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Reconstruction of sparse Legendre and Gegenbauer expansions

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Abstract We present a new deterministic approximate algorithm for the reconstruction of sparse Legendre expansions from a small number of given samples. Using asymptotic properties of Legendre polynomials, this reconstruction is based on Prony-like methods. The method proposed is robust with respect to noisy sampled data. Furthermore we show that the suggested method can be extended to the reconstruction of sparse Gegenbauer expansions of low positive order.

Keywords Legendre polynomials · Sparse Legendre expansions · Gegenbauer polynomials · Ultraspherical polynomials · Sparse Gegenbauer expansions · Sparse recovering · Sparse Legendre interpolation · Sparse Gegenbauer interpolation · Asymptotic formula · Prony-like method

Mathematics Subject Classification 65D05 · 33C45 · 41A45 · 65F15

1 Introduction

In this paper, we present a new deterministic approximate approach to the reconstruction of sparse Legendre and Gegenbauer expansions, if relatively few samples on a special grid are given.

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Recently the reconstruction of sparse trigonometric polynomials has attained much attention. There exist recovery methods based on random sampling related to compressed sensing (see e.g. [5,6,11,18] and the references therein) and methods based on deterministic sampling related to Prony-like methods (see e.g. [16] and the references therein).

Both methods are already generalized to other polynomial systems. Rauhut and Ward [19] presented a recovery method of a polynomial of degree at most $N-1$ given in Legendre expansion with M nonzero terms, where $\mathcal{O}(M\,(\log N)^4)$ random samples are taken independently according to the Chebyshev probability measure of $[-1,\ 1]$. Some recovery algorithms in compressive sensing are based on (weighted) ℓ_1-minimization (see [19,20] and the references therein). Exact recovery of sparse functions can be ensured only with a certain probability.

Peter et al. [15] have presented a Prony-like method for the reconstruction of sparse Legendre expansions, where only $2M+1$ function resp. derivative values at one point are given. We mention that sparse Legendre expansions are important for the analysis of spherical functions which are represented by expansions of spherical harmonics. In [1], it was observed that the equilibrium state admits a sparse representation in spherical harmonics (see [1, Section 4.3]), i.e., the equilibrium state can be represented by a very short Legendre expansion (see [1, Figure 4.5]).

Recently, the authors have described a unified approach to Prony-like methods in [16] and applied it to the recovery of sparse expansions of Chebyshev polynomials of first and second kind in [17]. Similar sparse interpolation problems for special polynomial systems have been formerly explored in [3,4,8,12] and also solved by Prony-like methods. A very general approach for the reconstruction of sparse expansions of eigenfunctions of suitable linear operators was suggested by Peter and Plonka [14]. New reconstruction formulas for M-sparse expansions of orthogonal polynomials using the Sturm–Liouville operator, were presented. However one has to use sampling points and derivative values.

In this paper we present a new method for the reconstruction of sparse Legendre expansions which is based on a local approximation of Legendre polynomials by cosine functions. Therefore this algorithm is closely related to [17]. Note that fast algorithms for the computation of Fourier–Legendre coefficients in a Legendre expansion (see [2,7]) are based on similar asymptotic formulas of the Legendre polynomials. However the key idea is that the conveniently scaled Legendre polynomials behave similar to the cosine functions near zero. Therefore we use a sampling grid located near zero. Finally we generalize this method to sparse Gegenbauer expansions.

The outline of this paper is as follows. In Sect. 2, we collect some useful properties of Legendre polynomials. In Sect. 3, we present the new reconstruction method for sparse Legendre expansions. We extend our recovery method in Sect. 4 to the case of sparse Gegenbauer expansions of low positive order. Finally we show some results of numerical experiments in Sect. 5. In Example 5.5, it is shown that the method proposed is robust with respect to noisy sampled data.

2 Properties of Legendre polynomials

As known, the Legendre polynomials P_n are special Gegenbauer polynomials $C_n^{(\alpha)}$ of order $\alpha = \frac{1}{2}$ so that the properties of P_n follow from corresponding properties of the Gegenbauer polynomials (see e.g. [21, pp. 80–84]). For each $n \in \mathbb{N}_0$, the Legendre polynomials P_n can be recursively defined by

$$P_{n+2}(x) := \frac{2n+3}{n+2}\, x\, P_{n+1}(x) - \frac{n+1}{n+2}\, P_n(x) \quad (x \in \mathbb{R})$$

with $P_0(x) := 1$ and $P_1(x) := x$ (see [21, p. 81]). The Legendre polynomial P_n can be represented in the explicit form

$$P_n(x) = \frac{1}{2^n} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j\,(2n-2j)!}{j!\,(n-j)!\,(n-2j)!}\, x^{n-2j}$$

(see [21, p. 84]). Hence it follows that for $m \in \mathbb{N}_0$

$$P_{2m}(0) = \frac{(-1)^m\,(2m)!}{2^{2m}\,(m!)^2}, \quad P_{2m+1}(0) = 0\,, \tag{2.1}$$

$$P'_{2m+1}(0) = \frac{(-1)^m\,(2m+1)!}{2^{2m}\,(m!)^2}, \quad P'_{2m}(0) = 0. \tag{2.2}$$

Further, Legendre polynomials of even degree are even and Legendre polynomials of odd degree are odd, i.e.

$$P_n(-x) = (-1)^n\, P_n(x). \tag{2.3}$$

Moreover, the Legendre polynomial P_n satisfies the following homogeneous linear differential equation of second order

$$(1-x^2)\, P''_n(x) - 2x\, P'_n(x) + n\,(n+1)\, P_n(x) = 0 \tag{2.4}$$

(see [21, p. 80]). In the Hilbert space $L^2_{1/2}([-1,\ 1])$ with the constant weight $\frac{1}{2}$, the *normalized Legendre polynomials*

$$L_n(x) := \sqrt{2n+1}\, P_n(x) \quad (n \in \mathbb{N}_0) \tag{2.5}$$

form an orthonormal basis, since

$$\frac{1}{2} \int_{-1}^{1} L_n(x)\, L_m(x)\, \mathrm{d}x = \delta_{n-m} \quad (m,\ n \in \mathbb{N}_0),$$

where δ_n denotes the Kronecker symbol (see [21, p. 81]). Note that the uniform norm

$$\max_{x \in [-1,1]} |L_n(x)| = |L_n(-1)| = |L_n(1)| = (2n+1)^{1/2}$$

is increasing with respect to n (see [21, p. 164]).

Let M be a positive integer. A polynomial

$$H(x) := \sum_{k=0}^{d} b_k\, L_k(x) \tag{2.6}$$

of degree d with $d \gg M$ is called *M-sparse in the Legendre basis* or simply a *sparse Legendre expansion*, if M coefficients b_k are nonzero and if the other $d - M + 1$ coefficients vanish. Then such an M-sparse polynomial H can be represented in the form

$$H(x) = \sum_{j=1}^{M_0} c_{0,j}\, L_{n_{0,j}}(x) + \sum_{k=1}^{M_1} c_{1,k}\, L_{n_{1,k}}(x) \tag{2.7}$$

with $c_{0,j} := b_{n_{0,j}} \neq 0$ for all even $n_{0,j}$ with $0 \le n_{0,1} < n_{0,2} < \cdots < n_{0,M_0}$ and with $c_{1,k} := b_{n_{1,k}} \neq 0$ for all odd $n_{1,k}$ with $1 \le n_{1,1} < n_{1,2} < \cdots < n_{1,M_1}$. The positive integer $M = M_0 + M_1$ is called the *Legendre sparsity* of the polynomial H. The numbers $M_0, M_1 \in \mathbb{N}_0$ are the *even* and *odd Legendre sparsities*, respectively.

Remark 2.1 The sparsity of a polynomial depends essentially on the chosen polynomial basis. If

$$T_n(x) := \cos(n\ \arccos x) \quad (x \in [-1,\ 1])$$

denotes the nth Chebyshev polynomial of first kind, then the nth Legendre polynomial P_n can be represented in the Chebyshev basis by

$$P_n(x) = \frac{1}{2^{2n}} \sum_{j=0}^{\lfloor n/2 \rfloor} (2 - \delta_{n-2j})\, \frac{(2j)!\,(2n-2j)!}{(j!)^2\,((n-j)!)^2}\, T_{n-2j}(x)$$

(see [21, p. 90]). Thus a sparse polynomial in the Legendre basis is in general not a sparse polynomial in the Chebyshev basis. In other words, one has to solve the reconstruction problem of a sparse Legendre expansion without change of the Legendre basis. □

As in [19], we transform the Legendre polynomial system $\{L_n;\ n \in \mathbb{N}_0\}$ into a uniformly bounded orthonormal system. We introduce the functions $Q_n : [-1,\ 1] \to \mathbb{R}$ for each $n \in \mathbb{N}_0$ by

$$Q_n(x) := \sqrt{\frac{\pi}{2}}\, \sqrt[4]{1-x^2}\, L_n(x) = \sqrt{\frac{(2n+1)\pi}{2}}\, \sqrt[4]{1-x^2}\, P_n(x). \tag{2.8}$$

Note that these functions Q_n have the same symmetry properties (2.3) as the Legendre polynomials, namely

$$Q_n(-x) = (-1)^n\, Q_n(x) \quad (x \in [-1,\ 1]). \tag{2.9}$$

Further the functions Q_n are orthonormal in the Hilbert space $L^2_w([-1,\ 1])$ with the Chebyshev weight $w(x) := \frac{1}{\pi}\,(1-x^2)^{-1/2}$, since for all $m,\ n \in \mathbb{N}_0$

$$\int_{-1}^{1} Q_n(x)\, Q_m(x)\, w(x)\, \mathrm{d}x = \frac{1}{2} \int_{-1}^{1} L_n(x)\, L_m(x)\, \mathrm{d}x = \delta_{n-m}.$$

Note that $\int_{-1}^{1} w(x)\, \mathrm{d}x = 1$. In the following, we use the standard substitution $x = \cos\theta$ $(\theta \in [0,\ \pi])$ and obtain

$$Q_n(\cos\theta) = \sqrt{\frac{\pi}{2}}\, \sqrt{\sin\theta}\, L_n(\cos\theta) = \sqrt{\frac{(2n+1)\pi}{2}}\, \sqrt{\sin\theta}\, P_n(\cos\theta) \quad (\theta \in [0,\ \pi]).$$

Lemma 2.1 *For all $n \in \mathbb{N}_0$, the functions $Q_n(\cos\theta)$ are uniformly bounded on the interval $[0,\ \pi]$, i.e.*

$$|Q_n(\cos\theta)| < 2 \quad (\theta \in [0,\ \pi]). \tag{2.10}$$

Since Lemma 2.1 is a special case of Lemma 4.1 for $\alpha = \frac{1}{2}$, we abstain here from a separate proof.

Now we describe the asymptotic properties of the Legendre polynomials.

Theorem 2.1 *For each $n \in \mathbb{N}_0$, the function $Q_n(\cos\theta)$ can be represented by the asymptotic formula*

$$Q_n(\cos\theta) = \lambda_n\, \cos\left[\left(n + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right] + R_n(\theta) \quad (\theta \in [0,\ \pi]) \tag{2.11}$$

with the scaling factor

$$\lambda_n := \begin{cases} \sqrt{\frac{(4m+1)\pi}{2}}\ \frac{(2m)!}{2^{2m}\,(m!)^2} & n = 2m, \\ \sqrt{\frac{\pi}{4m+3}}\ \frac{(2m+1)!}{2^{2m}\,(m!)^2} & n = 2m+1 \end{cases}$$

and the error term

$$R_n(\theta) := -\frac{1}{4n+2} \int_{\pi/2}^{\theta} \frac{\sin\left[\left(n+\frac{1}{2}\right)(\theta-\tau)\right]}{(\sin\tau)^2}\, Q_n(\cos\tau)\, \mathrm{d}\tau \quad (\theta \in (0,\ \pi)). \tag{2.12}$$

The error term $R_n(\theta)$ fulfills the conditions $R_n(\frac{\pi}{2}) = R_n'(\frac{\pi}{2}) = 0$ and has the symmetry property

$$R_n(\pi - \theta) = (-1)^n\, R_n(\theta). \tag{2.13}$$

Further, the error term can be estimated by

$$|R_n(\theta)| \le \frac{1}{2n+1}\, |\cot\theta|. \tag{2.14}$$

Since Theorem 2.1 is a special case of Theorem 4.1 for $\alpha = \frac{1}{2}$, we leave out a separate proof.

Remark 2.2 From Theorem 2.1 it follows the asymptotic formula of Laplace (see [21, p. 194]) that for $\theta \in (0, \pi)$

$$Q_n(\cos\theta) = \sqrt{2}\,\cos\left[\left(n + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right] + \mathcal{O}(n^{-1}) \quad (n \to \infty). \tag{2.15}$$

The error bound holds uniformly in $[\varepsilon, \pi - \varepsilon]$ with $\varepsilon \in (0, \frac{\pi}{2})$. □

Remark 2.3 For arbitrary $m \in \mathbb{N}_0$, the scaling factors λ_n in (2.11) can be expressed in the following form

$$\lambda_n = \begin{cases} \sqrt{\frac{(4m+1)\pi}{2}}\,\alpha_m & n = 2m, \\ \sqrt{\frac{\pi}{4m+3}}\,(2m+2)\,\alpha_{m+1} & n = 2m+1 \end{cases}$$

with

$$\alpha_0 := 1, \quad \alpha_n := \frac{1 \cdot 3 \cdots (2n+1)}{2 \cdot 4 \cdots (2n)} \quad (n \in \mathbb{N}).$$

□

In Fig. 1, we plot the expression $|\tan(\theta)\,R_n(\theta)|$ for some polynomial degrees n.

We have proved the estimate (2.14) in Theorem 2.1. In the Table 1, one can see that the maximum value of $|\tan(\theta)\,R_n(\theta)|$ on the interval $[0, \pi]$ is much smaller than $\frac{1}{2n+1}$.

We observe that the upper bound (2.14) of $|R_n(\theta)|$ is very accurate in a small neighborhood of $\theta = \frac{\pi}{2}$. Substituting $t = \theta - \frac{\pi}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we obtain by (2.9) and (2.11) that

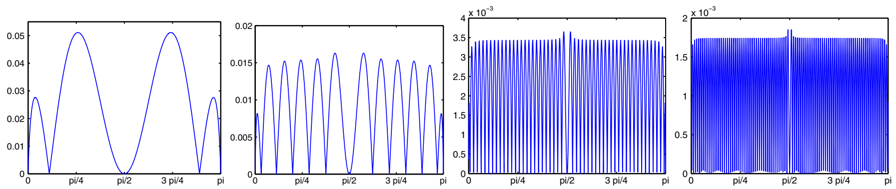

Fig. 1 Expression $|\tan(\theta)\,R_n(\theta)|$ for $n \in \{3, 11, 51, 101\}$

Table 1 Maximum values of $|\tan(\theta)\,R_n(\theta)|$ for some polynomial degrees n

n	3	5	7	9	11	13
$\max_{\theta\in[0,\pi]} \lvert\tan(\theta) R_n(\theta)\rvert$	0.0511	0.0336	0.0249	0.0197	0.0163	0.0139

$$\begin{aligned} Q_n(\sin t) &= (-1)^n\,\lambda_n\,\cos\left[\left(n+\frac{1}{2}\right)t+\frac{n\pi}{2}\right]+(-1)^n\,R_n\left(t+\frac{\pi}{2}\right)\\ &= (-1)^n\,\lambda_n\,\cos\left(\frac{n\pi}{2}\right)\cos\left[\left(n+\frac{1}{2}\right)t\right]-(-1)^n\,\lambda_n\,\sin\left(\frac{n\pi}{2}\right)\\ &\quad\times\sin\left[\left(n+\frac{1}{2}\right)t\right]+\,(-1)^n\,R_n\left(t+\frac{\pi}{2}\right). \end{aligned} \tag{2.16}$$

3 Prony-like method

In a first step we determine the even and odd indexes $n_{0,j}$, $n_{1,k}$ in (2.7), similar to [17]. We use (2.8) and consider the function

$$\sqrt{\frac{\pi}{2}}\,\sqrt[4]{1-x^2}\,H(x)=\sum_{j=1}^{M_0}c_{0,j}\,Q_{n_{0,j}}(x)+\sum_{k=1}^{M_1}c_{1,k}\,Q_{n_{1,k}}(x) \tag{3.1}$$

on $[-1,\ 1]$. In the following we use the asymptotic formulas of $Q_{n_{0,j}}$ and $Q_{n_{1,k}}$ from Theorem 2.1. We substitute $x=\cos(t+\frac{\pi}{2})=-\sin t$ for $t\in[-\frac{\pi}{2},\ \frac{\pi}{2}]$. Now we have to determine indexes $n_{0,j}$ and $n_{1,k}$ from sampling values of the function

$$\sqrt{\frac{\pi}{2}}\,\sqrt{\cos t}\,H(-\sin t)\quad\left(t\in\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]\right). \tag{3.2}$$

We introduce the function

$$F(t):=\sum_{j=1}^{M_0}d_{0,j}\,\cos\left[\left(n_{0,j}+\frac{1}{2}\right)t\right]+\sum_{k=1}^{M_1}d_{1,k}\sin\left[\left(n_{1,k}+\frac{1}{2}\right)t\right]\quad\left(t\in\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]\right) \tag{3.3}$$

with the coefficients

$$d_{0,j}:=(-1)^{n_{0,j}/2}\,c_{0,j}\,\lambda_{n_{0,j}}\,,\quad d_{1,k}:=(-1)^{(n_{1,k}+1)/2}\,c_{1,k}\,\lambda_{n_{1,k}}\,.$$

By (2.9) and (2.16), the new function (3.3) approximates the sampled function (3.2) in a small neighborhood of $t=0$. Then we form

$$\frac{F(t)+F(-t)}{2}=\sum_{j=1}^{M_0}d_{0,j}\,\cos\left[\left(n_{0,j}+\frac{1}{2}\right)t\right] \tag{3.4}$$

and

$$\frac{F(t)-F(-t)}{2}=\sum_{k=1}^{M_1}d_{1,k}\,\sin\left[\left(n_{1,k}+\frac{1}{2}\right)t\right]. \tag{3.5}$$

Now we proceed similar to [17], but we use only sampling points near 0, due to the small values of the error term $R_n(t+\frac{\pi}{2})$ in a small neighborhood of $t=0$ [see (2.14)].

Let $N \in \mathbb{N}$ be sufficiently large such that $N > M$ and $2N-1$ is an upper bound of the degree of the polynomial (2.6). For $u_N := \sin \frac{\pi}{2N-1}$ we form the nonequidistant sine–grid $\{u_{N,k} := \sin \frac{k\pi}{2N-1};\ k = 1-2M, \ldots, 2M-1\}$ in the interval $[-1,\ 1]$.

We consider the following problem of *sparse Legendre interpolation*: For given sampled data

$$h_k := \sqrt{\frac{\pi}{2}} \sqrt{\cos \frac{k\pi}{2N-1}}\, H\left(-\sin \frac{k\pi}{2N-1}\right) \quad (k = 1-2M, \ldots, 2M-1)$$

determine all parameters $n_{0,j}$ $(j = 1, \ldots, M_0)$ of the sparse cosine sum (3.4), determine all parameters $n_{1,k}$ $(k = 1, \ldots, M_1)$ of the sparse sine sum (3.5) and finally determine all coefficients $c_{0,j}$ $(j = 1, \ldots, M_0)$ and $c_{1,k}$ $(k = 1, \ldots, M_1)$ of the sparse Legendre expansion (2.7).

3.1 Sparse even Legendre interpolation

For a moment, we assume that the even Legendre sparsity M_0 of the polynomial (2.7) is known. Then we see that the above interpolation problem is closely related to the *interpolation problem of the sparse, even trigonometric polynomial*

$$\frac{h_k + h_{-k}}{2} \approx f_k := \sum_{j=1}^{M_0} d_{0,j} \cos \frac{(n_{0,j} + 1/2)k\pi}{2N-1} \quad (k = 1-2M_0, \ldots, 2M_0-1), \tag{3.6}$$

where the sampled values f_k $(k = 1-2M_0, \ldots, 2M_0-1)$ are approximately given by $\frac{h_k + h_{-k}}{2}$. Note that $f_{-k} = f_k$ $(k = 0, \ldots, 2M_0-1)$.

We introduce the *Prony polynomial* Π_0 of degree M_0 with the leading coefficient 2^{M_0-1}, whose roots are $\cos \frac{(n_{0,j}+1/2)\pi}{2N-1}$ $(j = 1, \ldots, M_0)$, i.e.

$$\Pi_0(x) = 2^{M_0-1} \prod_{j=1}^{M_0} \left(x - \cos \frac{(n_{0,j}+1/2)\pi}{2N-1}\right).$$

Then the Prony polynomial Π_0 can be represented in the Chebyshev basis by

$$\Pi_0(x) = \sum_{\ell=0}^{M_0} p_{0,\ell}\, T_\ell(x) \quad (p_{0,M_0} := 1). \tag{3.7}$$

The coefficients $p_{0,\ell}$ of the Prony polynomial (3.7) can be characterized as follows:

Lemma 3.1 (See [17, Lemma 3.2]) *For all $k = 0,\ \ldots,\ M_0 - 1$, the sampled data f_k and the coefficients $p_{0,\ell}$ of the Prony polynomial* (3.7) *satisfy the equations*

$$\sum_{\ell=0}^{M_0-1} (f_{k+\ell} + f_{\ell-k})\, p_{0,\ell} = -(f_{k+M_0} + f_{M_0-k}).$$

Using Lemma 3.1, one obtains immediately a Prony method for sparse even Legendre interpolation in the case of known even Legendre sparsity. This algorithm is similar to [17, Algorithm 2.7] and omitted here.

In practice, the even/odd Legendre sparsities M_0, M_1 of the polynomial (2.7) of degree at most $2N-1$ are unknown. Then we can apply the same technique as in [17, Section 3]. We assume that an upper bound $L \in \mathbb{N}$ of $\max\{M_0, M_1\}$ is known, where $N \in \mathbb{N}$ is sufficiently large with $\max\{M_0, M_1\} \le L \le N$. In order to improve the numerical stability, we allow to choose more sampling points. Therefore we introduce an additional parameter K with $L \le K \le N$ such that we use $K+L$ sampling points of (2.7), more precisely we assume that sampled data f_k $(k = 0, \ldots, L+K-1)$ from (3.6) are given. With the $L+K$ sampled data $f_k \in \mathbb{R}$ $(k = 0, \ldots, L+K-1)$, we form the *rectangular Toeplitz-plus-Hankel matrix*

$$H^{(0)}_{K,L+1} := (f_{k+\ell} + f_{\ell-k})_{k,\ell=0}^{K-1,L}\,. \tag{3.8}$$

Note that $H^{(0)}_{K,L+1}$ is rank deficient with rank M_0 (see [17, Lemma 3.1]).

3.2 Sparse odd Legendre interpolation

First we assume that the odd Legendre sparsity M_1 of the polynomial (2.7) is known. Then we see that the above interpolation problem is closely related to the *interpolation problem of the sparse, odd trigonometric polynomial*

$$\frac{h_k - h_{-k}}{2} \approx g_k := \sum_{j=1}^{M_1} d_{1,j}\, \sin \frac{(n_{1,j}+1/2)k\pi}{2N-1} \quad (k = 1-2M_1, \ldots, 2M_1-1), \tag{3.9}$$

where the sampled values g_k $(k = 0, \ldots, 2M_1-1)$ are approximately given by $\frac{h_k - h_{-k}}{2}$. Note that $g_{-k} = -g_k$ $(k = 0, \ldots, 2M_1-1)$.

We introduce the *Prony polynomial* Π_1 of degree M_1 with the leading coefficient 2^{M_1-1}, whose roots are $\cos \frac{(n_{1,j}+1/2)\pi}{2N-1}$ $(j = 1, \ldots, M_1)$, i.e.

$$\Pi_1(x) = 2^{M_1-1} \prod_{j=1}^{M_1} \left(x - \cos \frac{(n_{1,j}+1/2)\pi}{2N-1} \right).$$

Then the Prony polynomial Π_1 can be represented in the Chebyshev basis by

$$\Pi_1(x) = \sum_{\ell=0}^{M_1} p_{1,\ell}\, T_\ell(x) \quad (p_{1,M_1} := 1). \tag{3.10}$$

The coefficients $p_{1,\ell}$ of the Prony polynomial (3.10) can be characterized as follows:

Lemma 3.2 *For all $k = 0, \ldots, M_1 - 1$, the sampled data g_k and the coefficients $p_{1,\ell}$ of the Prony polynomial (3.10) satisfy the equations*

$$\sum_{\ell=0}^{M_1-1} (g_{k+\ell} + g_{k-\ell})\, p_{1,\ell} = -(g_{k+M_1} + g_{k-M_1}). \tag{3.11}$$

Proof Using $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\,\sin\alpha\,\cos\beta$, we obtain by (3.9) that

$$\begin{aligned} g_{k+\ell} + g_{k-\ell} &= \sum_{j=1}^{M_1} d_{1,j} \left(\sin \frac{(n_{1,j}+1/2)(k+\ell)\pi}{2N-1} + \sin \frac{(n_{1,j}+1/2)(k-\ell)\pi}{2N-1} \right) \\ &= 2 \sum_{j=1}^{M_1} d_{1,j}\, \sin \frac{(n_{1,j}+1/2)k\pi}{2N-1}\, \cos \frac{(n_{1,j}+1/2)\ell\pi}{2N-1}. \end{aligned}$$

Thus we conclude that

$$\begin{aligned} \sum_{\ell=0}^{M_1} (g_{k+\ell}+g_{k-\ell})\, p_{1,\ell} &= 2 \sum_{j=1}^{M_1} d_{1,j}\, \sin \frac{(n_{1,j}+1/2)k\pi}{2N-1} \sum_{\ell=0}^{M_1} p_{1,\ell}\, \cos \frac{(n_{1,j}+1/2)\ell\pi}{2N-1} \\ &= 2 \sum_{j=1}^{M_1} d_{1,j}\, \sin \frac{(n_{1,j}+1/2)k\pi}{2N-1}\, \Pi_1 \left(\cos \frac{(n_{1,j}+1/2)\pi}{2N-1} \right) = 0. \end{aligned}$$

By $p_{1,M_1} = 1$, this implies the assertion (3.11). □

Using Lemma 3.2, one can formulate a Prony method for sparse odd Legendre interpolation in the case of known odd Legendre sparsity. This algorithm is similar to [17, Algorithm 2.7] and omitted here.

In general, the even/odd Legendre sparsities M_0 and M_1 of the polynomial (2.7) of degree at most $2N-1$ are unknown. Similarly to Sect. 3.1, let $L \in \mathbb{N}$ be a convenient upper bound of $\max\{M_0, M_1\}$, where $N \in \mathbb{N}$ is sufficiently large with $\max\{M_0, M_1\} \le L \le N$. In order to improve the numerical stability, we allow to choose more sampling points. Therefore we introduce an additional parameter K with $L \le K \le N$ such that we use $K + L$ sampling points of (2.7), more precisely we assume that sampled data g_k $(k = 0, \ldots, L+K-1)$ from (3.9) are given. With the $L+K$ sampled data $g_k \in \mathbb{R}$ $(k = 0, \ldots, L+K-1)$ we form the *rectangular Toeplitz-plus-Hankel matrix*

$$H^{(1)}_{K,L+1} := (g_{k+\ell} + g_{k-\ell})_{k,\ell=0}^{K-1,L}\,. \tag{3.12}$$

Note that $H^{(1)}_{K,L+1}$ is rank deficient with rank M_1. This is an analogous result to [17, Lemma 3.1].

3.3 Sparse Legendre interpolation

In this subsection, we sketch a Prony-like method for the computation of the polynomial degrees $n_{0,j}$ and $n_{1,k}$ of the sparse Legendre expansion (2.7). Mainly we use singular value decompositions (SVD) of the Toeplitz-plus-Hankel matrices (3.8) and (3.12). For details of this method see [17, Section 3]. We start with the singular value factorizations

$$H^{(0)}_{K,L+1} = U^{(0)}_K\, D^{(0)}_{K,L+1}\, W^{(0)}_{L+1},$$
$$H^{(1)}_{K,L+1} = U^{(1)}_K\, D^{(1)}_{K,L+1}\, W^{(1)}_{L+1},$$

where $U^{(0)}_K$, $U^{(1)}_K$, $W^{(0)}_{L+1}$ and $W^{(1)}_{L+1}$ are orthogonal matrices and where $D^{(0)}_{K,L+1}$ and $D^{(1)}_{K,L+1}$ are rectangular diagonal matrices. The diagonal entries of $D^{(0)}_{K,L+1}$ are the singular values of (3.8) arranged in nonincreasing order

$$\sigma^{(0)}_1 \ge \sigma^{(0)}_2 \ge \dots \ge \sigma^{(0)}_{M_0} \ge \sigma^{(0)}_{M_0+1} \ge \dots \ge \sigma^{(0)}_{L+1} \ge 0.$$

We determine the largest M_0 such that $\sigma^{(0)}_{M_0}/\sigma^{(0)}_1 > \varepsilon$, which is approximately the rank of the matrix (3.8) and which coincides with the even Legendre sparsity M_0 of the polynomial (2.7).

Similarly, the diagonal entries of $D^{(1)}_{K,L+1}$ are the singular values of (3.12) arranged in nonincreasing order

$$\sigma^{(1)}_1 \ge \sigma^{(1)}_2 \ge \dots \ge \sigma^{(1)}_{M_1} \ge \sigma^{(1)}_{M_1+1} \ge \dots \ge \sigma^{(1)}_{L+1} \ge 0.$$

We determine the largest M_1 such that $\sigma^{(1)}_{M_1}/\sigma^{(1)}_1 > \varepsilon$, which is approximately the rank of the matrix (3.12) and which coincides with the odd Legendre sparsity M_1 of the polynomial (2.7). Note that there is often a gap in the singular values, such that we can choose $\varepsilon = 10^{-8}$ in general.

Introducing the matrices

$$D^{(0)}_{K,M_0} := D^{(0)}_{K,L+1}(1:K,\ 1:M_0) = \begin{pmatrix} \operatorname{diag}\,(\sigma^{(0)}_j)^{M_0}_{j=1} \\ O_{K-M_0,M_0} \end{pmatrix},$$
$$W^{(0)}_{M_0,L+1} := W^{(0)}_{L+1}(1:M_0,\ 1:L+1),$$
$$D^{(1)}_{K,M_1} := D^{(1)}_{K,L+1}(1:K,\ 1:M_1) = \begin{pmatrix} \operatorname{diag}\,(\sigma^{(1)}_j)^{M_1}_{j=1} \\ O_{K-M_1,M_1} \end{pmatrix},$$
$$W^{(1)}_{M_1,L+1} := W^{(1)}_{L+1}(1:M_1,\ 1:L+1),$$

we can simplify the SVD of the Toeplitz-plus-Hankel matrices (3.8) and (3.12) as follows

$$H^{(0)}_{K,L+1} = U^{(0)}_K\, D^{(0)}_{K,M_0}\, W^{(0)}_{M_0,L+1}, \qquad H^{(1)}_{K,L+1} = U^{(1)}_K\, D^{(1)}_{K,M_1}\, W^{(1)}_{M_1,L+1}.$$

Using the known submatrix notation and setting

$$W^{(0)}_{M_0,L}(s) := W^{(0)}_{M_0,L+1}(1 : M_0,\ 1+s : L+s) \quad (s = 0,\ 1), \tag{3.13}$$

$$W^{(1)}_{M_1,L}(s) := W^{(1)}_{M_1,L+1}(1 : M_1,\ 1+s : L+s) \quad (s = 0,\ 1), \tag{3.14}$$

we form the matrices

$$F^{(0)}_{M_0} := \left(W^{(0)}_{M_0,L}(0)\right)^{\dagger}\ W^{(0)}_{M_0,L}(1), \tag{3.15}$$

$$F^{(1)}_{M_1} := \left(W^{(1)}_{M_1,L}(0)\right)^{\dagger}\ W^{(1)}_{M_1,L}(1), \tag{3.16}$$

where $(W^{(0)}_{M_0,L}(0))^{\dagger}$ denotes the Moore–Penrose pseudoinverse of $W^{(0)}_{M_0,L}(0)$.

Finally we determine the nodes $x_{0,j} \in [-1,\ 1]$ ($j = 1, \ldots, M_0$) and $x_{1,j} \in [-1,\ 1]$ ($j = 1, \ldots, M_1$) as eigenvalues of the matrix $F^{(0)}_{M_0}$ and $F^{(1)}_{M_1}$, respectively. Thus the algorithm reads as follows:

Algorithm 3.1 (Sparse Legendre interpolation based on SVD)

Input: $L, K, N \in \mathbb{N}$ ($N \gg 1,\ 3 \le L \le K \le N$), L is upper bound of $\max\{M_0, M_1\}$, sampled values $H(-\sin \frac{k\pi}{2N-1})$ ($k = 1-L-K, \ldots, L+K-1$) of the polynomial (2.7) of degree at most $2N-1$.

1. Compute

$$h_k := \sqrt{\frac{\pi}{2}}\sqrt{\cos\frac{k\pi}{2N-1}}\,H\Big(-\sin\frac{k\pi}{2N-1}\Big) \quad (k = 1-L-K, \ldots, L+K-1)$$

and form

$$f_k := \frac{h_k + h_{-k}}{2}, \quad g_k := \frac{h_k - h_{-k}}{2} \quad (k = 1-L-K, \ldots, L+K-1).$$

2. Compute the SVD of the rectangular Toeplitz–plus–Hankel matrices (3.8) and (3.12). Determine the approximate rank M_0 of (3.8) such that $\sigma^{(0)}_{M_0}/\sigma^{(0)}_1 > 10^{-8}$ and form the matrix (3.13). Determine the approximate rank M_1 of (3.12) such that $\sigma^{(1)}_{M_1}/\sigma^{(1)}_1 > 10^{-8}$ and form the matrix (3.14).

3. Compute all eigenvalues $x_{0,j} \in [-1,\ 1]$ ($j = 1, \ldots, M_0$) of the square matrix (3.15). Assume that the eigenvalues are ordered in the following form $1 \ge x_{0,1} > x_{0,2} > \cdots > x_{0,M_0} \ge -1$. Calculate $n_{0,j} := [\frac{2N-1}{\pi}\arccos x_{0,j} - \frac{1}{2}]$ ($j = 1, \ldots, M_0$), where $[x] := \lfloor x + 0.5 \rfloor$ means rounding of $x \in \mathbb{R}$ to the nearest integer.

4. Compute all eigenvalues $x_{1,j} \in [-1,\ 1]$ ($j = 1, \ldots, M_1$) of the square matrix (3.16). Assume that the eigenvalues are ordered in the following form $1 \ge x_{1,1} > x_{1,2} > \cdots > x_{1,M_1} \ge -1$. Calculate $n_{1,j} := [\frac{2N-1}{\pi}\arccos x_{1,j} - \frac{1}{2}]$ ($j = 1, \ldots, M_1$).

5. Compute the coefficients $c_{0,j} \in \mathbb{R}$ ($j = 1, \ldots, M_0$) and $c_{1,j} \in \mathbb{R}$ ($j = 1, \ldots, M_1$) as least squares solutions of the overdetermined linear Vandermonde–like systems

$$\sum_{j=1}^{M_0} c_{0,j}\, Q_{n_{0,j}}\left(\sin \frac{k\pi}{2N-1}\right) = f_k \quad (k = 0, \ldots, L+K-1)\,,$$
$$\sum_{j=1}^{M_1} c_{1,j}\, Q_{n_{1,j}}\left(\sin \frac{k\pi}{2N-1}\right) = g_{-k} \quad (k = 0, \ldots, L+K-1).$$

Output: $M_0 \in \mathbb{N}_0$, $n_{0,j} \in \mathbb{N}_0$ $(0 \le n_{0,1} < n_{0,2} < \cdots < n_{0,M_0} < 2N)$, $c_{0,j} \in \mathbb{R}$ $(j = 1, \ldots, M_0)$. $M_1 \in \mathbb{N}_0$, $n_{1,j} \in \mathbb{N}$ $(1 \le n_{1,1} < n_{1,2} < \cdots < n_{1,M_1} < 2N)$, $c_{1,j} \in \mathbb{R}$ $(j = 1, \ldots, M_1)$.

Remark 3.1 The Algorithm 3.1 is very similar to [17, Algorithm 3.5]. Note that one can also use the QR decomposition of the rectangular Toeplitz-plus-Hankel matrices (3.8) and (3.12) instead of the SVD. In that case one obtains an algorithm similar to [17, Algorithm 3.4]. □

4 Extension to Gegenbauer polynomials

In this section we show that our reconstruction method can be generalized to sparse Gegenbauer expansions of low positive order. The *Gegenbauer polynomials* $C_n^{(\alpha)}$ of degree $n \in \mathbb{N}_0$ and fixed order $\alpha > 0$ can be defined by the recursion relation (see [21, p. 81]):

$$C_{n+2}^{(\alpha)}(x) := \frac{2\alpha + 2n + 2}{n+2}\, x\, C_{n+1}^{(\alpha)}(x) - \frac{2\alpha + n}{n+2}\, C_n^{(\alpha)}(x) \quad (n \in \mathbb{N}_0)$$

with $C_0^{(\alpha)}(x) := 1$ and $C_1^{(\alpha)}(x) := 2\alpha\, x$. Sometimes, $C_n^{(\alpha)}$ are called *ultraspherical polynomials* too. In the case $\alpha = \frac{1}{2}$, one obtains again the Legendre polynomials $P_n = C_n^{(1/2)}$. By [21, p. 84], an explicit representation of the Gegenbauer polynomial $C_n^{(\alpha)}$ reads as follows

$$C_n^{(\alpha)}(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j\, \Gamma(n-j+\alpha)}{\Gamma(\alpha)\, \Gamma(j+1)\, \Gamma(n-2j+1)}\, (2x)^{n-2j}.$$

Thus the Gegenbauer polynomials satisfy the symmetry relations

$$C_n^{(\alpha)}(-x) = (-1)^n\, C_n^{(\alpha)}(x). \tag{4.1}$$

Further one obtains that for $m \in \mathbb{N}_0$

$$C_{2m}^{(\alpha)}(0) = \frac{(-1)^m\, \Gamma\left(m + \frac{1}{2}\right)}{\Gamma(\alpha)\, \Gamma(m+1)}, \quad C_{2m+1}^{(\alpha)}(0) = 0, \tag{4.2}$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\, C_{2m+1}^{(\alpha)}\right)(0) = \frac{2\,(-1)^m\, \Gamma(\alpha + m + 1)}{\Gamma(\alpha)\, \Gamma(m+1)}, \quad \left(\frac{\mathrm{d}}{\mathrm{d}x}\, C_{2m}^{(\alpha)}\right)(0) = 0. \tag{4.3}$$

Moreover, the Gegenbauer polynomial $C_n^{(\alpha)}$ satisfies the following homogeneous linear differential equation of second order (see [21, p. 80])

$$(1-x^2)\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,C_n^{(\alpha)}(x) - (2\alpha+1)\,x\,\frac{\mathrm{d}}{\mathrm{d}x}\,C_n^{(\alpha)}(x) + n\,(n+2\alpha)\,C_n^{(\alpha)}(x) = 0. \tag{4.4}$$

Further, the Gegenbauer polynomials are orthogonal over the interval $[-1,\ 1]$ with respect to the weight function

$$w^{(\alpha)}(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\,(1-x^2)^{\alpha-1/2} \quad (x \in (-1,\ 1)),$$

i.e. more precisely by [21, p. 81]

$$\int_{-1}^{1} C_m^{(\alpha)}(x)\,C_n^{(\alpha)}(x)\,w^{(\alpha)}(x)\,\mathrm{d}x = \frac{\alpha\,\Gamma(2\alpha+n)}{(n+\alpha)\,\Gamma(n+1)\,\Gamma(2\alpha)}\,\delta_{m-n} \quad (m,\ n \in \mathbb{N}_0).$$

Note that the weight function $w^{(\alpha)}$ is normalized by

$$\int_{-1}^{1} w^{(\alpha)}(x)\,\mathrm{d}x = 1.$$

Then the *normalized Gegenbauer polynomials*

$$L_n^{(\alpha)}(x) := \sqrt{\frac{(n+\alpha)\,\Gamma(n+1)\,\Gamma(2\alpha)}{\alpha\,\Gamma(2\alpha+n)}}\,C_n^{(\alpha)}(x) \quad (n \in \mathbb{N}_0) \tag{4.5}$$

form an orthonormal basis in the weighted Hilbert space $L_{w^{(\alpha)}}([-1,\ 1])$.

Let M be a positive integer. A polynomial

$$H(x) := \sum_{k=0}^{d} b_k\,L_k^{(\alpha)}(x)$$

of degree d with $d \gg M$ is called *M-sparse in the Gegenbauer basis* or simply a *sparse Gegenbauer expansion*, if M coefficients b_k are nonzero and if the other $d-M+1$ coefficients vanish. Then such an M-sparse polynomial H can be represented in the form

$$H(x) = \sum_{j=1}^{M_0} c_{0,j}\,L_{n_{0,j}}^{(\alpha)}(x) + \sum_{k=1}^{M_1} c_{1,k}\,L_{n_{1,k}}^{(\alpha)}(x) \tag{4.6}$$

with $c_{0,j} := b_{n_{0,j}} \neq 0$ for all even $n_{0,j}$ with $0 \le n_{0,1} < n_{0,2} < \cdots < n_{0,M_0}$ and with $c_{1,k} := b_{n_{1,k}} \neq 0$ for all odd $n_{1,k}$ with $1 \le n_{1,1} < n_{1,2} < \cdots < n_{1,M_1}$. The positive integer $M = M_0 + M_1$ is called the *Gegenbauer sparsity* of the polynomial H. The integers M_0, M_1 are the *even* and *odd Gegenbauer sparsities*, respectively.

Now for each $n \in \mathbb{N}_0$, we introduce the functions $Q_n^{(\alpha)}$ by

$$Q_n^{(\alpha)}(x) := \sqrt{\frac{\Gamma(\alpha+1)\sqrt{\pi}}{\Gamma\left(\alpha+\frac{1}{2}\right)}}\,(1-x^2)^{\alpha/2}\,L_n^{(\alpha)}(x) \quad (x \in [-1,\,1]) \tag{4.7}$$

These functions $Q_n^{(\alpha)}$ possess the same symmetry properties (4.1) as the Gegenbauer polynomials, namely

$$Q_n^{(\alpha)}(-x) = (-1)^n\, Q_n^{(\alpha)}(x) \quad (x \in [-1,\,1]). \tag{4.8}$$

Further the functions $Q_n^{(\alpha)}$ are orthonormal in the weighted Hilbert space $L_w^2([-1,\,1])$ with the Chebyshev weight $w(x) = \frac{1}{\pi}\,(1-x^2)^{-1/2}$, since for all $m,\ n \in \mathbb{N}_0$

$$\int_{-1}^{1} Q_m^{(\alpha)}(x)\, Q_n^{(\alpha)}(x)\, w(x)\,\mathrm{d}x = \int_{-1}^{1} L_m^{(\alpha)}(x)\, L_n^{(\alpha)}(x)\, w^{(\alpha)}(x)\,\mathrm{d}x = \delta_{m-n}.$$

In the following, we use the standard substitution $x = \cos\theta$ ($\theta \in [0,\,\pi]$) and obtain

$$Q_n^{(\alpha)}(\cos\theta) := \sqrt{\frac{\Gamma(\alpha+1)\sqrt{\pi}}{\Gamma(\alpha+\frac{1}{2})}}\,(\sin\theta)^{\alpha}\,L_n^{(\alpha)}(\cos\theta) \quad (\theta \in [0,\,\pi]).$$

Lemma 4.1 *For all $n \in \mathbb{N}_0$ and $\alpha \in (0,\,1)$, the functions $Q_n^{(\alpha)}(\cos\theta)$ are uniformly bounded on the interval $[0,\,\pi]$, i.e.*

$$|Q_n^{(\alpha)}(\cos\theta)| < 2 \quad (\theta \in [0,\,\pi]). \tag{4.9}$$

Proof For $n \in \mathbb{N}_0$ and $\alpha \in (0,\,1)$, we know by [13] that for all $\theta \in [0,\,\pi]$

$$(\sin\theta)^{\alpha}\,|C_n^{(\alpha)}(\cos\theta)| < \frac{2^{1-\alpha}}{\Gamma(\alpha)}\,(n+\alpha)^{\alpha-1}.$$

Then for the normalized Gegenbauer polynomials $L_n^{(\alpha)}$, we obtain the estimate

$$(\sin\theta)^{\alpha}\,|L_n^{(\alpha)}(\cos\theta)| < \frac{2^{1-\alpha}}{\Gamma(\alpha)}\sqrt{\frac{(n+\alpha)\ \Gamma(n+1)\,\Gamma(2\alpha)}{\alpha\,\Gamma(2\alpha+n)}}\,(n+\alpha)^{\alpha-1}.$$

Using the duplication formula of the Gamma function

$$\Gamma(\alpha)\,\Gamma\left(\alpha+\frac{1}{2}\right) = 2^{1-2\alpha}\,\sqrt{\pi}\,\Gamma(2\alpha), \tag{4.10}$$

we can estimate

$$|Q_n^{(\alpha)}(\cos\theta)| < \sqrt{2}\sqrt{\frac{\Gamma(n+1)}{\Gamma(2\alpha+n)}}\,(n+\alpha)^{\alpha-1/2}.$$

For $\alpha = \frac{1}{2}$, we obtain $|Q_n^{(1/2)}(\cos\theta)| < \sqrt{2}$. In the following, we use the inequalities (see [9])

$$\left(n+\frac{\sigma}{2}\right)^{1-\sigma} < \frac{\Gamma(n+1)}{\Gamma(n+\sigma)} < \left(n-\frac{1}{2}+\sqrt{\sigma+\frac{1}{4}}\right)^{1-\sigma} \tag{4.11}$$

for all $n \in \mathbb{N}$ and $\sigma \in (0,\ 1)$.

In the case $0 < \alpha < \frac{1}{2}$, the estimate (4.11) with $\sigma = 2\alpha$ implies that

$$|Q_n^{(\alpha)}(\cos\theta)| < \sqrt{2}\left(\frac{n-\frac{1}{2}+\sqrt{2\alpha+\frac{1}{4}}}{n+\alpha}\right)^{-\alpha+1/2}.$$

Since $n - \frac{1}{2} + \sqrt{2\alpha+\frac{1}{4}} < 2\,(n+\alpha)$ for all $n \in \mathbb{N}$, we conclude that

$$|Q_n^{(\alpha)}(\cos\theta)| < 2^{1-\alpha}.$$

In the case $\frac{1}{2} < \alpha < 1$, we set $\beta := 1 - 2\,\alpha \in (0,\ 1)$. By (4.11) with $\sigma = \beta$, we can estimate

$$\frac{\Gamma(n+1)}{\Gamma(n+2\alpha)} = \frac{\Gamma(n+1)}{(n+\beta)\,\Gamma(n+\beta)} < \frac{1}{n+\beta}\left(n-\frac{1}{2}+\sqrt{\beta+\frac{1}{4}}\right)^{1-\beta}.$$

Hence we obtain by $n - \frac{1}{2} + \sqrt{\beta+\frac{1}{4}} < n+\beta$ that

$$\begin{aligned} |Q_n^{(\alpha)}(\cos\theta)| &< \frac{\sqrt{2}}{\sqrt{n+\beta}}\left(n-\frac{1}{2}+\sqrt{\beta+\frac{1}{4}}\right)^{(1-\beta)/2}(n+\alpha)^{\alpha-1/2} \\ &< \sqrt{2}\left(n-\frac{1}{2}+\sqrt{\beta+\frac{1}{4}}\right)^{-\beta/2}\left(n+\frac{1-\beta}{2}\right)^{\beta/2} < \sqrt{2}. \end{aligned}$$

Finally, by

$$Q_0^{(\alpha)}(\cos\theta) = \sqrt{\frac{\alpha\,\Gamma(\alpha)\,\sqrt{\pi}}{\Gamma\left(\alpha+\frac{1}{2}\right)}}\,(\sin\theta)^{\alpha}$$

and

$$|Q_0^{(\alpha)}(\cos\theta)| \le \sqrt[4]{\pi},$$

we see that the estimate (4.9) is also true for $n = 0$. □

By (4.4), the function $Q_n^{(\alpha)}(\cos\theta)$ satisfies the following linear differential equation of second order (see [21, p. 81])

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\, Q_n^{(\alpha)}(\cos\theta) + \left((n+\alpha)^2 + \frac{\alpha(1-\alpha)}{(\sin\theta)^2}\right) Q_n^{(\alpha)}(\cos\theta) = 0 \quad (\theta \in (0,\,\pi)). \tag{4.12}$$

By the method of Liouville–Stekloff, see [21, pp. 210–212], we show that for *arbitrary* $n \in \mathbb{N}_0$, the function $Q_n^{(\alpha)}(\cos\theta)$ is approximately equal to some multiple of $\cos[(n+\alpha)\,\theta - \frac{\alpha\pi}{2}]$ in a small neighborhood of $\theta = \frac{\pi}{2}$.

Theorem 4.1 *For each $n \in \mathbb{N}_0$ and $\alpha \in (0,\,1)$, the function $Q_n^{(\alpha)}(\cos\theta)$ can be represented by the asymptotic formula*

$$Q_n^{(\alpha)}(\cos\theta) = \lambda_n\, \cos\left[(n+\alpha)\,\theta - \frac{\alpha\pi}{2}\right] + R_n^{(\alpha)}(\theta) \quad (\theta \in [0,\,\pi]) \tag{4.13}$$

with the scaling factor

$$\lambda_n := \begin{cases} \sqrt{\frac{(2m+\alpha)\,\Gamma(2m+1)}{\Gamma(2\alpha+2m)}}\, \frac{2^{\alpha-1/2}\,\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)} & n = 2m, \\ \sqrt{\frac{\Gamma(2m+2)}{(2m+1+\alpha)\,\Gamma(2\alpha+2m+1)}}\, \frac{2^{\alpha+1/2}\,\Gamma(\alpha+m+1)}{\Gamma(m+1)} & n = 2m+1 \end{cases}$$

and the error term

$$R_n^{(\alpha)}(\theta) := -\frac{\alpha(1-\alpha)}{n+\alpha} \int_{\pi/2}^{\theta} \frac{\sin[(n+\alpha)(\theta-\tau)]}{(\sin\tau)^2}\, Q_n^{(\alpha)}(\cos\tau)\,\mathrm{d}\tau \quad (\theta \in (0,\,\pi)). \tag{4.14}$$

The error term $R_n^{(\alpha)}(\theta)$ satisfies the conditions

$$R_n^{(\alpha)}\left(\frac{\pi}{2}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\, R_n^{(\alpha)}\right)\left(\frac{\pi}{2}\right) = 0$$

and has the symmetry property

$$R_n^{(\alpha)}(\pi-\theta) = (-1)^n\, R_n^{(\alpha)}(\theta). \tag{4.15}$$

Further, the error term can be estimated by

$$|R_n^{(\alpha)}(\theta)| \le \frac{2\alpha(1-\alpha)}{n+\alpha}\,|\cot\theta|. \tag{4.16}$$

Proof 1. Using the method of Liouville–Stekloff (see [21, pp. 210 – 212]), we derive the asymptotic formula (4.13) from the differential equation (4.12), which can be written in the form

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\, Q_n^{(\alpha)}(\cos\theta) + (n+\alpha)^2\, Q_n^{(\alpha)}(\cos\theta) = -\frac{\alpha(1-\alpha)}{(\sin\theta)^2}\, Q_n^{(\alpha)}(\cos\theta) \quad (\theta \in (0,\,\pi)). \tag{4.17}$$

Since the homogeneous linear differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\, X(\theta) + (n+\alpha)^2\, X(\theta) = 0$$

has the fundamental system

$$\cos\left[(n+\alpha)\,\theta - \frac{\alpha\pi}{2}\right], \quad \sin\left[(n+\alpha)\,\theta - \frac{\alpha\pi}{2}\right],$$

the differential equation (4.17) can be transformed into the Volterra integral equation

$$\begin{aligned} Q_n^{(\alpha)}(\cos\theta) &= \lambda_n \, \cos\left[(n+\alpha)\,\theta - \frac{\alpha\pi}{2}\right] + \mu_n \, \sin\left[(n+\alpha)\,\theta - \frac{\alpha\pi}{2}\right] \\ &\quad - \frac{\alpha(1-\alpha)}{n+\alpha} \int_{\pi/2}^{\theta} \frac{\sin[(n+\alpha)(\theta-\tau)]}{(\sin\tau)^2}\, Q_n^{(\alpha)}(\cos\tau)\,\mathrm{d}\tau \quad (\theta \in (0,\, \pi)) \end{aligned}$$

with certain real constants λ_n and μ_n. Introducing $R_n^{(\alpha)}(\theta)$ by (4.14), we see immediately that $R_n^{(\alpha)}(\frac{\pi}{2}) = 0$. From

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\, R_n^{(\alpha)}(\theta) = -\alpha\,(1-\alpha) \int_{\pi/2}^{\theta} \frac{\cos[(n+\alpha)(\theta-\tau)]}{(\sin\tau)^2}\, Q_n^{(\alpha)}(\cos\tau)\,\mathrm{d}\tau$$

it follows that $(\frac{\mathrm{d}}{\mathrm{d}\theta}\, R_n^{(\alpha)})(\frac{\pi}{2}) = 0$.

2. Now we determine the constants λ_n and μ_n. For arbitrary even $n = 2m$ ($m \in \mathbb{N}_0$), the function $Q_{2m}^{(\alpha)}(\cos\theta)$ can be represented in the form

$$Q_{2m}^{(\alpha)}(\cos\theta) = \lambda_{2m}\, \cos\left[(2m+\alpha)\,\theta - \frac{\alpha\pi}{2}\right] + \mu_{2m}\, \sin\left[(2m+\alpha)\,\theta - \frac{\alpha\pi}{2}\right] + R_{2m}^{(\alpha)}(\theta)$$

Hence the condition $R_{2m}^{(\alpha)}(\frac{\pi}{2}) = 0$ means that $Q_{2m}^{(\alpha)}(0) = (-1)^m \lambda_{2m}$. Using (4.7), (4.5), (4.2), and the duplication formula (4.10), we obtain that

$$\lambda_{2m} = \sqrt{\frac{(2m+\alpha)\,\Gamma(2m+1)}{\Gamma(2\alpha+2m)}}\, \frac{2^{\alpha-1/2}\,\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)}.$$

From $(\frac{\mathrm{d}}{\mathrm{d}x}\, C_{2m}^{(\alpha)})(0) = 0$ by (4.3) it follows that the derivative of $Q_{2m}^{(\alpha)}(\cos\theta)$ vanishes for $\theta = \frac{\pi}{2}$. Thus the second condition $(\frac{\mathrm{d}}{\mathrm{d}\theta}\, R_{2m}^{(\alpha)})(\frac{\pi}{2}) = 0$ implies that

$$0 = \mu_{2m}\,(2m+\alpha)\,(-1)^m,$$

i.e. $\mu_{2m} = 0$.

3. If $n = 2m+1$ ($m \in \mathbb{N}_0$) is odd, then

$$Q_{2m+1}^{(\alpha)}(\cos\theta) = \lambda_{2m+1}\, \cos\left[(2m+1+\alpha)\theta - \frac{\alpha\pi}{2}\right] \\ + \mu_{2m+1}\, \sin\left[(2m+1+\alpha)\,\theta - \frac{\alpha\pi}{2}\right] + R_{2m+1}^{(\alpha)}(\theta).$$

Hence the condition $R_{2m+1}^{(\alpha)}(\frac{\pi}{2}) = 0$ implies by $C_{2m+1}^{(\alpha)}(0) = 0$ [see (4.2)] that

$$0 = \mu_{2m+1}\,(-1)^m,$$

i.e. $\mu_{2m+1} = 0$. The second condition $(\frac{\mathrm{d}}{\mathrm{d}\theta}\, R_{2m+1}^{(\alpha)})(\frac{\pi}{2}) = 0$ reads as follows

$$-\sqrt{\frac{(n+\alpha)\,\Gamma(\alpha)\,\Gamma(2\alpha)\,\Gamma(2m+2)\,\sqrt{\pi}}{\Gamma(\alpha+\frac{1}{2})\,\Gamma(2\alpha+2m+1)\,\Gamma(\alpha)}}\,\left(\frac{\mathrm{d}}{\mathrm{d}x}\,C_{2m+1}^{(\alpha)}\right)(0) \\ = -\lambda_{2m+1}\,(2m+1+\alpha)\,(-1)^m.$$

Thus we obtain by (4.3) and the duplication formula (4.10) that

$$\lambda_{2m+1} = \sqrt{\frac{\Gamma(2m+2)}{(2m+1+\alpha)\,\Gamma(2\alpha+2m+1)}}\,\frac{2^{\alpha+1/2}\,\Gamma(\alpha+m+1)}{\Gamma(m+1)}.$$

4. As shown, the error term $R_n^{(\alpha)}(\theta)$ has the explicit representation (4.14). Using (4.9), we estimate this integral and obtain

$$|R_n^{(\alpha)}(\theta)| \le \frac{2\alpha(1-\alpha)}{n+\alpha}\,\left|\int_{\pi/2}^{\theta} \frac{1}{(\sin\tau)^2}\,\mathrm{d}\tau\right| = \frac{2\alpha(1-\alpha)}{n+\alpha}\,|\cot\theta|.$$

The symmetry property (4.15) of the error term

$$R_n^{(\alpha)}(\theta) = Q_n^{(\alpha)}(\cos\theta) - \lambda_n\, \cos\left[(n+\alpha)\theta - \frac{\alpha\pi}{2}\right]$$

follows from the fact that $Q_n^{(\alpha)}(\cos\theta)$ and $\cos[(n+\alpha)\theta - \frac{\alpha\pi}{2}]$ possess the same symmetry properties as (4.8). This completes the proof. □

Remark 4.1 The following result is stated in [10]: If $\alpha \ge \frac{1}{2}$, then

$$(\sin\theta)^\alpha\,|L_n^{(\alpha)}(\cos\theta)| \le 22\sqrt{\frac{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha+1)}}\,\left(\alpha-\frac{1}{2}\right)^{1/6}\,\left(1+\frac{2\alpha-1}{2n}\right)^{1/12}$$

for all $n \ge 6$ and $\theta \in [0, \pi]$. Using (4.11), we can see that $(\sin\theta)^\alpha\,|L_n^{(\alpha)}(\cos\theta)|$ is uniformly bounded for all $\alpha \ge \frac{1}{2}$, $n \ge 6$ and $\theta \in [0, \pi]$. Using above estimate, one can extend Theorem 4.1 to the case of moderately sized order $\alpha \ge \frac{1}{2}$. □

We observe that the upper bound (4.16) of $|R_n^{(\alpha)}(\theta)|$ is very accurate in a small neighborhood of $\theta = \frac{\pi}{2}$. By the substitution $t = \theta - \frac{\pi}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and (4.8), we obtain

$$\begin{aligned} Q_n^{(\alpha)}(\sin t) &= (-1)^n \lambda_n \cos\left[(n+\alpha)\,t + \frac{n\pi}{2}\right] + (-1)^n R_n\left(t + \frac{\pi}{2}\right) \\ &= (-1)^n \lambda_n \cos\left(\frac{n\pi}{2}\right) \cos[(n+\alpha)\,t] - (-1)^n \lambda_n \sin\left(\frac{n\pi}{2}\right) \sin[(n+\alpha)\,t] \\ &\quad + (-1)^n R_n\left(t + \frac{\pi}{2}\right). \end{aligned}$$

Now the Algorithm 3.1 can be straightforward generalized to the case of a sparse Gegenbauer expansion (4.6):

Algorithm 4.2 (Sparse Gegenbauer interpolation based on SVD)
Input: $L, K, N \in \mathbb{N}$ $(N \gg 1,\ 3 \le L \le K \le N)$, L is upper bound of $\max\{M_0, M_1\}$, sampled values $H(-\sin \frac{k\pi}{2N-1})$ $(k = 1-L-K, \ldots, L+K-1)$ of polynomial (4.6) of degree at most $2N-1$ and of low order $\alpha > 0$.

1. Compute for $k = 1-L-K, \ldots, L+K-1$

$$h_k := \sqrt{\frac{\Gamma(\alpha+1)\sqrt{\pi}}{\Gamma(\alpha+\frac{1}{2})}} \left(\cos \frac{k\pi}{2N-1}\right)^{\alpha} H\left(-\sin \frac{k\pi}{2N-1}\right)$$

   and form

$$f_k := \frac{h_k + h_{-k}}{2}, \quad g_k := \frac{h_k - h_{-k}}{2}.$$

2. Compute the SVD of the rectangular Toeplitz–plus–Hankel matrices (3.8) and (3.12). Determine the approximate rank M_0 of (3.8) such that $\sigma_{M_0}^{(0)}/\sigma_1^{(0)} > 10^{-8}$ and form the matrix (3.13). Determine the approximate rank M_1 of (3.12) such that $\sigma_{M_1}^{(1)}/\sigma_1^{(1)} > 10^{-8}$ and form the matrix (3.14).
3. Compute all eigenvalues $x_{0,j} \in [-1, 1]$ $(j = 1, \ldots, M_0)$ of the square matrix (3.15). Assume that the eigenvalues are ordered in the following form $1 \ge x_{0,1} > x_{0,2} > \cdots > x_{0,M_0} \ge -1$. Calculate $n_{0,j} := [\frac{2N-1}{\pi} \arccos x_{0,j} - \alpha]$ $(j = 1, \ldots, M_0)$.
4. Compute all eigenvalues $x_{1,j} \in [-1, 1]$ $(j = 1, \ldots, M_1)$ of the square matrix (3.16). Assume that the eigenvalues are ordered in the following form $1 \ge x_{1,1} > x_{1,2} > \cdots > x_{1,M_1} \ge -1$. Calculate $n_{1,j} := [\frac{2N-1}{\pi} \arccos x_{1,j} - \alpha]$ $(j = 1, \ldots, M_1)$.
5. Compute the coefficients $c_{0,j} \in \mathbb{R}$ $(j = 1, \ldots, M_0)$ and $c_{1,j} \in \mathbb{R}$ $(j = 1, \ldots, M_1)$ as least squares solutions of the overdetermined linear Vandermonde–like systems

$$\sum_{j=1}^{M_0} c_{0,j}\, Q_{n_{0,j}}^{(\alpha)}\left(\sin\frac{k\pi}{2N-1}\right) = f_k \quad (k = 0, \ldots, L+K-1)\,,$$

$$\sum_{j=1}^{M_1} c_{1,j}\, Q_{n_{1,j}}^{(\alpha)}\left(\sin\frac{k\pi}{2N-1}\right) = g_{-k} \quad (k = 0, \ldots, L+K-1).$$

Output: $M_0 \in \mathbb{N}_0$, $n_{0,j} \in \mathbb{N}_0$ $(0 \le n_{0,1} < n_{0,2} < \cdots < n_{0,M_0} < 2N)$, $c_{0,j} \in \mathbb{R}$ $(j = 1, \ldots, M_0)$. $M_1 \in \mathbb{N}_0$, $n_{1,j} \in \mathbb{N}$ $(1 \le n_{1,1} < n_{1,2} < \cdots < n_{1,M_1} < 2N)$, $c_{1,j} \in \mathbb{R}$ $(j = 1, \ldots, M_1)$.

5 Numerical examples

Now we illustrate the behavior and the limits of the suggested algorithms. Using IEEE standard floating point arithmetic with double precision, we have implemented our algorithms in MATLAB. In Example 5.1, an M-sparse Legendre expansion is given in the form (2.7) with normed Legendre polynomials of even degree $n_{0,j}$ $(j = 1, \ldots, M_0)$ and odd degree $n_{1,k}$ $(k = 1, \ldots, M_1)$, respectively, and corresponding real non-vanishing coefficients $c_{0,j}$ and $c_{1,k}$, respectively. In Examples 5.3 and 5.4, an M-sparse Gegenbauer expansion is given in the form (4.6) with normed Gegenbauer polynomials (of even/odd degree $n_{0,j}$ resp. $n_{1,k}$ and order $\alpha > 0$) and corresponding real non-vanishing coefficients $c_{0,j}$ resp. $c_{1,k}$. We compute the absolute error of the coefficients by

$$e(\boldsymbol{c}) := \max_{\substack{j=1,\ldots,M_0\\ k=1,\ldots,M_1}} \{|c_{0,j} - \tilde{c}_{0,j}|,\ |c_{1,k} - \tilde{c}_{1,k}|\}$$
$$(c := (c_{0,1}, \ldots, c_{0,M_0}, c_{1,1}, \ldots, c_{1,M_1})^{\mathrm{T}}),$$

where $\tilde{c}_{0,j}$ and $\tilde{c}_{1,k}$ are the coefficients computed by our algorithms. The symbol + in the Tables 2, 3, 4 and 5 means that all degrees n_j are correctly reconstructed, the

Table 2 Results of Example 5.1 for exact sampled data

N	K	L	Algorithm 3.1	$e(\boldsymbol{c})$
101	5	5	+	3.3307e−15
200	5	5	+	5.5511e−16
300	5	5	+	1.5876e−14
400	5	5	−	–
400	6	5	+	1.6209e−14
500	6	5	−	–
500	7	5	−	–
500	9	5	+	2.4780e−13

symbol – indicates that the reconstruction of the degrees fails. We present the error $e(\boldsymbol{c})$ in the last column of the tables.

Example 5.1 We start with the reconstruction of a 5-sparse Legendre expansion (2.7) which is a polynomial of degree 200. We choose the even degrees $n_{0,1} = 6, n_{0,2} = 12$, $n_{0,3} = 200$ and the odd degrees $n_{1,1} = 175$, $n_{1,2} = 177$ in (2.7). The corresponding coefficients $c_{0,j}$ and $c_{1,k}$ are equal to 1. Note that for the parameters $N = 400$ and $K = L = 5$, due to roundoff errors, some eigenvalues $\tilde{x}_{0,j}$ resp. $\tilde{x}_{1,k}$ are not contained in $[-1,\ 1]$. But we can improve the stability by choosing more sampling values. In the case $N = 500$, $K = 9$ and $L = 5$, we need only $2\,(K + L) - 1 = 27$ sampled values of (3.1) for the exact reconstruction of the 5-sparse Legendre expansion (2.7). □

Example 5.2 Now we show that the Algorithm 3.1 is robust with respect to noisy sampled data. We reconstruct a 5-sparse Legendre expansion (2.7) with the even degrees $n_{0,1} = 12$, $n_{0,2} = 150$ and the odd degrees $n_{1,1} = 75$, $n_{1,2} = 277$ and $n_{1,3} = 313$, where the corresponding coefficients $c_{0,j}$ and $c_{1,k}$ are equal to 1. Then we add noise of size $10^{-\delta}\eta$, where η is uniformly distributed in $[-1, 1]$, to each sampling value. The corresponding results of Algorithm 3.1 are shown in Table 3. Note that for certain parameters, some eigenvalues $\tilde{x}_{0,j}$ resp. $\tilde{x}_{1,k}$ are not contained in $[-1,\ 1]$. But we can improve the stability of Algorithm 3.1 by choosing more sampling values (see Table 3). Note that we can also reconstruct the polynomials for higher noisy level, if we replace the identification of the rank M_0 of (3.8) and M_1 of (3.12) in step 2 of Algorithm 3.1 by a gap condition. For the results see the last four lines of Table 3. □

Example 5.3 We consider now the reconstruction of a 5-sparse Gegenbauer expansion (4.6) of order $\alpha > 0$ which is a polynomial of degree 200. Similar as in Example 5.1, we choose the even degrees $n_{0,1} = 6$, $n_{0,2} = 12$, $n_{0,3} = 200$ and the odd degrees $n_{1,1} = 175, n_{1,2} = 177$ in (4.6). The corresponding coefficients $c_{0,j}$ and $c_{1,k}$ are equal to 1. Here we use only $2\,(L + K) - 1 = 19$ sampled values for the exact recovery of the 5-sparse Gegenbauer expansion (4.6) of degree 200. Despite the fact, that we show in Theorem 4.1 only results for $\alpha \in (0, 1)$, we show also some examples for $\alpha > 1$. But the suggested method fails for $\alpha = 3.5$. In this case our algorithm cannot exactly detect the smallest degrees $n_{0,1} = 6$ and $n_{0,2} = 12$, but all the higher degrees are exactly detected. □

Example 5.4 We consider the reconstruction of a 5-sparse Gegenbauer expansion (4.6) of order $\alpha > 0$ which does not consist of Gegenbauer polynomials of low degrees. Thus we choose the even degrees $n_{0,1} = 60$, $n_{0,2} = 120$, $n_{0,3} = 200$ and the odd degrees $n_{1,1} = 175$, $n_{1,2} = 177$ in (4.6). The corresponding coefficients $c_{0,j}$ and $c_{1,k}$ are equal to 1. In Table 5, we show also some examples for $\alpha \geq 2.5$. But the suggested method fails for $\alpha = 8$. In this case, Algorithm 4.2 cannot exactly detect the smallest degree $n_{0,1} = 60$, but all the higher degrees are exactly recovered. This observation is in perfect accordance with the very good local approximation near $\theta = \pi/2$, see Theorem 4.1 and Remark 4.1. □

Table 3 Results of Example 5.2 for noisy sampled data

N	K	L	δ	Algorithm 3.1	$e(\boldsymbol{c})$
200	5	5	5	−	–
200	9	9	5	+	1.6020e−05
200	25	25	5	+	7.9357e−06
200	65	65	5	+	3.3771e−07
200	100	30	3	+	5.1114e−03
200	110	30	3	+	1.2290e−03
200	110	40	3	+	1.2226e−03
200	100	50	3	+	5.6290e−04

Table 4 Results of Example 5.3

α	N	K	L	Algorithm 4.2	$e(\boldsymbol{c})$
0.1	101	5	5	+	5.5511e−16
0.2	101	5	5	+	2.2204e−16
0.4	101	5	5	−	–
0.4	200	5	5	+	1.0769e−14
0.5	200	5	5	+	8.8818e−16
0.9	200	5	5	+	7.5835e−16
1.5	200	5	5	+	1.3323e−15
2.5	200	5	5	+	1.1102e−16
3.5	200	5	5	−	–

Table 5 Results of Example 5.4

α	N	K	L	Algorithm 4.2	$e(\boldsymbol{c})$
0.1	101	5	5	+	1.2879e−14
0.2	101	5	5	+	1.1879e−14
0.4	101	5	5	−	–
0.4	200	5	5	+	3.1086e−15
0.9	200	5	5	+	1.3323e−14
2.5	200	5	5	+	7.7716e−16
3.5	200	5	5	+	5.4401e−15
4.5	200	5	5	+	3.3862e−14
7.0	200	5	5	+	2.2204e−16
7.5	200	5	5	+	3.3307e−16
8.0	200	5	5	−	–
9.0	200	5	5	−	–

Example 5.5 We stress again that the Prony-like methods are very powerful tools for the recovery of a sparse exponential sum

$$S(x) := \sum_{j=1}^{M} c_j \, \mathrm{e}^{f_j x} \quad (x \geq 0)$$

with distinct numbers $f_j \in [-\delta, 0] + \mathrm{i}\,[-\pi, \pi)$ $(0 \le \delta \ll 1)$ and complex non–vanishing coefficients c_j, if only finitely many sampled data of S are given. In [17, Example 5.5], we have presented a method to reconstruct functions of the form

$$F(\theta) = \sum_{j=1}^{M} \big(c_j\, \cos(\nu_j \theta) + d_j\, \sin(\mu_j \theta)\big) \quad (\theta \in [0,\, \pi]).$$

with real coefficients c_j, d_j and distinct frequencies $\nu_j, \mu_j > 0$ by sampling the function F.

Now we reconstruct a sum of sparse Legendre and Chebyshev expansions

$$H(x) := \sum_{j=1}^{M} c_j\, L_{n_j}(x) + \sum_{k=1}^{M'} d_k\, T_{m_k}(x) \quad (x \in [-1,\ 1]). \tag{5.1}$$

Here we choose $c_j = d_k = 1$, $M = M' = 5$ and $(n_j)_{j=1}^5 = (6, 13, 165, 168, 190)^{\mathrm{T}}$ and $(m_k)_{k=1}^5 = (60, 120, 175, 178, 200)^{\mathrm{T}}$. Substituting $x = -\sin t$ for $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we obtain

$$\sqrt{\frac{\pi}{2}}\,\sqrt{\cos t}\; H(-\sin t) = \sum_{j=1}^{M} c_j\, Q_{n_j}(-\sin t) + \sum_{k=1}^{M'} d_k \sqrt{\frac{\pi}{2}}\,\sqrt{\cos t}\, \cos\left(m_k t + \frac{m_k \pi}{2}\right). \tag{5.2}$$

By Theorem 2.1, the function

$$\sum_{j=1}^{M} c_j\, \lambda_{n_j}\, \cos\left[\left(n_j + \frac{1}{2}\right) t + \frac{n_j \pi}{2}\right] + \sum_{k=1}^{M'} d_k \sqrt{\frac{\pi}{2}}\, \cos\left(m_k t + \frac{m_k \pi}{2}\right) \tag{5.3}$$

approximates (5.2) in the near of 0. We apply Algorithm 3.1 with $N = 200$ and $K = L = 20$. In step 3 of Algorithm 3.1, we calculate all frequencies $n_j + \frac{1}{2}$ and m_k of (5.3), if n_j and m_k are even:

$$\left(\frac{2N-1}{\pi} \arccos x_{0,j}\right)_{j=1}^{7} = (199.99,\ 190.50,\ 178.00,\ 168.49,\ 120.00,\ 60.000,\ 6.52)^{\mathrm{T}}.$$

In step 4 of Algorithm 3.1, we determine all frequencies $n_j + \frac{1}{2}$ and m_k of (5.3), if n_j and m_k are odd:

$$\left(\frac{2N-1}{\pi} \arccos x_{1,j}\right)_{j=1}^{3} = (165.500,\ 175.000,\ 13.509)^{\mathrm{T}}.$$

Thus the Legendre polynomials $L_{n_j}(x)$ in (5.1) have the even degrees 6, 168, and 190 and the odd degrees 13 and 165. Thus the Chebyshev polynomials $T_{m_k}(x)$ in (5.1) possess the even degrees 60, 120, 178, and 200 and the odd degree 175. Finally, one can compute the coefficients c_j and d_k. $\square$

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