mathrm{CL}(\boldsymbol{z}, M)$, where the generating vector $\boldsymbol{z} := (n, n+1)^\top$ and the size parameter $M := n(n+1)$. As discussed in [10, Section 2], the Padua point set $\mathcal{A}_n$ only consists of $\binom{n+2}{2} = \frac{n^2}{2} + \frac{3}{2}n + 1$ distinct points, whereas $M = n^2 + n$.

**Lemma IV.1.** *Let the index set $I = I_n^2 := \{\boldsymbol{k} \in \mathbb{N}_0^2 \colon k_1 + k_2 \leq n\}$, $n \in \mathbb{N}_0$, be the $\ell_1$-ball. Then, condition* (7) *is fulfilled and we can exactly reconstruct the coefficients $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, of a polynomial $p$ from* (1) *from sampling values at the nodes of the Padua point set $\mathcal{A}_n$ using* (6).

*Proof.* The assertion follows from the Lagrange interpolation formula [11, (7c)]. Alternatively, condition (9) from Corollary III.3 can be verified. □

In [4], an extensive search for higher-rank Chebyshev lattices allowing for the reconstruction of polynomials $p$ from (1) with $\ell_1$-ball index sets $I := I_n^d$ was performed and numerical results for the cases $d = 3, 4, 5$ were presented.

### B. Reconstructing rank-1 lattices of multivariate trigonometric polynomials

In the following, we briefly show the relation to reconstructing rank-1 lattices of multivariate trigonometric polynomials from [7].

**Theorem IV.2.** *Let $I \subset \mathbb{N}_0^d$ be an arbitrary index set of finite cardinality, $|I| < \infty$. Moreover, let $\Lambda(\boldsymbol{z}, \hat{M}) := \{\boldsymbol{y}_j := \frac{j}{\hat{M}}\boldsymbol{z} \bmod \boldsymbol{1} \colon j = 0, \ldots, \hat{M}-1\}$ be a reconstructing rank-1 lattice with generating vector $\boldsymbol{z} \in \mathbb{N}_0^d$ and even rank-1 lattice size $\hat{M} \in 2\mathbb{N}$ for the extended symmetric index set $\mathcal{M}(I)$, i.e.,*

$$\boldsymbol{h}\cdot\boldsymbol{z} \not\equiv \boldsymbol{h'}\cdot\boldsymbol{z} \pmod{\hat{M}} \text{ for all } \boldsymbol{h}, \boldsymbol{h'} \in \mathcal{M}(I),\ \boldsymbol{h} \neq \boldsymbol{h'}. \tag{10}$$

*Then, the rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, \frac{\hat{M}}{2})$ fulfills condition* (7)*, i.e., we are able to exactly reconstruct the coefficients of a polynomial from* (1) *using samples at the nodes of* $\mathrm{CL}(\boldsymbol{z}, \frac{\hat{M}}{2})$.

*Proof.* We consider the values

$$\boldsymbol{h}\cdot\boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2} = \begin{cases} \boldsymbol{h}\cdot\boldsymbol{z} \bmod \hat{M}, & \boldsymbol{h}\cdot\boldsymbol{z} \bmod \hat{M} \leq \frac{\hat{M}}{2}, \\ \hat{M} - (\boldsymbol{h}\cdot\boldsymbol{z} \bmod \hat{M}) & \text{else}, \end{cases}$$

for $\boldsymbol{h} \in \mathcal{M}(I)$. Due to property (10), all values $\boldsymbol{h}\cdot\boldsymbol{z} \bmod \hat{M}$ are distinct for $\boldsymbol{h} \in \mathcal{M}(I)$ and we obtain for each $l \in I^1_{\hat{M}/2}$ that one of the following three cases may occur: Either

1. exactly two distinct frequencies $\boldsymbol{h}, \boldsymbol{h'} \in \mathcal{M}(I)$ exist such that $\boldsymbol{h}\cdot\boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2} = \boldsymbol{h'}\cdot\boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2} = l$, or
2. exactly one frequency $\boldsymbol{h} \in \mathcal{M}(I)$ exists such that $\boldsymbol{h}\cdot\boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2} = l$, or
3. such a frequency does not exist for $l$.

In the first case, $\boldsymbol{h'} = -\boldsymbol{h}$ follows, since for each $\boldsymbol{h} \in \mathcal{M}(I)\backslash\{\boldsymbol{0}\}$, also the frequency $-\boldsymbol{h} \in \mathcal{M}(I)\backslash\{\boldsymbol{0}\}$ and we have (8) with $M := \frac{\hat{M}}{2}$, i.e., $(-\boldsymbol{h})\cdot\boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2} = \boldsymbol{h}\cdot\boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2} = l$. The second case can only occur for $\boldsymbol{h} = \boldsymbol{0}$, since otherwise the (non-zero) frequency $-\boldsymbol{h} \in \mathcal{M}(I) \setminus \{\boldsymbol{0}\}$, $-\boldsymbol{h} \neq \boldsymbol{h}$, and this would yield $(-\boldsymbol{h})\cdot\boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2} = \boldsymbol{h}\cdot\boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2}$ which corresponds to the first case. In total, we obtain $\boldsymbol{h}\cdot\boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2} \neq \boldsymbol{h'}\cdot\boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2}$ for all $\boldsymbol{h}, \boldsymbol{h'} \in \mathcal{M}(I)$, $(|h_1|, \ldots, |h_d|)^\top \neq (|h'_1|, \ldots, |h'_d|)^\top$, implying condition (7). □

**Remark IV.3.** *Condition* (7) *and* (10) *with $\hat{M} = 2M$ are not equivalent in general. For instance, the generating vector $\boldsymbol{z} := (8, 9)^\top$ and size parameter $M := 72$ from Fig. 2a fulfill condition* (7) *for $I = I_8^2$ but not condition* (10) *with $\hat{M} = 2M$.*

We note that there exist special cases where both the conditions (7) and (10) are fulfilled, see e.g. the examples in Fig. 2b and 2c, which fulfill both conditions for $I = I_8^2$.

**Remark IV.4.** *There always exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, \hat{M})$ for $\mathcal{M}(I)$ with even rank-1 lattice size*

$$\hat{M} \leq 2 \max\left\{\frac{2}{3}(|\mathcal{M}(I)|^2 - |\mathcal{M}(I)| + 8), \max_{\boldsymbol{k}\in I} 3\|\boldsymbol{k}\|_\infty\right\}$$

*and consequently a rank-1 Chebyshev lattice $\mathrm{CL}(\boldsymbol{z}, M)$ with size parameter $M := \hat{M}/2$. This result is due to [8, Theorem 2.1] which is a direct consequence of the results from [7].*

### C. Tent-transformed rank-1 lattices for cosine polynomials

In [12], [13], tent-transformed rank-1 lattices $P_\phi(\boldsymbol{z}, \hat{M}) := \{\phi(j\boldsymbol{z}/\hat{M} \bmod \boldsymbol{1}) \colon j = 0, \ldots, \hat{M}-1\}$, fulfilling a condition equivalent to (10) are used, where $\boldsymbol{z} \in \mathbb{N}_0^d$, $\hat{M} \in \mathbb{N}$ and the tent transform $\phi \colon [0,1] \to [0,1]$, $\phi(x) := 1 - |2x-1|$, is applied component-wise. Then, the exact reconstruction of cosine polynomials $\tilde{p} \colon [0,1] \to \mathbb{R}$, $\tilde{p}(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \tilde{a}_{\boldsymbol{k}} \prod_{t=1}^d \cos(\pi k_t x_t)$, $I \subset \mathbb{N}_0^d$, can be performed by applying a fast Fourier transform to samples at these nodes, cf. [13]. Note that these polynomials $\tilde{p}$ are not algebraic polynomials in general.

## V. Numerical results

Using the algorithm in Fig. 5, we construct rank-1 Chebyshev lattices $\mathrm{CL}(\boldsymbol{z}, M)$ fulfilling condition (7) for the $\ell_1$-ball index sets $I := I_n^d$ for various refinements $n \in \mathbb{N}$ and dimensions $d$. The corresponding size parameters $M$ and oversampling factors $(M+1)/|I_n^d|$ are shown in Table I. Additionally, we apply [7, Algorithm 1 and 2] to the extended symmetric index sets $\mathcal{M}(I_n^d)$ with the modification that an even rank-1 lattice size $\hat{M} \in 2\mathbb{N}$ is returned. We obtain reconstructing rank-1 lattices $\Lambda(\boldsymbol{z}, \hat{M})$ for $\mathcal{M}(I_n^d)$ and consequently rank-1 Chebyshev lattices $\mathrm{CL}(\boldsymbol{z}, \hat{M}/2)$ fulfilling condition (7) for $I_n^d$ due to Theorem IV.2. For the dimensions $d$

TABLE I

CARDINALITIES OF $\ell_1$-BALL INDEX SETS $I_n^d$ AS WELL AS SIZE PARAMETERS $M$ OF CORRESPONDING RANK-1 CHEBYSHEV LATTICES $\mathrm{CL}(\boldsymbol{z}, M)$, WHERE $M$ FULFILLS CONDITION (7) AND $\hat{M} = 2M$ CONDITION (10) FOR $I := I_n^d$, RESPECTIVELY.

| Parameters | | Cardinalities | | Condition (7) / (9) / (10) | |
|---|---|---|---|---|---|
| $d$ | $n$ | $\lvert I_n^d\rvert$ | $\lvert\mathcal{M}_1(I_n^d)\rvert$ | $M = \frac{\hat{M}}{2}$ | $\frac{M+1}{\lvert I_n^d\rvert}$ |
| 2 | 64 | 2 145 | 4 225 | 4 192 | 1.95 |
| 2 | 128 | 8 385 | 16 641 | 16 576 | 1.98 |
| 2 | 256 | 33 153 | 66 049 | 65 920 | 1.99 |
| 3 | 16 | 969 | 3 281 | 4 265 | 4.40 |
| 3 | 32 | 6 545 | 23 969 | 33 361 | 5.10 |
| 3 | 64 | 47 905 | 183 105 | 264 353 | 5.52 |
| 6 | 4 | 210 | 985 | 1 461 | 6.96 |
| 6 | 8 | 3 003 | 26 577 | 63 369 | 21.10 |
| 6 | 16 | 74 613 | 1 110 049 | 3 242 322 | 43.46 |
| 7 | 4 | 330 | 1 765 | 2 777 | 8.42 |
| 7 | 8 | 6 435 | 74 313 | 223 332 | 34.71 |
| 7 | 16 | 245 157 | 5 529 233 | 21 254 517 | 86.70 |
| 10 | 2 | 66 | 201 | 202 | 3.08 |
| 10 | 4 | 1 001 | 7 001 | 19 423 | 19.40 |
| 10 | 8 | 43 758 | 927 441 | 5 912 807 | 135.13 |

TABLE II

CARDINALITIES OF HYPERBOLIC CROSS INDEX SETS $H_n^d$ AS WELL AS SIZE PARAMETERS $M := \widetilde{M}$ AND $M := \hat{M}/2$ OF CORRESPONDING RANK-1 CHEBYSHEV LATTICES $\mathrm{CL}(\boldsymbol{z}, M)$ FULFILLING CONDITION (7) AND (10) FOR $I := H_n^d$, RESPECTIVELY.

| Parameters | | Card. | Condition (7) / (9) | | Condition (10) |
|---|---|---|---|---|---|
| $d$ | $n$ | $\lvert H_n^d\rvert$ | $\widetilde{M}$ | $\frac{\widetilde{M}+1}{\lvert H_n^d\rvert}$ | $\hat{M}/2$ |
| 2 | 256 | 1 979 | 66 050 | 33.38 | 66 050 |
| 2 | 512 | 4 305 | 263 170 | 61.13 | 263 170 |
| 2 | 1 024 | 9 311 | 1 050 626 | 112.84 | 1 050 626 |
| 3 | 256 | 10 303 | 302 883 | 29.40 | 359 075 |
| 3 | 512 | 23 976 | 1 424 613 | 59.42 | 1 424 662 |
| 3 | 1 024 | 55 202 | 4 600 672 | 83.34 | 5 560 838 |
| 6 | 16 | 8 684 | 303 396 | 34.94 | 557 773 |
| 6 | 32 | 26 088 | 1 751 513 | 67.14 | 2 867 903 |
| 6 | 64 | 76 433 | 8 979 932 | 117.49 | 13 603 339 |
| 7 | 8 | 7 184 | 291 267 | 40.54 | 529 877 |
| 7 | 16 | 23 816 | 1 659 143 | 69.67 | 3 575 914 |
| 7 | 32 | 75 532 | 10 375 340 | 137.36 | 21 375 543 |
| 10 | 2 | 6 144 | 495 451 | 80.64 | 2 157 672 |
| 10 | 4 | 27 904 | 3 083 988 | 110.52 | 15 390 479 |
| 10 | 8 | 109 824 | 25 099 619 | 228.54 | 88 580 127 |

and refinements $n$ in Table I except the case $d = 7$ and $n = 16$, these rank-1 Chebyshev lattices are identical to the ones constructed by the algorithm in Fig. 5. In the mentioned case, the algorithm in Fig. 5 yielded a slightly larger size parameter $M = 21\,344\,934$. The reason for this is the greedy search for the generating vector $\boldsymbol{z}$ with fixed initial size parameter $M = M_{\mathrm{start}}$ and both approaches returned a distinct generating vector $\boldsymbol{z}$. If we run the algorithm in Fig. 5 setting $M_{\mathrm{start}} := 21\,254\,517$, then both approaches yield an identical rank-1 Chebyshev lattice.

Moreover, we consider hyperbolic cross index sets $I := H_n^d$. Again, we apply both algorithms for the construction of rank-1 Chebyshev lattices $\mathrm{CL}(\boldsymbol{z}, M)$ suitable for reconstruction. The results of these construction processes are shown in Table II. We remark that the size parameters $M$ of the rank-1 Chebyshev lattices $\mathrm{CL}(\boldsymbol{z}, M)$ are distinctly larger for $d \geq 3$ when using [7, Algorithm 1 and 2], which itself uses condition (10).

## VI. CONCLUSION

In this paper, we considered the fast evaluation as well as the fast and exact reconstruction of arbitrary high-dimensional multivariate algebraic polynomials in Chebyshev form. To this end, we used the nodes of rank-1 Chebyshev lattices. Moreover, we presented an algorithm for the construction of rank-1 Chebyshev lattices suitable for reconstruction based on ideas for the CBC construction in the periodic case.

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## REFERENCES


[1] J. Shen, T. Tang, and L.-L. Wang, *Spectral Methods*, ser. Springer Ser. Comput. Math. Berlin: Springer-Verlag Berlin Heidelberg, 2011, vol. 41.

[2] R. Cools and K. Poppe, "Chebyshev lattices, a unifying framework for cubature with Chebyshev weight function," *BIT Numerical Mathematics*, vol. 51, pp. 275 – 288, 2011.

[3] K. Poppe and R. Cools, "CHEBINT: A MATLAB/Octave Toolbox for Fast Multivariate Integration and Interpolation Based on Chebyshev Approximations over Hypercubes," *ACM Trans. Math. Softw.*, vol. 40, no. 1, pp. 2:1–2:13, Oct. 2013.

[4] ——, "In Search for Good Chebyshev Lattices," in *Monte Carlo and Quasi-Monte Carlo Methods 2010*, ser. Springer Proceedings in Mathematics & Statistics, L. Plaskota and H. Woźniakowski, Eds. Springer Berlin Heidelberg, 2012, vol. 23, pp. 639 – 654.

[5] I. H. Sloan and A. V. Reztsov, "Component-by-component construction of good lattice rules," *Math. Comp.*, vol. 71, pp. 263 – 273, 2002.

[6] R. Cools and D. Nuyens, "Fast algorithms for component-by-component construction of rank-1 lattice rules in shift-invariant reproducing kernel Hilbert spaces," *Math. Comp.*, vol. 75, pp. 903 – 920, 2004.

[7] L. Kämmerer, "Reconstructing multivariate trigonometric polynomials from samples along rank-1 lattices," in *Approximation Theory XIV: San Antonio 2013*, G. E. Fasshauer and L. L. Schumaker, Eds. Springer International Publishing, 2014, pp. 255 – 271.

[8] D. Potts and T. Volkmer, "Sparse high-dimensional FFT based on rank-1 lattice sampling," *Preprint 171, DFG Priority Program 1324*, 2014.

[9] K. R. Rao and P. Yip, *Discrete Cosine Transform: Algorithms, Advantages, Applications*. Boston: Academic Press, 1990.

[10] L. Bos, M. Caliari, S. De Marchi, M. Vianello, and Y. Xu, "Bivariate Lagrange interpolation at the Padua points: The generating curve approach," *J. Approx. Theory*, vol. 143, pp. 15 – 25, 2006, special Issue on Foundations of Computational Mathematics.

[11] M. Caliari, S. De Marchi, and M. Vianello, "Bivariate Lagrange interpolation at the Padua points: Computational aspects," *J. Comput. Appl. Math.*, vol. 221, pp. 284 – 292, 2008, Special Issue: Recent Progress in Spline and Wavelet Approximation.

[12] J. Dick, D. Nuyens, and F. Pillichshammer, "Lattice rules for nonperiodic smooth integrands," *Numerische Mathematik*, vol. 126, pp. 259 – 291, 2014.

[13] D. Nuyens, "Approximation in cosine space using tent transformed lattice rules," 2014, Slides of talk given at ICERM Semester Program, September 15–19 2014, Brown University, Providence, RI.