Efficient multivariate approximation on the cube

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For the approximation of multivariate non-periodic functions h on the high-dimensional cube $\left[-\frac{1}{2},\frac{1}{2}\right]^d$ we combine a periodization strategy for weighted L_2-integrands with efficient approximation methods. We prove sufficient conditions on d-variate torus-to-cube transformations $\psi : \left[-\frac{1}{2},\frac{1}{2}\right]^d \to \left[-\frac{1}{2},\frac{1}{2}\right]^d$ and on the non-negative weight function ω such that the composition of a possibly non-periodic function with a transformation ψ yields a smooth function in the Sobolev space $\mathcal{H}^m(\mathbb{T}^d)$. In this framework we adapt certain $L_\infty(\mathbb{T}^d)$- and $L_2(\mathbb{T}^d)$-approximation error estimates for single rank-1 lattice approximation methods as well as algorithms for the evaluation and reconstruction of multivariate trigonometric polynomials on the torus to the non-periodic setting. Various numerical tests in up to dimension $d = 5$ confirm the obtained theoretical results for the transformed approximation methods.

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1 Introduction

In this paper we discuss a general framework for the approximation of non-periodic multivariate functions h on the d-dimensional cube $\left[-\frac{1}{2},\frac{1}{2}\right]^d$ in which we combine a particular periodization strategy for weighted L_2-integrands with approximation methods based on rank-1 lattices. At first we consider univariate transformations $\psi : \left[-\frac{1}{2},\frac{1}{2}\right] \to \left[-\frac{1}{2},\frac{1}{2}\right]$ that are increasing, continuously differentiable and have some number $k \in \mathbb{N}$ of derivatives $\psi_j^{(k)}$ that vanish at the boundary points $\left\{-\frac{1}{2},\frac{1}{2}\right\}$. In one dimension the application of such a change of variables $y = \psi(x)$ to any $h \in L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right],\omega\right)$ with a non-negative weight function ω yields

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |h(y)|^2\,\omega(y)\,\mathrm{d}y = \int_{-\frac{1}{2}}^{\frac{1}{2}} |h(\psi(x))|^2\,\omega(\psi(x))\,\psi'(x)\,\mathrm{d}x = \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^2\,\mathrm{d}x \tag{1.1}$$

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with $f(x) = h(\psi(x))\sqrt{\omega(\psi(x))\,\psi'(x)}$. For dimensions $d \geq 2$ a multivariate generalization yields similar d-variate functions f. We prove that if the non-periodic function h has certain smoothness properties and we assume certain boundary conditions on both the weight function ω and the transformation ψ, then the transformed function f is continuously extendable on the torus $\mathbb{T}^d$ and has some guaranteed minimal degree of Sobolev-smoothness. This enables us to rewrite the involved objects, algorithms and approximation error bounds by means of the inverse transformation ψ^{-1}. The approximation of functions $f \in L_2(\mathbb{T}^d)$ with respect to the Fourier system $\{\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\circ}\}_{\mathbf{k}\in\mathbb{Z}^d}$ by a Fourier partial sum $S_I f := \sum_{\mathbf{k}\in I} \hat{f}_{\mathbf{k}}\,\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\circ}$ translates into the approximation of functions $h \in L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d, \omega\right)$ by a transformed Fourier partial sum of the form $\sum_{\mathbf{k}\in I} \hat{h}_{\mathbf{k}}\,\varphi_{\mathbf{k}}$ with respect to the orthonormal system $\left\{\varphi_{\mathbf{k}} := \sqrt{\frac{(\psi^{-1})'(\circ)}{\omega(\circ)}}\,\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\psi^{-1}(\circ)}\right\}_{\mathbf{k}\in\mathbb{Z}^d}$ and the $\mathbf{k}$-th Fourier coefficient of h is given by $\hat{h}_{\mathbf{k}} := (h,\varphi_{\mathbf{k}})_{L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d,\omega\right)}$.

The outlined periodization strategy furthermore allows us to apply existing approximation methods for smooth functions defined on the torus $\mathbb{T}^d$. We focus on approximation theory concerned with the Wiener algebra $\mathcal{A}(\mathbb{T}^d)$ containing all $L_1(\mathbb{T}^d)$-functions with absolutely summable Fourier coefficients $\hat{f}_{\mathbf{k}}$ and $\mathbf{k} = (k_1,\ldots,k_d)^\top \in \mathbb{Z}^d$, see [39, 11]. Considering the weight function

$$\omega_{\mathrm{hc}}(\mathbf{k}) := \prod_{j=1}^{d} \max(1,|k_j|), \tag{1.2}$$

there are for $\beta \geq 0$ the subspaces of the Wiener algebra $\mathcal{A}(\mathbb{T}^d)$ in form of

$$\mathcal{A}^\beta(\mathbb{T}^d) := \left\{ f \in L_1(\mathbb{T}^d) : \|f\|_{\mathcal{A}^\beta(\mathbb{T}^d)} := \sum_{\mathbf{k}\in\mathbb{Z}^d} \omega_{\mathrm{hc}}(\mathbf{k})^\beta |\hat{f}_{\mathbf{k}}| < \infty \right\} \tag{1.3}$$

and the Hilbert spaces

$$\mathcal{H}^\beta(\mathbb{T}^d) := \left\{ f \in L_2(\mathbb{T}^d) : \|f\|_{\mathcal{H}^\beta(\mathbb{T}^d)} := \left(\sum_{\mathbf{k}\in\mathbb{Z}^d} \omega_{\mathrm{hc}}(\mathbf{k})^{2\beta} |\hat{f}_{\mathbf{k}}|^2 \right)^{\frac{1}{2}} < \infty \right\}, \tag{1.4}$$

whose norms contain information about the decay rate of the Fourier coefficients $\hat{f}_{\mathbf{k}}$ with respect to the weight function ω_{hc}. For the hyperbolic crosses $I_N^d := \{\mathbf{k} \in \mathbb{Z}^d : \omega_{\mathrm{hc}}(\mathbf{k}) \leq N\} \subset \mathbb{Z}^d$ with $N \in \mathbb{N}$ and the approximated Fourier partial sums of the form $S_I^\Lambda f := \sum_{\mathbf{k}\in I} \hat{f}_{\mathbf{k}}^\Lambda\,\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\circ}$ with only approximated Fourier coefficients $\hat{f}_{\mathbf{k}}^\Lambda \approx \hat{f}_{\mathbf{k}}$ there are approximation error bounds when using single rank-1 lattices. It was shown in [21, Theorem 3.3] that the error of approximating a continuous function $f \in \mathcal{A}^\beta(\mathbb{T}^d)$ by the approximated Fourier partial sum $S_{I_N^d}^\Lambda f$ measured in the $L_\infty(\mathbb{T}^d)$-norm is bounded above by $N^{-\beta}\|f\|_{\mathcal{A}^\beta(\mathbb{T}^d)}$. Approximating a continuous function $f \in \mathcal{H}^\beta(\mathbb{T}^d)$ by the approximated Fourier partial sum $S_{I_N^d}^\Lambda f$ measured in the $L_2(\mathbb{T}^d)$-norm is bounded above by $C_{d,\beta}N^{-\beta}(\log N)^{(d-1)/2}\|f\|_{\mathcal{H}^\beta(\mathbb{T}^d)}$ with some constant $C_{d,\beta} = C(d,\beta) > 0$ as shown in [41, Theorem 2.30]. The approximation of functions in the Hilbert spaces $\mathcal{H}^\beta(\mathbb{T}^d)$ was also investigated by V. N. Temlyakov, see e.g. [38, 21].

A major problem is that in general it's hard to calculate the Fourier coefficients $\hat{f}_{\mathbf{k}}$ in order to determine if they are absolutely or square summable. Instead we utilize certain norm equivalences to get information about the decay rate of the Fourier coefficients $\hat{f}_{\mathbf{k}}$. Given a multi-index $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)^\top \in \mathbb{N}_0^d$ with $\|\boldsymbol{\alpha}\|_{\ell_\infty} := \max(|\alpha_1|, \ldots, |\alpha_d|)$ we define for $\Omega \in \{\mathbb{T}^d, \left[-\frac{1}{2}, \frac{1}{2}\right]^d\}$ the norm

$$\|f\|_{H^m_{\mathrm{mix}}(\Omega)} := \left(\sum_{\|\boldsymbol{\alpha}\|_{\ell_\infty} \leq m} \|D^{\boldsymbol{\alpha}}[f]\|^2_{L_2(\Omega)} \right)^{1/2} \tag{1.5}$$

of the Sobolev space $H^m_{\mathrm{mix}}(\Omega)$ of functions $f \in L_2(\Omega)$ with mixed natural smoothness $m \in \mathbb{N}_0$, that were discussed in [34, 40, 42]. Similarly, we define the norm

$$\|f\|_{\mathcal{C}^m_{\mathrm{mix}}(\Omega)} := \sum_{\|\boldsymbol{\alpha}\|_{\ell_\infty} \leq m} \|D^{\boldsymbol{\alpha}}[f]\|_{L_\infty(\Omega)} \tag{1.6}$$

of the space $\mathcal{C}^m_{\mathrm{mix}}(\Omega)$ of functions with mixed continuous differentiability order $m \in \mathbb{N}_0$, given in [34]. As shown in [26, Lemma 2.3], the norms $\|\cdot\|_{H^m_{\mathrm{mix}}(\mathbb{T}^d)}$ and $\|\cdot\|_{\mathcal{H}^\beta(\mathbb{T}^d)}$ are equivalent for $\beta = m \in \mathbb{N}$. Furthermore, for all $\beta \geq 0$ and all $\lambda > \frac{1}{2}$ we have the continuous embedding $\mathcal{H}^{\beta+\lambda}(\mathbb{T}^d) \hookrightarrow \mathcal{A}^\beta(\mathbb{T}^d)$ as shown in [21, Lemma 2.2].

For functions f obtained by the change of variables (1.1) we provide a set of sufficient L_∞-conditions so that such an f is in $\mathcal{A}^m(\mathbb{T}^d)$ or $\mathcal{H}^m(\mathbb{T}^d)$, that are easier to check than calculating either of the equivalent norms $\|\cdot\|_{H^m_{\mathrm{mix}}(\mathbb{T}^d)} \sim \|\cdot\|_{\mathcal{H}^m(\mathbb{T}^d)}$. At first we prove these conditions for all possible transformations ψ and weight functions ω. Later on we consider families of parameterized transformations $\psi(\circ) = \psi(\circ, \boldsymbol{\eta})$ and families of weight functions $\omega(\circ) = \omega(\circ, \boldsymbol{\mu})$ with $\boldsymbol{\eta}, \boldsymbol{\mu} \in \mathbb{R}^d_+$. Then we have parameterized transformed functions $f(\circ) = f(\circ, \boldsymbol{\eta}, \boldsymbol{\mu}) \in L_2(\mathbb{T}^d)$ and both parameters may impact the smoothness of these functions. With the sufficient L_∞-smoothness conditions we calculate lower bounds for $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ such that the smoothness degree m of a function $h \in L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d, \omega(\circ, \boldsymbol{\mu})\right) \cap \mathcal{C}^m_{\mathrm{mix}}\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right)$ remains in a certain sense under composition with a family of transformations $\psi(\circ, \boldsymbol{\eta})$ so that we end up with $f \in \mathcal{H}^m(\mathbb{T}^d)$.

Based on these facts we are able to transfer well-known results from the torus $\mathbb{T}^d$ to the non-periodic setting on the cube $\left[-\frac{1}{2}, \frac{1}{2}\right]^d$ by means of the inverse transformation ψ^{-1}, which includes:

- some good L_2- and L_∞-approximation results as in [21, 3],
- fast algorithms in [21, 17] based on rank-1 lattice approximation which are suitable for high-dimensional approximation.

Generally the change of variables is a versatile and powerful tool in numerical analysis. An excellent overview is found in [1, Chapter 16 and 17] which contains many practical aspects of the mapped methods. In recent years they were repeatedly used for the numerical integration and approximation of non-periodic functions in Chebyshev spaces [32] as well as in half-periodic cosine spaces and Korobov spaces by means of tent-transformed lattice rules [10, 6, 13, 27]. In particular for numerical integration certain strategies to periodize integrands have been discussed in [28]. For sampling purposes besides single and multiple rank-1 lattice rules [21, 17], there are sampling methods on sparse grids [14, 2, 15], randomized least square sampling approaches [16, 24] and also interlaced scrambled polynomial lattice rules [12, 8].

An introduction to lattice rules can be found in [31, 36, 9]. These rules were used also for the approximation of functions on the torus, see [39]. Recently, efficient algorithms based on component-by-component methods [7, 5] were presented in order to compute high-dimensional integrals. For the approximation of high-dimensional functions there are efficient algorithms using sampling schemes based on rank-1 lattices [21, 17], and furthermore these schemes provide good approximation properties, see also [3, 27]. We adapt these algorithms to the non-periodic setting and incorporate the outlined use of transformations. Furthermore, we present numerical examples.

The outline of the paper is as follows: In Section 2 we establish the basic notions from classical Fourier approximation theory on the torus $\mathbb{T}^d$, the corresponding function spaces and important convergence properties. We introduce the Sobolev spaces $H^m_{\text{mix}}(\mathbb{T}^d)$ of mixed natural smoothness order $m \in \mathbb{N}_0$ and the Wiener Algebra $\mathcal{A}(\mathbb{T}^d)$ of functions with absolutely summable Fourier coefficients. Furthermore, we discuss certain properties of the subspaces $\mathcal{A}^\beta(\mathbb{T}^d)$ and $\mathcal{H}^\beta(\mathbb{T}^d)$ of the Wiener Algebra, in particular we highlight the norm equivalence of $\|\cdot\|_{\mathcal{H}^m(\mathbb{T}^d)}$ and $\|\cdot\|_{H^m_{\text{mix}}(\mathbb{T}^d)}$ for all $m \in \mathbb{N}$, see [26]. Then we define rank-1 lattices as introduced in [23], discuss their importance in the context of Fourier approximation and recall two important approximation error bounds on the torus in Theorems 2.2 and 2.3. In Section 3 we define the notion of a torus-to-cube transformation $\psi : \left[-\frac{1}{2}, \frac{1}{2}\right]^d \to \left[-\frac{1}{2}, \frac{1}{2}\right]^d$ and provide a couple of examples that we will use later on. Then we introduce weight functions $\omega : \left[-\frac{1}{2}, \frac{1}{2}\right]^d \to [0, \infty)$ and describe the structure of the weighted Hilbert spaces $L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d, \omega\right)$, the corresponding weighted scalar product $(\cdot, \cdot)_{L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d, \omega\right)}$ and the Fourier coefficients $\hat{h}_{\mathbf{k}}$. Afterwards we prove sufficient L_∞-conditions on the transformation ψ and weight function ω, such that a function $h \in L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d, \omega\right) \cap \mathcal{C}^m_{\text{mix}}\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right)$ is transformed under composition with ψ into a smooth function $f \in \mathcal{H}^m(\mathbb{T}^d)$. Then we are able to prove upper bounds for the approximation error $\left\|h - S_I^\Lambda h\right\|$ measured in weighted L_2- and L_∞-norms on $\left[-\frac{1}{2}, \frac{1}{2}\right]^d$ in Theorems 3.4 and 3.5 based on the theorems on the torus $\mathbb{T}^d$ that were recalled in Section 2. In Section 4 we incorporate the usage of transformations ψ into the algorithms [17, Algorithm 3.1 and 3.2] for the evaluation and the reconstruction of multivariate functions in Algorithms 4.1 and 4.2 based on transformed rank-1 lattices. In Section 5 we discuss examples in which we use the logarithmic transformation (3.9) and the sine transformation (3.11). For both univariate and multivariate test functions h we use a constant weight function $\omega \equiv 1$. With the sufficient L_∞-conditions from Section 3 we then calculate explicit bounds for $\boldsymbol{\eta} \in \mathbb{R}^d_+$ that determine the degree of smoothness $m \in \mathbb{N}$ of h that is preserved under composition with the family of transformations $\psi(\circ, \boldsymbol{\eta})$. Then we use the algorithms of the previous section, compare the decay of the discretized weighted L_∞-approximation error given in (4.3) and observe the proposed approximation error decay caused by increasing the parameter $\boldsymbol{\eta} \in \mathbb{R}^d_+$ in up to dimension $d = 5$.

2 Fourier approximation

At first we introduce weighted L_2-function spaces and Sobolev spaces of mixed smoothness, recall some definitions of classical Fourier approximation theory and define a space of functions that have absolute square-summable Fourier coefficients. Finally, we reflect the ideas of rank-1 lattices from [37, 5, 17], the corresponding Fourier approximation methods, and approximation error bounds that were discussed in e.g., [38, 21, 3].

2.1 Preliminaries

Let $\Omega \in \left\{\mathbb{T}^d, \left[-\frac{1}{2}, \frac{1}{2}\right]^d\right\}$ with $\mathbb{T}^d \simeq [-\frac{1}{2}, \frac{1}{2})^d$ being the d-dimensional torus. The space $\left(\mathcal{C}(\Omega), \|\cdot\|_{L_\infty(\Omega)}\right)$ denotes the collection of all continuous multivariate functions $f : \Omega \to \mathbb{C}$, and $\left(\mathcal{C}_0\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right), \|\cdot\|_{L_\infty\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right)}\right)$ denotes the space of all continuous functions defined on $\left[-\frac{1}{2}, \frac{1}{2}\right]^d$ that vanish at the boundary points $[-\frac{1}{2}, \frac{1}{2}]^d \setminus (-\frac{1}{2}, \frac{1}{2})^d$. For multi-indices $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ and the differential operator

$$D^{\boldsymbol{\alpha}}[f](\mathbf{x}) = D^{(\alpha_1,\ldots,\alpha_d)}[f](x_1, \ldots, x_d) := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}[f](x_1, \ldots, x_d),$$

we define the *space of mixed continuous differentiability* of order $m \in \mathbb{N}$, see [34, page 132], as

$$\mathcal{C}^m_{\mathrm{mix}}(\Omega) := \left\{f \in \mathcal{C}(\Omega) : \|f\|_{\mathcal{C}^m_{\mathrm{mix}}(\Omega)} < \infty\right\}$$

with $\|\cdot\|_{\mathcal{C}^m_{\mathrm{mix}}(\Omega)}$ given in (1.6). The respective univariate spaces are denoted by $\mathcal{C}^m(\Omega)$.

The weighted function spaces $L_2(\Omega, \omega)$ with an integrable weight function $\omega : \Omega \to [0, \infty)$ are defined as

$$L_2(\Omega, \omega) := \left\{h \in L_2(\Omega) : \|h\|_{L_2(\Omega,\omega)} := \left(\int_\Omega |h(\mathbf{x})|^2\, \omega(\mathbf{x})\, \mathrm{d}\mathbf{x}\right)^{\frac{1}{2}} < \infty\right\}. \tag{2.1}$$

For the constant weight function $\omega(\mathbf{x}) \equiv 1$ we have $L_2(\Omega, \omega) = L_2(\Omega)$. For functions f and g in the Hilbert space $L_2(\mathbb{T}^d)$ we have the scalar product

$$(f, g)_{L_2(\mathbb{T}^d)} := \int_{\mathbb{T}^d} f(\mathbf{x})\, \overline{g(\mathbf{x})}\, \mathrm{d}\mathbf{x}.$$

For any frequency set $I \subset \mathbb{Z}^d$ of finite cardinality $|I| < \infty$ we denote the space of all multivariate trigonometric polynomials supported on I by

$$\Pi_I := \mathrm{span}\{\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\circ} : \mathbf{k} \in I\}.$$

The functions $\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}} = \prod_{j=1}^d \mathrm{e}^{2\pi\mathrm{i}k_j x_j}$ with $\mathbf{k} \in \mathbb{Z}^d$ and $\mathbf{x} \in \mathbb{T}^d$ are orthogonal with respect to the $L_2(\mathbb{T}^d)$-scalar product. For all $\mathbf{k} \in \mathbb{Z}^d$ we denote the *Fourier coefficients* $\hat{f}_{\mathbf{k}}$ by

$$\hat{f}_{\mathbf{k}} = (f, \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\circ})_{L_2(\mathbb{T}^d)} = \int_{\mathbb{T}^d} f(\mathbf{x})\, \mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\, \mathrm{d}\mathbf{x},$$

and the corresponding *Fourier partial sum* by $S_I f(\mathbf{x}) = \sum_{\mathbf{k}\in I} \hat{f}_{\mathbf{k}}\, \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$. For all $f \in L_2(\mathbb{T}^d)$ we have

$$\|f - S_I f\|_{L_2(\mathbb{T}^d)} \to 0 \quad \text{for} \quad |I| \to \infty,$$

where $|I| \to \infty$ means $\min(|k_1|, \ldots, |k_d|) \to \infty$ for $\mathbf{k} = (k_1, \ldots, k_d)^\top \in I$, see [43, Theorem 4.1].

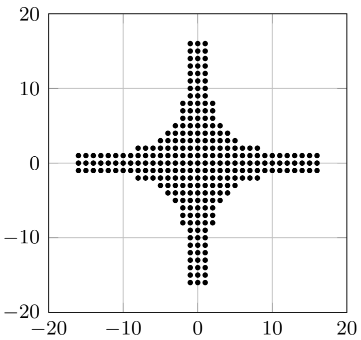

Figure 2.1: The hyperbolic cross I_N^d for $N = 16$ and $d = 2$.

Finally, we define the *Sobolev spaces of mixed natural smoothness* of $L_2(\Omega)$-functions with smoothness order $m \in \mathbb{N}_0$, see [34, 40, 42], as

$$H^m_{\mathrm{mix}}(\Omega) := \Big\{ f \in L_2(\Omega) : \|f\|_{H^m_{\mathrm{mix}}(\Omega)} < \infty \Big\}$$

with $\|\cdot\|_{H^m_{\mathrm{mix}}(\Omega)}$ as given in (1.5). The respective univariate spaces are denoted by $H^m(\Omega)$.

Based on the weight function $\omega_{\mathrm{hc}}(\mathbf{k})$ given in (1.2) we define *hyperbolic crosses* I_N^d as

$$I_N^d := \Big\{ \mathbf{k} \in \mathbb{Z}^d : \omega_{\mathrm{hc}}(\mathbf{k}) \le N \Big\}, \tag{2.2}$$

illustrated for $N = 16$ in two dimensions in Figure 2.1. Furthermore, for $\beta \ge 0$ we form the Hilbert space $\mathcal{H}^\beta(\mathbb{T}^d)$ consisting of functions $f \in L_2(\mathbb{T}^d)$ with absolutely square-summable weighted Fourier coefficients $\omega_{\mathrm{hc}}(\mathbf{k})\hat{f}_{\mathbf{k}}$, as defined in (1.4). In [26] it was shown that

$$\|\cdot\|_{\mathcal{H}^m(\mathbb{T}^d)} \sim \|\cdot\|_{H^m_{\mathrm{mix}}(\mathbb{T}^d)} \tag{2.3}$$

for all $m \in \mathbb{N}$. Closely related are the function spaces $\mathcal{A}^\beta(\mathbb{T}^d), \beta \ge 0$ of $L_1(\mathbb{T}^d)$-functions with absolutely summable Fourier coefficients, as defined in (1.3). For $\beta = 0$ and the constant weight function $\omega_{\mathrm{hc}}(\mathbf{k}) \equiv 1$ we call the space $\mathcal{A}(\mathbb{T}^d) := \mathcal{A}^0(\mathbb{T}^d)$ the *Wiener Algebra.* As shown in [21, Lemma 2.2], for $\beta \ge 0, \lambda > \frac{1}{2}$ and fixed $d \in \mathbb{N}$ there are the continuous embeddings

$$\mathcal{H}^{\beta+\lambda}(\mathbb{T}^d) \hookrightarrow \mathcal{A}^\beta(\mathbb{T}^d) \hookrightarrow \mathcal{A}(\mathbb{T}^d) \tag{2.4}$$

and for $f \in \mathcal{A}^\beta(\mathbb{T}^d)$ we have

$$\|f\|_{\mathcal{A}^\beta(\mathbb{T}^d)} \le C_{d,\lambda} \|f\|_{\mathcal{H}^{\beta+\lambda}(\mathbb{T}^d)} \tag{2.5}$$

with a constant $C_{d,\lambda} := C(d, \lambda) > 1$. Additionally, for each function in $\mathcal{A}(\mathbb{T}^d)$ there exists a continuous representative, as proven in [17, Lemma 2.1]. Later on, when we sample functions $f \in \mathcal{H}^{\beta+\lambda}(\mathbb{T}^d)$ we identify them with their continuous representatives given by their Fourier series $\sum_{\mathbf{k}\in\mathbb{Z}^d} \hat{f}_{\mathbf{k}}\, \mathrm{e}^{2\pi \mathrm{i}\mathbf{k}\cdot\circ}$ and this identification will be denoted by $f \in \mathcal{H}^{\beta+\lambda}(\mathbb{T}^d) \cap \mathcal{C}(\mathbb{T}^d)$.

2.2 Rank-1 lattices and reconstructing rank-1 lattices

Before discussing the approximation of functions $f \in \mathcal{H}^\beta(\mathbb{T}^d) \cap \mathcal{C}(\mathbb{T}^d)$ we recollect some related objects and observations from [37, 5, 17]. For each frequency set $I \subset \mathbb{Z}^d$ there is the *difference set*

$$\mathcal{D}(I) := \{\mathbf{k} \in \mathbb{Z}^d : \mathbf{k} = \mathbf{k}_1 - \mathbf{k}_2 \text{ with } \mathbf{k}_1, \mathbf{k}_2 \in I\}.$$

Furthermore, the set

$$\Lambda(\mathbf{z}, M) := \left\{\mathbf{x}_j := \left(\frac{j}{M}\mathbf{z} \bmod \mathbf{1}\right) \in \mathbb{T}^d : j = 0, 1, \ldots M-1\right\} \tag{2.6}$$

is called *rank-*1 *lattice* with the *generating vector* $\mathbf{z} \in \mathbb{Z}^d$ and the *lattice size* $M \in \mathbb{N}$, where $\mathbf{1} := (1, \ldots, 1)^\top \in \mathbb{Z}^d$. A *reconstructing rank-*1 *lattice* $\Lambda(\mathbf{z}, M, I)$ is a rank-1 lattice $\Lambda(\mathbf{z}, M)$ for which the condition

$$\mathbf{t} \cdot \mathbf{z} \not\equiv 0 \,(\operatorname{mod} M) \quad \text{for all } \mathbf{t} \in \mathcal{D}(I) \setminus \{\mathbf{0}\}$$

holds. Given a reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, I)$, we have exact integration for all multivariate trigonometric polynomials $g \in \Pi_{\mathcal{D}(I)}$, see [37], so that

$$\int_{\mathbb{T}^d} g(\mathbf{x}) \,\mathrm{d}\mathbf{x} = \frac{1}{M} \sum_{j=0}^{M-1} g(\mathbf{x}_j), \quad \mathbf{x}_j \in \Lambda(\mathbf{z}, M, I).$$

In particular, for $f \in \Pi_I$ and $\mathbf{k} \in I$ we have $f(\circ)\,\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\circ} \in \Pi_{\mathcal{D}(I)}$ and

$$\hat{f}_{\mathbf{k}} = \int_{\mathbb{T}^d} f(\mathbf{x})\,\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \,\mathrm{d}\mathbf{x} = \frac{1}{M} \sum_{j=0}^{M-1} f(\mathbf{x}_j)\,\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}_j}, \quad \mathbf{x}_j \in \Lambda(\mathbf{z}, M, I). \tag{2.7}$$

For an arbitrary function $f \in \mathcal{H}^\beta(\mathbb{T}^d) \cap \mathcal{C}(\mathbb{T}^d)$ and lattice points $\mathbf{x}_j \in \Lambda(\mathbf{z}, M, I)$ we lose the former mentioned exactness and get *approximated Fourier coefficients* $\hat{f}_{\mathbf{k}}^\Lambda$ of the form

$$\hat{f}_{\mathbf{k}} \approx \hat{f}_{\mathbf{k}}^\Lambda := \frac{1}{M} \sum_{j=0}^{M-1} f(\mathbf{x}_j)\,\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}_j}$$

leading to the *approximated Fourier partial sum* $S_I^\Lambda f$ given by

$$S_I f(\mathbf{x}) \approx S_I^\Lambda f(\mathbf{x}) := \sum_{\mathbf{k} \in I} \hat{f}_{\mathbf{k}}^\Lambda \,\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}.$$

2.3 Lattice based approximation on the torus

We discuss upper bounds for certain approximation errors $\left\| f - S_{I_N^d}^\Lambda f \right\|$ of functions f in $\mathcal{A}^\beta(\mathbb{T}^d) \cap \mathcal{C}(\mathbb{T}^d)$ and $\mathcal{H}^\beta(\mathbb{T}^d) \cap \mathcal{C}(\mathbb{T}^d)$. For this matter the existence of reconstructing rank-1 lattices is secured by the arguments provided in [18, Corollary 1] and [21, Theorem 2.1].

Remark 2.1. *We note that techniques using multiple rank-*1 *lattices were recently suggested in [19, 20] and methods for the dimension incremental construction of unknown frequency set* $I \subset \mathbb{Z}^d$ *are presented in [33]. Futhermore, a dimensional incremental support identification technique based on randomly chosen sampling points, that was recently developed in [4]. Even though the transformation method is easily incorporated into both the multiple rank-*1 *lattice methods as well as the component-by-component construction method, they won't be discussed any further in this work.* □

Now it's possible to prove an upper error bound for the L_∞-approximation of functions in the subspace $\mathcal{A}^\beta(\mathbb{T}^d)$ of the Wiener Algebra, as seen in [21, Theorem 3.3]:

Theorem 2.2. *Let $f \in \mathcal{A}^\beta(\mathbb{T}^d) \cap \mathcal{C}(\mathbb{T}^d)$ with $\beta \geq 0$ and $d \in \mathbb{N}$, a hyperbolic cross I_N^d with $|I_N^d| < \infty$ and $N \in \mathbb{N}$, and a reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, I_N^d)$ be given. The approximation of f by the approximated Fourier partial sum $S_{I_N^d}^\Lambda f$ leads to an approximation error that is estimated by*

$$\left\| f - S_{I_N^d}^\Lambda f \right\|_{L_\infty(\mathbb{T}^d)} \leq 2N^{-\beta} \|f\|_{\mathcal{A}^\beta(\mathbb{T}^d)}. \tag{2.8}$$

The approximation of functions in the Hilbert spaces $\mathcal{H}^\beta(\mathbb{T}^d)$ was investigated by V. N. Temlyakov, see [38, 21]. He showed that for $\beta > 1$ there exists a reconstructing rank-1 lattice generated by a vector of Korobov form $\mathbf{z} := (1, z, z^2, \ldots, z^{d-1})^\top \in \mathbb{Z}^d$ such that the L_2-truncation error is bounded above by

$$\left\| f - S_{I_N^d}^\Lambda f \right\|_{L_2(\mathbb{T}^d)} \leq N^{-\beta} (\log N)^{(d-1)/2} \|f\|_{\mathcal{H}^\beta(\mathbb{T}^d)}.$$

A generalization of this estimate as well as an upper bound for the corresponding aliasing error can be found in [3, Theorem 2], where they are stated in terms of dyadic hyperbolic cross frequency sets and where they use a component-by-component approach to construct the generating vector $\mathbf{z} \in \mathbb{Z}^d$, which generally isn't of Korobov form anymore. However, every dyadic hyperbolic cross is embedded in a non-dyadic one, see [41, Lemma 2.29]. Thus, the error estimates are easily translated in terms of non-dyadic hyperbolic crosses I_N^d, see [41, Theorem 2.30], and we are particularly interested in the following special case:

Theorem 2.3. *Let $\beta > \frac{1}{2}$, the dimension $d \in \mathbb{N}$, a function $f \in \mathcal{H}^\beta(\mathbb{T}^d) \cap \mathcal{C}(\mathbb{T}^d)$, a hyperbolic cross I_N^d with $N \geq 2^{d+1}$, and a reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, I_N^d)$ be given. Then we have*

$$\left\| f - S_{I_N^d}^\Lambda f \right\|_{L_2(\mathbb{T}^d)} \leq C_{d,\beta} N^{-\beta} (\log N)^{(d-1)/2} \|f\|_{\mathcal{H}^\beta(\mathbb{T}^d)} \tag{2.9}$$

with some constant $C_{d,\beta} := C(d, \beta) > 0$.

Later on, when we use these approximation error bounds for classes of transformed functions we will heavily utilize the norm equivalence $\|\cdot\|_{\mathcal{H}^\beta(\mathbb{T}^d)} \sim \|\cdot\|_{H^m_{\text{mix}}(\mathbb{T}^d)}$ for all $m \in \mathbb{N}$ as highlighted in (2.3) and the fact that the smoothness of functions in the Sobolev space $\mathcal{H}^\beta(\mathbb{T}^d)$ can be characterized by their derivatives, too.

3 Torus-to-cube transformation mappings

Change of variables were discussed for example in [1, 35] and were used for high dimensional integration in e.g. [29, 25]. In this chapter we define torus-to-cube transformations $\psi : \left[-\frac{1}{2}, \frac{1}{2}\right]^d \to \left[-\frac{1}{2}, \frac{1}{2}\right]^d$ and discuss some examples. Furthermore, we discuss special parameterized families of such torus-to-interval transformations, some of which are induced by transformations $\tilde{\psi} : (-\frac{1}{2}, \frac{1}{2})^d \to \mathbb{R}^d$ and their inverse $\tilde{\psi}^{-1} : \mathbb{R}^d \to (-\frac{1}{2}, \frac{1}{2})^d$ that were discussed in [1, 35, 30]. We provide examples that will reappear later in this paper. Afterwards we describe the weighted Hilbert spaces $L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d, \omega\right)$ with the integrable weight function $\omega : \left[-\frac{1}{2}, \frac{1}{2}\right]^d \to [0, \infty)$ and investigate their structure. Then we prove sufficient conditions on ψ and ω such that a test function $h \in L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d, \omega\right) \cap \mathcal{C}^m_{\text{mix}}\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right)$ with $m \in \mathbb{N}_0$

is transformed as in (1.1) by the transformation ψ into a function that is lying in a Sobolev space $\mathcal{H}^m(\mathbb{T}^d)$. Eventually we show that with an incorporated transformation ψ, we still have upper bounds for certain approximation errors on $\left[-\frac{1}{2}, \frac{1}{2}\right]^d$, which are based on the already established error bounds with respect to the $L_\infty(\mathbb{T}^d)$- and $L_2(\mathbb{T}^d)$-norms recalled in Theorems 2.2 and 2.3 respectively.

3.1 Torus-to-cube transformations

We call a mapping

$$\psi : \left[-\frac{1}{2}, \frac{1}{2}\right] \to \left[-\frac{1}{2}, \frac{1}{2}\right] \quad \text{with} \quad \lim_{x\to\pm\frac{1}{2}} \psi(x) = \pm\frac{1}{2} \tag{3.1}$$

a *torus-to-cube transformation* if it is continuously differentiable, increasing and has the first derivative $\psi'(x) := \frac{\mathrm{d}}{\mathrm{d}x}[\psi](x) \in \mathcal{C}(\mathbb{T})$. The respective inverse transformation is also continuously differentiable, increasing and is denoted by $\psi^{-1} : [-\frac{1}{2}, \frac{1}{2}] \to [-\frac{1}{2}, \frac{1}{2}]$ in the sense of $y = \psi(x) \Leftrightarrow x = \psi^{-1}(y)$ with $\psi^{-1}(y) \to \pm\frac{1}{2}$ as $y \to \pm\frac{1}{2}$. We call the derivative of the inverse transformation the *density function ϱ of ψ*, which is a non-negative L_1-function on the interval $[-\frac{1}{2}, \frac{1}{2}]$ and given by

$$\varrho(y) := (\psi^{-1})'(y) = \frac{1}{\psi'(\psi^{-1}(y))}. \tag{3.2}$$

For multivariate transformations we put

$$\psi(\mathbf{x}) := (\psi_1(x_1), \ldots, \psi_d(x_d))^\top \tag{3.3}$$

with $\mathbf{x} = (x_1, \ldots, x_d)^\top \in [-\frac{1}{2}, \frac{1}{2}]^d$, where we may use different univariate torus-to-cube transformations ψ_j in each coordinate. Similarly, for all $\mathbf{y} = (y_1, \ldots, y_d)^\top \in [-\frac{1}{2}, \frac{1}{2}]^d$ we put $\psi^{-1}(\mathbf{y}) := (\psi_1^{-1}(y_1), \ldots, \psi_d^{-1}(y_d))^\top$ and

$$\varrho(\mathbf{y}) := \prod_{j=1}^{d} \varrho_j(y_j). \tag{3.4}$$

Next we describe a particular family of parameterized torus-to-cube transformations as defined in (3.1) that are based on transformations $\tilde{\psi}$ to $\mathbb{R}$, whose definition is recalled from [30]. We call a continuously differentiable, increasing and odd mapping $\tilde{\psi} : (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}$ with $\tilde{\psi}(x) \to \pm\infty$ for $x \to \pm\frac{1}{2}$ a *transformation to* $\mathbb{R}$. Based on those, we obtain parameterized torus-to-cube transformations $\psi(\cdot, \eta) : [-\frac{1}{2}, \frac{1}{2}] \to [-\frac{1}{2}, \frac{1}{2}]$ with $\eta \in \mathbb{R}_+ := (0, \infty)$ given by

$$\psi(x, \eta) := \begin{cases} \tilde{\psi}^{-1}(\eta\, \tilde{\psi}(x)) & \text{for} \quad x \in \left(-\frac{1}{2}, \frac{1}{2}\right), \\ \pm\frac{1}{2} & \text{for} \quad x = \pm\frac{1}{2}. \end{cases} \tag{3.5}$$

These transformations form a subset of all torus-to-cube transformations and are in a natural way continuously differentiable and increasing. The respective first derivative and inverse torus-to-cube transformation are given by

$$\psi'(x, \eta) := \frac{\partial}{\partial x}[\psi](x, \eta) \quad \text{and} \quad \psi^{-1}(y, \eta) := \tilde{\psi}^{-1}\left(\frac{1}{\eta}\, \tilde{\psi}(y)\right).$$

The corresponding density functions $\varrho(\circ, \eta)$ and $\varrho(\circ, \boldsymbol{\eta})$ as well as the multivariate torus-to-cube transformation $\psi(\circ, \boldsymbol{\eta})$ and its inverse $\psi^{-1}(\circ, \boldsymbol{\eta})$ with $\boldsymbol{\eta} \in \mathbb{R}_+^d$ are simply parameterized versions of (3.1), (3.2) and (3.3) and share the same properties.

3.2 Exemplary transformations

In e.g. [1, Section 17.6], [35, Section 7.5] and [30] we find various suggestions for transformations to $\mathbb{R}$. We are particularly interested in the transformation

$$\tilde{\psi}(x) = \frac{1}{2}\log\left(\frac{1+2x}{1-2x}\right) = \tanh^{-1}(2x) \tag{3.6}$$

based on the log-function, and the transformation

$$\tilde{\psi}(x) = \operatorname{erf}^{-1}(2x) \tag{3.7}$$

based on the inverse of the error function

$$\operatorname{erf}(y) = \frac{1}{\sqrt{\pi}}\int_{-y}^{y} \mathrm{e}^{-t^2}\,\mathrm{d}t, \quad y \in \mathbb{R}. \tag{3.8}$$

Both (3.6) and (3.7) induce a parameterized torus-to-cube transformation $\psi\left(y, \frac{1}{\eta}\right)$ with $\eta > 0$ as defined in (3.5). We observe that $\psi^{-1}(y,\eta) = \psi\left(y, \frac{1}{\eta}\right)$ and $\varrho(y,\eta) = \psi'\left(y, \frac{1}{\eta}\right)$. For $x, y \in [-\frac{1}{2}, \frac{1}{2}]$ we have the following torus-to-cube transformations:

- *logarithmic transformation*:

$$\psi(x,\eta) = \frac{1}{2}\frac{(1+2x)^\eta - (1-2x)^\eta}{(1+2x)^\eta + (1-2x)^\eta}, \quad \psi'(x,\eta) = 4\eta\,\frac{(1-4x^2)^{\eta-1}}{\left((1+2x)^\eta + (1-2x)^\eta\right)^2}, \tag{3.9}$$

  and we observe that $\lim_{x\to\pm\frac{1}{2}} \psi'(x,\eta) = 0$ for $\eta > 1$.

- *error function transformation*:

$$\psi(x,\eta) = \frac{1}{2}\operatorname{erf}(\eta \operatorname{erf}^{-1}(2x)), \quad \psi'(x,\eta) = \eta\,\mathrm{e}^{(1-\eta^2)(\operatorname{erf}^{-1}(2x))^2} \tag{3.10}$$

  with the error function $\operatorname{erf}(\circ)$ as given in (3.8), and erf^{-1} denoting the inverse error function. Again we observe that $\lim_{x\to\pm\frac{1}{2}} \psi'(x,\eta) = 0$ for $\eta > 1$.

Additionally, we list an example for a torus-to-cube transformation $\psi : [-\frac{1}{2}, \frac{1}{2}] \to [-\frac{1}{2}, \frac{1}{2}]$ as defined in (3.1) that isn't induced by a transformation to $\mathbb{R}$:

- *sine transformation*:

$$\psi(x) = \frac{1}{2}\sin(\pi x) = \frac{1}{2}\cos\left(\pi\left(x - \frac{1}{2}\right)\right), \quad \psi'(x) = \frac{\pi}{2}\cos(\pi x). \tag{3.11}$$

Later on we compare the highly limited smoothening effect of this particular transformation on any given test function $h \in L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d, \omega\right) \cap \mathcal{C}^m_{\text{mix}}\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right)$ with the logarithmic transformation (3.9) for which we can achieve much more smoothness if the parameter $\eta \in \mathbb{R}_+$ is large enough. In Figure 3.1 we compare the transformation mapping, the inverse and their derivatives of the logarithmic transformation (3.9) for $\eta \in \{2, 4\}$ with the sine transformation (3.11).

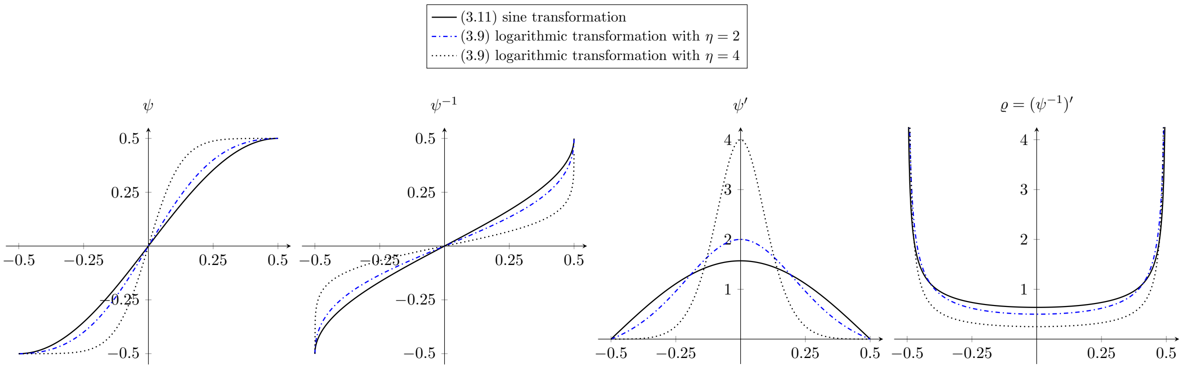

Figure 3.1: Comparison of the logarithmic transformation (3.9) with $\eta \in \{2, 4\}$ and the sine transformation (3.11).

3.3 Weighted Hilbert spaces on the cube

We describe the structure of the univariate weighted function spaces $L_2\left([-\frac{1}{2}, \frac{1}{2}], \omega\right)$ as defined in (2.1). In this section the weight function $\omega : [-\frac{1}{2}, \frac{1}{2}] \to [0, \infty)$ remains unspecified. Later on we may consider families of parameterized integrable weight functions $\omega(\circ, \mu)$ with $\mu \in \mathbb{R}_+$ to control the smoothness of functions in $L_2\left([-\frac{1}{2}, \frac{1}{2}], \omega(\circ, \mu)\right) \cap \mathcal{C}^m\left([-\frac{1}{2}, \frac{1}{2}]\right)$ and of the corresponding transformed functions as in (1.1) on the torus $\mathbb{T}$. Families of multivariate parameterized weight functions are defined as

$$\omega(\mathbf{y}, \boldsymbol{\mu}) := \prod_{j=1}^{d} \omega_j(y_j, \mu_j), \quad \mathbf{y} \in \left[-\frac{1}{2}, \frac{1}{2}\right]^d, \boldsymbol{\mu} \in \mathbb{R}_+^d, \tag{3.12}$$

with univariate weight functions $\omega_j(\circ, \mu_j) : [-\frac{1}{2}, \frac{1}{2}] \to [0, \infty)$. For now we simplify the notation of the transformation, the weight function, and all related functions by omitting any parameter and just writing $\psi(\circ), \omega(\circ)$, etc.

We remain in the univariate setting. The system $\{\varphi_k\}_{k \in \mathbb{Z}}$ of weighted exponential functions

$$\varphi_k(y) := \sqrt{\frac{\varrho(y)}{\omega(y)}}\, \mathrm{e}^{2\pi \mathrm{i} k \psi^{-1}(y)}, \quad y \in \left[-\frac{1}{2}, \frac{1}{2}\right] \tag{3.13}$$

forms an orthogonal system with respect to the scalar product

$$(h_1, h_2)_{L_2([-\frac{1}{2}, \frac{1}{2}], \omega)} := \int_{-\frac{1}{2}}^{\frac{1}{2}} h_1(y)\, \overline{h_2(y)}\, \omega(y)\, \mathrm{d}y \tag{3.14}$$

and for $k_1, k_2 \in \mathbb{Z}$ we have

$$(\varphi_{k_1}, \varphi_{k_2})_{L_2([-\frac{1}{2}, \frac{1}{2}], \omega)} = \delta_{k_1, k_2}.$$

The weighted scalar product (3.14) induces the norm

$$\|h\|_{L_2([-\frac{1}{2}, \frac{1}{2}], \omega)} := \sqrt{(h, h)_{L_2([-\frac{1}{2}, \frac{1}{2}], \omega)}}$$

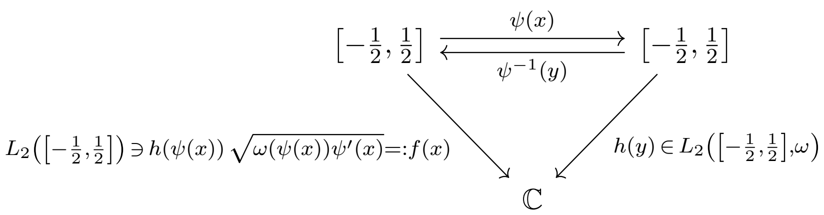

Figure 3.2: Scheme of the relation between f and h caused by a transformation ψ.

and in a natural way we have Fourier coefficients of the form

$$\hat{h}_k := (h, \varphi_k)_{L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right],\omega\right)} = \int_{-\frac{1}{2}}^{\frac{1}{2}} h(y)\,\sqrt{\varrho(y)\,\omega(y)}\,\mathrm{e}^{-2\pi\mathrm{i}k\psi^{-1}(y)}\,\mathrm{d}y, \tag{3.15}$$

as well as the respective Fourier partial sum for $I \subset \mathbb{Z}$ given by

$$S_I h(y) := \sum_{k\in I} \hat{h}_k\,\varphi_k(y). \tag{3.16}$$

3.4 Smoothness properties of transformed functions in $\mathcal{H}^m(\mathbb{T}^d)$

In this section we discuss the smoothness properties of functions h defined on $\left[-\frac{1}{2}, \frac{1}{2}\right]^d$ and of their corresponding transformed versions f as in (3.21) on $\left[-\frac{1}{2}, \frac{1}{2}\right]^d$ after the application of a torus-to-cube transformation ψ as in (3.5), as well as the possiblity to continuously extend transformed functions f to the torus $\mathbb{T}^d$. We propose specific sufficient conditions for ψ and ω such that the eventual transformed functions f are in $\mathcal{H}^m\left(\mathbb{T}^d\right)$ with $m \in \mathbb{N}_0$. These conditions are stated for both univariate and multivariate functions. Afterwards we utilize the embedding $\mathcal{H}^{\beta+\lambda}(\mathbb{T}^d) \hookrightarrow \mathcal{A}^{\beta}(\mathbb{T}^d)$ in (2.4) for all $\lambda > \frac{1}{2}$ in order to discuss high dimensional approximation problems in which we apply rank-1 lattice based fast Fourier approximation methods. Throughout this section we still omit the parameters $\boldsymbol{\eta}, \boldsymbol{\mu} \in \mathbb{R}_+^d$ in the notation of the torus-to-cube transformations ψ and the weight functions ω.

For now we consider univariate transformed functions $f \in L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$ of the form

$$f(x) := h(\psi(x))\,\sqrt{\omega(\psi(x))\,\psi'(x)}, \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right], \tag{3.17}$$

that are the result of applying a torus-to-cube transformation $y = \psi(x)$ as defined in (3.1) to the $L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right], \omega\right)$-norm of the given function h so that we have the identity

$$\|h\|^2_{L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right],\omega\right)} = \int_{-\frac{1}{2}}^{\frac{1}{2}} |h(y)|^2\,\omega(y)\,\mathrm{d}y = \int_{-\frac{1}{2}}^{\frac{1}{2}} |h(\psi(x))|^2\,\omega(\psi(x))\,\psi'(x)\,\mathrm{d}x = \|f\|^2_{L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]\right)},$$

schematically shown in Figure 3.2.

It is generally rather difficult to check if such transformed functions f are smooth and lie in $H^m\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$ for some fixed $m \in \mathbb{N}_0$ by calculating the individual $L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$-norms within the Sobolev norm $\|f\|_{H^m\left(\left[-\frac{1}{2},\frac{1}{2}\right]\right)}$. Therefore we propose a certain set of sufficient conditions such that $f \in H^m\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$ with $m \in \mathbb{N}_0$, that eliminates the necessity to evaluate

L_2-integrals of various derivatives of f by utilizing the product structure of the functions f in (3.17). Furthermore, once we consider particular parameterized families of torus-to-cube transformations $\psi(\circ, \eta)$ and families of weight functions $\omega(\circ, \mu)$, these conditions enable us for each smoothness order $m \in \mathbb{N}_0$ to explicitly calculate how large the parameters $\eta, \mu \in \mathbb{R}_+$ have to be in order to preserve the fixed degree of smoothness m when transforming $h \in L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right], \omega\right) \cap \mathcal{C}^m\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$ into $f \in H^m\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$ via $\psi(\circ, \eta)$. By additionally assuming a certain vanishing behavior of the derivatives of the transformed weight function $\sqrt{(\omega(\psi(\circ))\, \psi'(\circ)}$ the transformed functions f are continuously extendable to the torus $\mathbb{T}$ and we finally have smooth transformed functions $f \in \mathcal{H}^m(\mathbb{T})$ due to the norm equivalence (2.3).

Remark 3.1. *In the univariate setting we need a continuous function f with $f(-\frac{1}{2}) = f(\frac{1}{2})$, which reads as*

$$0 = h(-\tfrac{1}{2})\sqrt{\omega(-\tfrac{1}{2})\psi'(-\tfrac{1}{2})} - h(\tfrac{1}{2})\sqrt{\omega(\tfrac{1}{2})\psi'(\tfrac{1}{2})},$$

after recalling the we have $\psi\left(\pm\frac{1}{2}\right) = \pm\frac{1}{2}$. One approach to achieve this equality is to choose transformations ψ whose first derivative ψ' converges to 0 at $x = \pm\frac{1}{2}$ fast enough that it isn't counteracted by the function h or the weight function ω. Hence, we assume that $\sqrt{\omega(\psi(\circ))\psi'(\circ)} \in \mathcal{C}_0\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$. We focus on this approach, even though there are obviously more ways to achieve the above equality. In higher dimensions we analogously assume $D^{\mathbf{m}}\left[\prod_{j=1}^d \sqrt{(\omega_j \circ \psi_j)\, \psi_j'}\right] \in \mathcal{C}_0\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right)$ for $\mathbf{m} \in \mathbb{N}_0^d$ with $\|\mathbf{m}\|_{\ell_\infty} \leq m$.

Later on we will repeatedly choose a constant weight function $\omega \equiv 1$ and make use of the logarithmic transformation (3.9) *or the error transformation* (3.10) *for the purpose of achieving this behavior of the transformed functions f at the boundary points. While their first derivatives $\psi'(\circ, \eta)$ are always 0 at the boundary points, the parameter η has to be sufficiently large to achieve the same property for higher derivatives.* □

We proceed by proposing a set of univariate sufficient conditions such that we obtain smooth transformed functions $f \in \mathcal{H}^m(\mathbb{T})$. We denote the k-th derivative of a function $f(x)$ with respect to x by one of the equivalent expressions $f^{(k)}(x) = \frac{\mathrm{d}^k}{\mathrm{d}x^k}[f](x)$, and for $k = 1$ we continue to use the notation $f'(x)$.

Theorem 3.2. *Let $m \in \mathbb{N}_0$, a torus-to-cube transformation $\psi : \left[-\frac{1}{2}, \frac{1}{2}\right] \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ as defined in* (3.1) *with the density function ϱ as in* (3.2)*, a function $h \in L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right], \omega\right) \cap \mathcal{C}^m\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$ with an integrable weight function $\omega : \left[-\frac{1}{2}, \frac{1}{2}\right] \to [0, \infty)$ and the corresponding transformed functions f of the form* (3.17) *be given.*

We have $f \in \mathcal{H}^m(\mathbb{T})$, if for all $n = 0, 1, \dots, m$ we have

$$\psi \in \mathcal{C}^m\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right) \quad \text{and} \quad \left(\sqrt{(\omega \circ \psi)\, \psi'}\right)^{(n)}(\circ) \in \mathcal{C}_0\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right). \tag{3.18}$$

Proof. For $h \in L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right], \omega\right) \cap \mathcal{C}^m\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$ with $m \in \mathbb{N}_0$ and a torus-to-cube transformation ψ as defined in (3.1) we consider the function f as given in (3.17). At first we check if $f \in H^m\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$ and have to show that $\left\|f^{(n)}(\circ)\right\|_{L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)} < \infty$ for all $n = 0, 1, \dots, m$.

We apply the *generalized Leibniz rule* for the n-th derivative of a product of functions

$$(f \cdot g)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)\, g^{(n-k)}(x)$$

to the Sobolev norm of f, which leads to

$$\begin{aligned}\|f\|_{H^m([-\frac{1}{2},\frac{1}{2}])} &= \left(\sum_{n=0}^{m} \|f^{(n)}(\circ)\|^2_{L_2([-\frac{1}{2},\frac{1}{2}])}\right)^{1/2} \\ &\le \left(\sum_{n=0}^{m}\left(\sum_{k=0}^{n}\binom{n}{k}\left\|(h\circ\psi)^{(k)}(\circ)\left(\sqrt{(\omega\circ\psi)\,\psi'}\right)^{(n-k)}(\circ)\right\|_{L_2([-\frac{1}{2},\frac{1}{2}])}\right)^2\right)^{1/2}\end{aligned} \tag{3.19}$$

We leave $h\circ\psi$ in the term corresponding to $k=0$ untouched for now. For $k=1,\ldots,m$ we use the *Faá di Bruno* formula to write the k-th derivative of the composition of functions h and ψ as

$$(h\circ\psi)^{(k)}(x) = \sum_{\ell=1}^{k} h^{(\ell)}(\psi(x))\, B_{k,\ell}(\psi'(x),\psi^{(2)}(x),\ldots,\psi^{(k-\ell+1)}(x)) \tag{3.20}$$

with $(h\circ\psi)^{(0)}(x)=h(\psi(x))$ and the well-known Bell polynomials $B_{k,\ell}$ for $k,\ell\in\mathbb{N}_0$ are given by

$$B_{k,\ell}(\mathbf{z}) := \sum_{\substack{j_1+j_2+\ldots+j_{k-\ell+1}=\ell,\\ j_1+2j_2+\ldots+(k-\ell+1)j_{k-\ell+1}=k}} \frac{\ell!}{j_1!\cdot\ldots\cdot j_{k-\ell+1}!}\prod_{m=1}^{k-\ell+1}\left(\frac{z_m}{m!}\right)^{j_m}$$

with $\mathbf{z}=(z_1,\ldots,z_{k-\ell+1})^\top$. By assumption all derivatives of ψ are bounded on the interval $[-\frac{1}{2},\frac{1}{2}]$, hence, each Bell polynomial $B_{k,\ell}$ in (3.20) is bounded, too. To simplify the notation we write $B_{k,\ell}(\psi(x)) := B_{k,\ell}(\psi'(x),\ldots,\psi^{(k-\ell+1)}(x))$. We insert (3.20) into (3.19) and estimate that

$$\begin{aligned}&\|f\|_{H^m([-\frac{1}{2},\frac{1}{2}])}\\ &\le \left(\sum_{n=0}^{m}\left(\sum_{k=0}^{n}\binom{n}{k}\left\|\sum_{\ell=1}^{k} h^{(\ell)}(\psi(\circ))\,B_{k,\ell}(\psi(\circ))\,\left(\sqrt{(\omega\circ\psi)\,\psi'}\right)^{(n-k)}(\circ)\right\|_{L_2([-\frac{1}{2},\frac{1}{2}])}\right)^2\right)^{1/2}.\end{aligned}$$

The appearing L_2-norms are estimated by their respective L_∞-norms, so that

$$\begin{aligned}&\left\|\sum_{\ell=1}^{k} h^{(\ell)}(\psi(\circ))\,B_{k,\ell}(\psi(\circ))\,\left(\sqrt{(\omega\circ\psi)\,\psi'}\right)^{(n-k)}(\circ)\right\|_{L_2([-\frac{1}{2},\frac{1}{2}])}\\ &\lesssim \sum_{\ell=1}^{k}\left\|\left(\sqrt{(\omega\circ\psi)\,\psi'}\right)^{(n-k)}(\circ)\right\|_{L_\infty([-\frac{1}{2},\frac{1}{2}])}.\end{aligned}$$

because of the boundedness of all appearing Bell polynomials $B_{k,\ell}$ and the assumption that h is m-times continuously differentiable. Thus, the norm $\|f\|_{H^m([-\frac{1}{2},\frac{1}{2}])}$ is finite if all L_∞-norms of the first m derivatives of $\sqrt{(\omega(\psi(\circ))\,\psi'(\circ)}$ exist.

Finally, the assumption that the first m derivatives of $\sqrt{(\omega(\psi(\circ))\,\psi'(\circ)}$ also vanish at the boundary points implies that the first m derivatives of the transformed function f vanish at the boundary points, too. Hence, f is in $\mathcal{H}^m(\mathbb{T})$ due to the norm equivalence (2.3). ∎

Next, we prove the multivariate version of Theorem 3.2. Similar to (3.17) we consider multivariate transformed functions $f \in L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)$ of the form

$$f(\mathbf{x}) = h(\psi_1(x_1),\ldots,\psi_d(x_d)) \prod_{k=1}^{d} \sqrt{\omega_k(\psi_k(x_k))\psi_k'(x_k)}, \quad \mathbf{x} \in \left[-\frac{1}{2},\frac{1}{2}\right]^d, \tag{3.21}$$

that are the result of applying the multivariate transformation

$$\mathbf{y} = (y_1,\ldots,y_d)^\top = (\psi_1(x_1),\ldots,\psi_d(x_d))^\top = \psi(\mathbf{x})$$

as defined in (3.3) to a function $h \in L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d, \omega\right)$ with a product weight ω as in (3.12) and for which we have the identity

$$\begin{aligned}\|h\|^2_{L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d,\omega\right)} &= \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} |h(\mathbf{y})|^2\, \omega(\mathbf{y})\, \mathrm{d}\mathbf{y} \\ &= \int_{\mathbb{T}^d} |(h\circ\psi)(\mathbf{x})|^2\, (\omega\circ\psi)(\mathbf{x}) \prod_{j=1}^{d} \psi_j'(x_j)\, \mathrm{d}\mathbf{x} = \|f\|^2_{L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)}.\end{aligned}$$

Again, we derive a set of sufficient L_∞-conditions on the multivariate transformation ψ and the product weight ω, that determine when a function $h \in L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d, \omega\right) \cap \mathcal{C}^m_{\mathrm{mix}}\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)$ can be transformed by ψ into an $f \in \mathcal{H}^m\left(\mathbb{T}^d\right)$ of form (3.21).

Theorem 3.3. *Let the dimension $d \in \mathbb{N}$, $m \in \mathbb{N}_0$, a d-variate torus-to-cube transformation $\psi : \left[-\frac{1}{2},\frac{1}{2}\right]^d \to \left[-\frac{1}{2},\frac{1}{2}\right]^d, \mathbf{x} \mapsto (\psi_1(x_1),\ldots,\psi_d(x_d))^\top$ as defined in (3.3) with the d-variate density function $\varrho(\mathbf{y}) = \prod_{j=1}^d \varrho_j(y_j)$ of ψ as in (3.4), a product weight function $\omega : \left[-\frac{1}{2},\frac{1}{2}\right]^d \to [0,\infty)$ as in (3.12), a d-variate function $h \in L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d, \omega\right) \cap \mathcal{C}^m_{\mathrm{mix}}\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)$ and the corresponding transformed functions f of the form (3.21) be given.*

We have $f \in \mathcal{H}^m\left(\mathbb{T}^d\right)$, if for all multi-indices $\mathbf{m} \in \mathbb{N}_0^d$, $\|\mathbf{m}\|_{\ell_\infty} \le m$, we have

$$\psi \in \mathcal{C}^m_{\mathrm{mix}}\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right) \quad \text{and} \quad D^{\mathbf{m}}\left[\prod_{k=1}^{d} \sqrt{(\omega_k\circ\psi_k)\psi_k'}\right] \in \mathcal{C}_0\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right). \tag{3.22}$$

Proof. For $h \in L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d, \omega\right) \cap \mathcal{C}^m_{\mathrm{mix}}\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)$ with $m \in \mathbb{N}_0$ and a multivariate torus-to-cube transformation ψ as defined in (3.3) we consider the function f as given in (3.21). At first we check if $f \in H^m_{\mathrm{mix}}\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)$ and have to show that for all multi-indices $\mathbf{m} \in \mathbb{N}_0^d$ with $\|\mathbf{m}\|_{\ell_\infty} \le m$ we have $\|D^{\mathbf{m}}[f]\|_{L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)} < \infty$.

Let $\mathbf{m} = (m_1,\ldots,m_d)^\top \in \mathbb{N}_0^d$ be any multi-index with $\|\mathbf{m}\|_{\ell_\infty} \le m$. For a multivariate transformed function f of the form (3.21) we have

$$\|D^{\mathbf{m}}[f](\mathbf{x})\|_{L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)} = \left(\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} \left| D^{\mathbf{m}}\left[(h\circ\psi) \prod_{k=1}^{d} \sqrt{(\omega_k\circ\psi_k)\psi_k'}\right](\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}. \tag{3.23}$$

Due to the product weight function in the transformed function f in (3.21) the componentwise application of the Leibniz formula as in (3.19) yields the expression

$$D^{\mathbf{m}}\left[(h\circ\psi)\prod_{k=1}^{d}\sqrt{(\omega_k\circ\psi_k)\psi_k'}\right](\mathbf{x}) \tag{3.24}$$
$$=\sum_{j_1=0}^{m_1}\binom{m_1}{j_1}\cdots\sum_{j_d=0}^{m_d}\binom{m_d}{j_d}D^{(j_1,\ldots,j_d)}[h\circ\psi](\mathbf{x})\,D^{(m_1-j_1,\ldots,m_d-j_d)}\left[\prod_{k=1}^{d}\sqrt{(\omega_k\circ\psi_k)\psi_k'}\right](\mathbf{x}).$$

Next, we apply the Faá di Bruno formula (3.20) to each univariate j_ℓ-th derivative occurring in the term $D^{(j_1,\ldots,j_d)}[h\circ\psi](\mathbf{x})$ in (3.24) so that for $\ell=1,\ldots,d$ we have

$$D^{(0,\ldots,0,j_\ell,0,\ldots,0)}[h\circ\psi](\mathbf{x})$$
$$=\begin{cases} h(\psi(\mathbf{x})) & \text{for } j_\ell=0,\\ \displaystyle\sum_{i_\ell=1}^{j_\ell} D^{(0,\ldots,0,i_\ell,0,\ldots,0)}[h](\psi(\mathbf{x}))B_{j_\ell,i_\ell}(\psi_\ell'(x_\ell),\ldots,\psi_\ell^{(j_\ell-i_\ell+1)}(x_\ell)) & \text{for } j_\ell\in\mathbb{N}.\end{cases} \tag{3.25}$$

We combine the norm $\|D^{\mathbf{m}}[f]\|_{L_2\left([-\frac{1}{2},\frac{1}{2}]^d\right)}$ in (3.23) with the expression resulting from applying the Leibniz formula to $D^{\mathbf{m}}[f]$ in (3.24) and the subsequent application of the Faá di Bruno formula in (3.25). Then we estimate the occurring summands by their L_2-norm, afterwards by their L_∞-norm and utilize the boundedness of the Bell polynomials B_{j_ℓ,i_ℓ} as well as the assumption that h is a $\mathcal{C}^m_{\text{mix}}$-function, so that we end up with

$$\begin{aligned}\|D^{\mathbf{m}}[f](\mathbf{x})\|_{L_2\left([-\frac{1}{2},\frac{1}{2}]^d\right)} &\leq \sum_{j_1=0,\ldots,j_d=0}^{m_1,\ldots,m_d}\ \sum_{i_1=1,\ldots,i_d=1}^{j_1,\ldots,j_d}\Bigg(\int_{[-\frac{1}{2},\frac{1}{2}]^d}|D^{(i_1,\ldots,i_d)}[h](\psi(\mathbf{x}))|^2\times\\ &\quad\times\prod_{\ell=1}^{d}\binom{m_\ell}{j_\ell}|B_{j_\ell,i_\ell}(\psi_\ell'(x_\ell),\ldots,\psi_\ell^{(j_\ell-i_\ell+1)}(x_\ell))|^2\times\\ &\quad\times\left|D^{(m_1-j_1,\ldots,m_d-j_d)}\left[\prod_{k=1}^{d}\sqrt{(\omega_k\circ\psi_k)\psi_k'}\right](\mathbf{x})\right|^2\mathrm{d}\mathbf{x}\Bigg)^{\frac{1}{2}}\\ &\lesssim\sum_{j_1=0,\ldots,j_d=0}^{m_1,\ldots,m_d}\left\|D^{(m_1-j_1,\ldots,m_d-j_d)}\left[\prod_{k=1}^{d}\sqrt{(\omega_k\circ\psi_k)\psi_k'}\right](\circ)\right\|_{L_\infty\left([-\frac{1}{2},\frac{1}{2}]^d\right)}.\end{aligned}$$

Finally, for all $\mathbf{m}\in\mathbb{N}_0^d$ with $\|\mathbf{m}\|_{\ell_\infty}\leq m$ the derivatives $D^{\mathbf{m}}\left[\prod_{k=1}^{d}\sqrt{(\omega_k\circ\psi_k)\psi_k'}\right]$ vanish at the boundary points by assumption, which implies that the derivatives $D^{\mathbf{m}}[f]$ of the transformed function f vanish at the boundary points, too. Hence, f is in $\mathcal{H}^m\left(\mathbb{T}^d\right)$ due to the equivalence (2.3). ■

3.5 Approximation of transformed functions

We establish two specific approximation error bounds for functions defined on $\left[-\frac{1}{2},\frac{1}{2}\right]^d$ based on the approximation error bounds on the torus $\mathbb{T}^d$ that we recalled in Theorems 2.2 and 2.3. The corresponding proofs rely heavily on the previously introduced sufficient conditions in

Theorem 3.3 with which we determine, when functions $h \in L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d,\omega\right) \cap \mathcal{C}^m_{\text{mix}}\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)$ are transformed into Sobolev functions of dominated mixed smoothness on $\mathbb{T}^d$ of the form (3.21) by multivariate transformations $\psi : \left[-\frac{1}{2},\frac{1}{2}\right]^d \to \left[-\frac{1}{2},\frac{1}{2}\right]^d$ as given in (3.3).

Beforehand, we fix some notation of certain multivariate objects. Based on the definition of a rank-1 lattice $\Lambda(\mathbf{z},M)$ in (2.6) we define a *transformed rank-*1 *lattice* as

$$\Lambda_\psi(\mathbf{z},M) := \{\mathbf{y}_j := \psi(\mathbf{x}_j) : \mathbf{x}_j \in \Lambda(\mathbf{z},M), j = 0,\dots,M-1\}. \tag{3.26}$$

Accordingly, we denote the *transformed reconstructing rank-*1 *lattice* by $\Lambda_\psi(\mathbf{z},M,I)$.

Besides the weight function ω, also the density ϱ of the transformation ψ is of product form as defined in (3.4), i.e., it is the product of univariate densities $\varrho_j(y_j), j = 1,\dots,d$. Hence, based on the functions φ_k given in (3.13) this product form extends to

$$\varphi_{\mathbf{k}}(\mathbf{y}) := \prod_{j=1}^d \varphi_{k_j}(y_j).$$

Similar to (3.14), the multivariate weighted $L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d,\omega\right)$-scalar product reads as

$$(h_1,h_2)_{L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d,\omega\right)} := \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} h_1(\mathbf{y})\,\overline{h_2(\mathbf{y})}\,\prod_{j=1}^d \omega_j(y_j)\,\mathrm{d}\mathbf{y}$$

and similar to (3.15) the multivariate Fourier coefficients $\hat{h}_{\mathbf{k}}$ are naturally given with respect to this scalar product as

$$\hat{h}_{\mathbf{k}} = (h,\varphi_{\mathbf{k}})_{L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d,\omega\right)}. \tag{3.27}$$

As before in (3.16) we define the multivariate Fourier partial sum for any $I \subset \mathbb{Z}^d$ as

$$S_I h(\mathbf{y}) := \sum_{\mathbf{k}\in I} \hat{h}_{\mathbf{k}}\,\varphi_{\mathbf{k}}(\mathbf{y}).$$

Suppose $f \in L_2\left(\mathbb{T}^d\right)$, then for each $I \subset \mathbb{Z}^d$ the system $\{\varphi_{\mathbf{k}}\}_{\mathbf{k}\in I}$ spans the space of transformed trigonometric polynomials

$$\Pi_{I,\psi} := \operatorname{span}\left\{\sqrt{\frac{\varrho(\circ)}{\omega(\circ)}}\,\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\psi^{-1}(\circ)} : \mathbf{k} \in I\right\}. \tag{3.28}$$

Similar to (2.7), for transformed trigonometric polynomials $h \in \Pi_{I,\psi}$, transformed lattice nodes $\mathbf{y}_j \in \Lambda_\psi(\mathbf{z},M,I)$ and $\mathbf{k} \in I$ we have the exact integration property of the form

$$\begin{aligned}\hat{h}_{\mathbf{k}} &= \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} h(\mathbf{y})\,\sqrt{\varrho(\mathbf{y})\,\omega(\mathbf{y})}\,\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\psi^{-1}(\mathbf{y})}\,\mathrm{d}\mathbf{y} = \int_{\mathbb{T}^d} f(\mathbf{x})\,\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x}\\ &= \frac{1}{M}\sum_{j=0}^{M-1} f(\mathbf{x}_j)\,\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}_j} = \frac{1}{M}\sum_{j=0}^{M-1} h(\mathbf{y}_j)\,\sqrt{\frac{\omega(\mathbf{y}_j)}{\varrho(\mathbf{y}_j)}}\,\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\psi^{-1}(\mathbf{y}_j)} = \hat{h}^{\Lambda}_{\mathbf{k}}.\end{aligned} \tag{3.29}$$

Generally, the multivariate approximated Fourier coefficients of the form

$$\hat{h}_{\mathbf{k}}^{\Lambda} = \frac{1}{M} \sum_{j=0}^{M-1} h(\mathbf{y}_j) \sqrt{\frac{\omega(\mathbf{y}_j)}{\varrho(\mathbf{y}_j)}}\, \mathrm{e}^{-2\pi \mathrm{i} \mathbf{k}\cdot\psi^{-1}(\mathbf{y}_j)} = \frac{1}{M} \sum_{j=0}^{M-1} \frac{\omega(\mathbf{y}_j)}{\varrho(\mathbf{y}_j)}\, h(\mathbf{y}_j)\, \overline{\varphi_{\mathbf{k}}(\mathbf{y}_j)}$$

only approximate the multivariate Fourier coefficients $\hat{h}_{\mathbf{k}}$. Finally, the multivariate version of the approximated Fourier partial sum is given by

$$S_I^{\Lambda} h(\mathbf{y}) := \sum_{\mathbf{k}\in I} \hat{h}_{\mathbf{k}}^{\Lambda}\, \varphi_{\mathbf{k}}(\mathbf{y}). \tag{3.30}$$

Similar to the Hilbert space $\mathcal{H}^{\beta}(\mathbb{T}^d)$ given in (1.4) we define such a space of L_2-functions on the cube with square summable Fourier coefficients $\hat{h}_{\mathbf{k}}$ as in (3.27) by

$$\mathcal{H}^{\beta}\left(\left[-\tfrac{1}{2},\tfrac{1}{2}\right]^d\right) := \left\{ h \in L_2\left(\left[-\tfrac{1}{2},\tfrac{1}{2}\right]^d, \omega\right) : \|h\|_{\mathcal{H}^{\beta}\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)} < \infty \right\}$$

with the norm

$$\|h\|_{\mathcal{H}^{\beta}\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)} := \left(\sum_{\mathbf{k}\in\mathbb{Z}^d} \omega_{\mathrm{hc}}(\mathbf{k})^{2\beta} |\hat{h}_{\mathbf{k}}|^2 \right)^{\frac{1}{2}}.$$

The existence of the Fourier coefficients $\hat{h}_{\mathbf{k}}$ becomes apparent after applying the well-known Cauchy-Schwarz-Inequality so that

$$\begin{aligned} |\hat{h}_{\mathbf{k}}| &= \left| \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} h(\mathbf{y}) \sqrt{\omega(\mathbf{y})\varrho(\mathbf{y})}\, \mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\psi^{-1}(\mathbf{y})}\, \mathrm{d}\mathbf{y} \right| \\ &\le \left(\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} |h(\mathbf{y})|^2\, \omega(\mathbf{y})\, \mathrm{d}\mathbf{y} \right)^{\frac{1}{2}} \left(\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} |\varrho(\mathbf{y})|\, \mathrm{d}\mathbf{y} \right)^{\frac{1}{2}} < \infty, \end{aligned}$$

which is due to the fact that $h \in L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d, \omega\right)$ and each univariate factor appearing in the multivariate density $\varrho(\mathbf{y})$ as defined in (3.4) is an L_1-function.

With these objects we transfer the approximation error bounds in Theorems 2.2 and 2.3 for functions defined on the torus to $\left[-\frac{1}{2},\frac{1}{2}\right]^d$.

3.5.1 L_∞-approximation error

Based on the $L_\infty(\mathbb{T}^d)$-approximation error bound (2.8) and the conditions proposed in Theorem 3.3 we prove a similar upper bound for the approximation error $\left\| h - S_{I_N^d}^{\Lambda} h \right\|$ in terms of a weighted L_∞-norm on $\left[-\frac{1}{2},\frac{1}{2}\right]^d$.

Theorem 3.4. *Let $d \in \mathbb{N}$, $m \in \mathbb{N}_0$, a hyperbolic cross I_N^d with $N \geq 2^{d+1}$ and a reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, I_N^d)$ be given. Let ω be a weight function as in (3.12), let $h \in L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d, \omega\right) \cap \mathcal{C}_{\mathrm{mix}}^m\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right)$, let ψ be a multivariate transformation as defined in (3.3), let ϱ be the corresponding density function in product form (3.4), and let $\lambda > \frac{1}{2}$. For all multi-indices $\mathbf{m} = (m_1, \ldots, m_d)^\top \in \mathbb{N}_0^d$ with $\|\mathbf{m}\|_{\ell_\infty} \leq m$ we assume that*

$$\psi \in \mathcal{C}_{\mathrm{mix}}^m\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right) \quad \text{and} \quad D^{\mathbf{m}}\left[\prod_{\ell=1}^d \sqrt{(\omega_\ell \circ \psi_\ell)\psi_\ell'}\right] \in \mathcal{C}_0\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right).$$

Then there is an approximation error estimate of the form

$$\left\|h - S_{I_N^d}^\Lambda h\right\|_{L_\infty\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d, \sqrt{\frac{\omega}{\varrho}}\right)} \lesssim 2N^{-m+\lambda}\|h\|_{\mathcal{H}^m\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right)}.$$

Proof. Let $m \in \mathbb{N}_0, d \in \mathbb{N}$ and $h \in L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d, \omega\right) \cap \mathcal{C}_{\mathrm{mix}}^m\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right)$. By assumption the criteria of Theorem 3.3 are fulfilled and the transformed function f of the form (3.21) is continuously extendable to the torus $\mathbb{T}^d$. Thus, we have $f \in \mathcal{H}^m(\mathbb{T}^d)$ and a continuous representative, because of the inclusion $\mathcal{H}^m(\mathbb{T}^d) \hookrightarrow \mathcal{A}^{m-\lambda}(\mathbb{T}^d) \hookrightarrow \mathcal{C}(\mathbb{T}^d)$ with $\lambda > \frac{1}{2}$ as in (2.4). Hence, for $f \in \mathcal{A}^{m-\lambda}(\mathbb{T}^d) \cap \mathcal{C}(\mathbb{T}^d)$ we have the approximation error bound

$$\left\|f - S_{I_N^d}^\Lambda f\right\|_{L_\infty(\mathbb{T}^d)} \leq 2N^{-m+\lambda}\|f\|_{\mathcal{A}^{m-\lambda}(\mathbb{T}^d)} \tag{3.31}$$

as stated in Theorem 2.2.

With the inverse transformation $\mathbf{x} = \psi^{-1}(\mathbf{y})$ we have

$$\hat{h}_{\mathbf{k}} = (h, \varphi_{\mathbf{k}})_{L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d, \omega\right)} = (f, \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\circ})_{L_2(\mathbb{T}^d)} = \hat{f}_{\mathbf{k}}$$

and

$$\|h\|_{\mathcal{H}^m\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right)}^2 = \sum_{\mathbf{k}\in\mathbb{Z}^d} \omega_{\mathrm{hc}}(\mathbf{k})^{2m}|\hat{h}_{\mathbf{k}}|^2 = \sum_{\mathbf{k}\in\mathbb{Z}^d} \omega_{\mathrm{hc}}(\mathbf{k})^{2m}|\hat{f}_{\mathbf{k}}|^2 = \|f\|_{\mathcal{H}^m(\mathbb{T}^d)}^2, \tag{3.32}$$

as well as

$$\begin{aligned}
\left\|h - S_{I_N^d} h\right\|_{L_\infty\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d, \sqrt{\frac{\omega}{\varrho}}\right)} &= \operatorname{ess\,sup}_{\mathbf{y}\in\left[-\frac{1}{2}, \frac{1}{2}\right]^d} \left|\sqrt{\frac{\omega(\mathbf{y})}{\varrho(\mathbf{y})}}\left(h(\mathbf{y}) - \sum_{\mathbf{k}\in I_N^d} \hat{h}_{\mathbf{k}}\,\varphi_{\mathbf{k}}(\mathbf{y})\right)\right| \\
&= \operatorname{ess\,sup}_{\mathbf{y}\in\left[-\frac{1}{2}, \frac{1}{2}\right]^d} \left|h(\mathbf{y})\sqrt{\frac{\omega(\mathbf{y})}{\varrho(\mathbf{y})}} - \sum_{\mathbf{k}\in I_N^d} \hat{h}_{\mathbf{k}}\,\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\psi^{-1}(\mathbf{y})}\right| \\
&= \operatorname{ess\,sup}_{\mathbf{x}\in\mathbb{T}^d} \left|h(\psi(\mathbf{x}))\sqrt{\omega(\psi(\mathbf{x}))\prod_{j=1}^d \psi_j'(x_j)} - \sum_{\mathbf{k}\in I_N^d} \hat{h}_{\mathbf{k}}\,\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\right| \\
&= \left\|f - S_{I_N^d} f\right\|_{L_\infty(\mathbb{T}^d)}
\end{aligned}$$

and

$$\left\| h - S_{I_N^d}^{\Lambda} h \right\|_{L_\infty\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d, \sqrt{\frac{\omega}{\varrho}}\right)} = \left\| f - S_{I_N^d}^{\Lambda} f \right\|_{L_\infty(\mathbb{T}^d)}. \tag{3.33}$$

In total, by combining (3.33), (3.31), (2.5), and (3.32) we estimated for $f \in \mathcal{H}^m(\mathbb{T}^d) \cap \mathcal{C}(\mathbb{T}^d)$ that

$$\begin{aligned} \left\| h - S_{I_N^d}^{\Lambda} h \right\|_{L_\infty\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d, \sqrt{\frac{\omega}{\varrho}}\right)} &= \left\| f - S_{I_N^d}^{\Lambda} f \right\|_{L_\infty(\mathbb{T}^d)} \leq 2N^{-m+\lambda} \|f\|_{\mathcal{A}^{m-\lambda}(\mathbb{T}^d)} \\ &\leq 2C_{d,\lambda} N^{-m+\lambda} \|f\|_{\mathcal{H}^m(\mathbb{T}^d)} = 2C_{d,\lambda} N^{-m+\lambda} \|h\|_{\mathcal{H}^m\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)} < \infty \end{aligned}$$

with $\lambda > \frac{1}{2}$ and some constant $C_{d,\lambda} > 1$. ■

3.5.2 L_2-approximation error

Similarly, based on the $L_2(\mathbb{T}^d)$-approximation error bound (2.9) and the conditions proposed in Theorem 3.3 we prove an upper bound for the approximation error $\left\| h - S_{I_N^d}^{\Lambda} h \right\|$ in terms of a weighted L_2-norm on $\left[-\frac{1}{2}, \frac{1}{2}\right]^d$.

Theorem 3.5. *Let $d \in \mathbb{N}$, $m \in \mathbb{N}_0$, a hyperbolic cross I_N^d with $N \geq 2^{d+1}$ and a reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, I_N^d)$ be given. Let ω be a weight function as in (3.12) and let $h \in L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d, \omega\right) \cap \mathcal{C}_{\text{mix}}^m\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right)$, and let ψ be a multivariate transformation as defined in (3.3). For all multi-indices $\mathbf{m} = (m_1, \ldots, m_d)^\top \in \mathbb{N}_0^d$ with $\|\mathbf{m}\|_{\ell_\infty} \leq m$ we assume that*

$$\psi \in \mathcal{C}_{\text{mix}}^m\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right) \quad \text{and} \quad D^{\mathbf{m}}\left[\prod_{\ell=1}^d \sqrt{(\omega_\ell \circ \psi_\ell)\psi'_\ell}\right] \in \mathcal{C}_0\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right).$$

Then there is an approximation error estimate of the form

$$\left\| h - S_{I_N^d}^{\Lambda} h \right\|_{L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d, \omega\right)} \lesssim N^{-m} (\log N)^{(d-1)/2} \|h\|_{\mathcal{H}^m\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)}.$$

Proof. Let $m \in \mathbb{N}_0, d \in \mathbb{N}$ and $h \in L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d, \omega\right) \cap \mathcal{C}_{\text{mix}}^m\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right)$. By assumption the criteria of Theorem 3.3 are fulfilled and the transformed function f of the form (3.21) is continuously extendable to the torus $\mathbb{T}^d$. Thus, we have $f \in \mathcal{H}^m(\mathbb{T}^d)$ and a continuous representative, because of the inclusion $\mathcal{H}^m(\mathbb{T}^d) \hookrightarrow \mathcal{C}(\mathbb{T}^d)$ as in (2.4). For $f \in \mathcal{H}^m(\mathbb{T}^d) \cap \mathcal{C}(\mathbb{T}^d)$ Theorem 2.3 yields the approximation error bound of the form

$$\left\| f - S_{I_N^d}^{\Lambda} f \right\|_{L_2(\mathbb{T}^d)} \leq C_{d,\beta} N^{-\beta} (\log N)^{(d-1)/2} \|f\|_{\mathcal{H}^\beta(\mathbb{T}^d)} \tag{3.34}$$

with some constant $C_{d,\beta} := C(d, \beta) > 0$. With the inverse transformation $\boldsymbol{x} = \psi^{-1}(\boldsymbol{y})$ we have

$$\hat{h}_{\mathbf{k}} = (h, \varphi_{\mathbf{k}})_{L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d, \omega\right)} = (f, \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\circ})_{L_2(\mathbb{T}^d)} = \hat{f}_{\mathbf{k}},$$

and

$$\|h\|^2_{\mathcal{H}^m\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)} = \sum_{\mathbf{k}\in\mathbb{Z}^d} \omega_{\mathrm{hc}}(\mathbf{k})^{2m}|\hat{h}_{\mathbf{k}}|^2 = \sum_{\mathbf{k}\in\mathbb{Z}^d} \omega_{\mathrm{hc}}(\mathbf{k})^{2m}|\hat{f}_{\mathbf{k}}|^2 = \|f\|^2_{\mathcal{H}^m(\mathbb{T}^d)}$$

as in (3.32), as well as

$$\left\|h - S_{I_N^d} h\right\|^2_{L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d,\omega\right)} = \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} \left| h(\mathbf{y}) - \sum_{\mathbf{k}\in I_N^d} \hat{h}_{\mathbf{k}}\,\varphi_{\mathbf{k}}(\mathbf{y})\right|^2 \omega(\mathbf{y})\,\mathrm{d}\mathbf{y} = \left\|f - S_{I_N^d} f\right\|^2_{L_2(\mathbb{T}^d)} \tag{3.35}$$

and $\left\|h - S^{\Lambda}_{I_N^d} h\right\|_{L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d,\omega\right)} = \left\|f - S^{\Lambda}_{I_N^d} f\right\|_{L_2(\mathbb{T}^d)}$. In total, by combining (3.35), (3.34), and (3.32) we estimated for $f \in \mathcal{H}^m(\mathbb{T}^d) \cap \mathcal{C}(\mathbb{T}^d)$ that

$$\begin{aligned}\left\|h - S^{\Lambda}_{I_N^d} h\right\|_{L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d,\omega\right)} = \left\|f - S^{\Lambda}_{I_N^d} f\right\|_{L_2(\mathbb{T}^d)} &\lesssim C_{d,\beta} N^{-\beta} (\log N)^{(d-1)/2} \|f\|_{\mathcal{H}^\beta(\mathbb{T}^d)} \\ &= C_{d,\beta} N^{-\beta} (\log N)^{(d-1)/2} \|h\|_{\mathcal{H}^m\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)} < \infty\end{aligned}$$

with some constant $C_{d,\beta} > 0$. ■

4 Algorithms

In this chapter we start denoting the parameters $\boldsymbol{\eta}, \boldsymbol{\mu} \in \mathbb{R}^d_+$. Hence, families of multivariate parameterized weight functions are denoted by $\omega(\circ, \boldsymbol{\mu})$ as in (3.12) and for families of multivariate torus-to-cube transformations we use the notation $\psi(\circ, \boldsymbol{\eta})$ to represent all possible torus-to-cube transformations in the sense of definition (3.3) and not just the parameterized transformations in (3.5). Furthermore, also all related functions and objects are now written with a parameter argument.

We adapt the algorithms described in [17, Algorithm 3.1 and 3.2] that are based on one-dimensional fast Fourier transforms (FFTs). They are used for the fast reconstruction of approximated Fourier coefficients $\hat{h}^{\Lambda}_{\mathbf{k}}$ and the evaluation of a transformed multivariate trigonometric polynomials, in particular the approximated Fourier series $S^{\Lambda}_I h$, both given in (3.30). This is denoted as matrix-vector-products of the form

$$\mathbf{h} = \mathbf{A}\hat{\mathbf{h}} \quad \text{and} \quad \hat{\mathbf{h}} = M^{-1}\mathbf{A}^*\mathbf{h}$$

with $\boldsymbol{\eta}, \boldsymbol{\mu} \in \mathbb{R}^d_+$, $\mathbf{h} := \left(h(\mathbf{y}_j)\sqrt{\frac{\omega(\mathbf{y}_j,\boldsymbol{\mu})}{\varrho(\mathbf{y}_j,\boldsymbol{\eta})}}\right)_{j=0,\ldots,M-1}$ for $\mathbf{y}_j \in \Lambda_{\psi(\circ,\boldsymbol{\eta})}(\mathbf{z}, M)$, $\hat{\mathbf{h}} := (\hat{h}_{\mathbf{k}})_{\mathbf{k}\in I_N}$ and the transformed Fourier matrices $\mathbf{A} \in \mathbb{C}^{M\times|I|}$ and $\mathbf{A}^* \in \mathbb{C}^{|I|\times M}$ given by

$$\mathbf{A} := \left(\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\psi^{-1}(\mathbf{y}_j,\boldsymbol{\eta})}\right)_{\mathbf{y}_j\in\Lambda_{\psi(\circ,\boldsymbol{\eta})}(\mathbf{z},M),\,\mathbf{k}\in I} = \left(\sqrt{\frac{\omega(\mathbf{y}_j,\boldsymbol{\mu})}{\varrho(\mathbf{y}_j,\boldsymbol{\eta})}}\,\varphi_{\mathbf{k}}(\mathbf{y}_j)\right)_{\mathbf{y}_j\in\Lambda_{\psi(\circ,\boldsymbol{\eta})}(\mathbf{z},M),\,\mathbf{k}\in I},$$

$$\mathbf{A}^* := \left(\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\psi^{-1}(\mathbf{y}_j,\boldsymbol{\eta})}\right)_{\mathbf{k}\in I,\,\mathbf{y}_j\in\Lambda_{\psi(\circ,\boldsymbol{\eta})}(\mathbf{z},M)} = \left(\sqrt{\frac{\omega(\mathbf{y}_j,\boldsymbol{\mu})}{\varrho(\mathbf{y}_j,\boldsymbol{\eta})}}\,\overline{\varphi_{\mathbf{k}}(\mathbf{y}_j)}\right)_{\mathbf{k}\in I,\mathbf{y}_j\in\Lambda_{\psi(\circ,\boldsymbol{\eta})}(\mathbf{z},M)}.$$

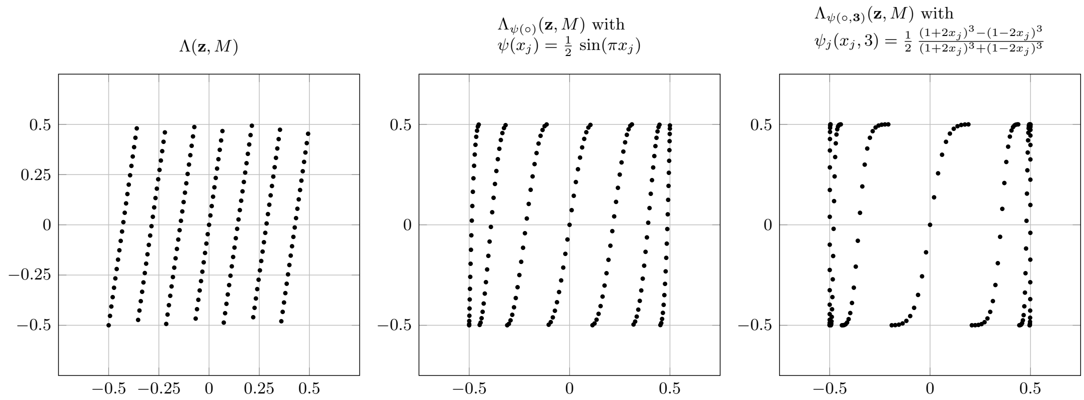

Figure 4.1: A two-dimensional lattice $\Lambda(\mathbf{z}, M)$ with $\mathbf{z} = (1,7)^\top, M = 150$ in the top left, the transformed lattice $\Lambda_{\psi(\circ,\boldsymbol{\eta})}(\mathbf{z}, M)$ for the sine transformation (3.11) in the bottom left, and resulting lattices from applying the logarithmic transformation (3.9) with $\boldsymbol{\eta} = \mathbf{3}$.

We incorporate the previously described idea that an $h \in L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d, \omega\right) \cap \mathcal{C}^m_{\text{mix}}\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)$ are transformed into functions f on the torus $\mathbb{T}^d$ that are of the form (3.21) via transformations $\mathbf{y}_j = \psi(\mathbf{x}_j, \boldsymbol{\eta})$ with $\mathbf{x}_j = (x_1^j, \ldots, x_d^j)$, so that we have samples of the form

$$h(\mathbf{y}_j)\sqrt{\frac{\omega(\mathbf{y}_j,\boldsymbol{\mu})}{\varrho(\mathbf{y}_j,\boldsymbol{\eta})}} = h(\psi(\mathbf{x}_j,\boldsymbol{\eta}))\sqrt{\omega(\psi(\mathbf{x}_j,\boldsymbol{\eta}),\boldsymbol{\mu})\prod_{k=1}^d \psi_k'(x_k^j,\eta_k)} = f(\mathbf{x}_j,\boldsymbol{\eta},\boldsymbol{\mu}) = f(\mathbf{x}_j), \quad (4.1)$$

depending on the particular choices for $\boldsymbol{\eta}, \boldsymbol{\mu} \in \mathbb{R}^d_+$.

Remark 4.1. *We identify $\mathbb{T}^d$ with different cubes. On one hand, when defining rank-1 lattices $\Lambda(\mathbf{z}, M)$ in (2.6) we identify it with $[0,1)^d$. On the other hand, in order to apply the transformations ψ we need to consider $\mathbb{T}^d \simeq [-\frac{1}{2},\frac{1}{2})^d$, which we achieve by reassigning all lattice points $\mathbf{x}_j \in \Lambda(\mathbf{z}, M)$ via*

$$\mathbf{x}_j \mapsto \left(\left(\mathbf{x}_j + \frac{\mathbf{1}}{2}\right) \bmod \mathbf{1}\right) - \frac{\mathbf{1}}{2}$$

for all $j = 0, \ldots, M-1$. □

In Figure 4.1 we show different two-dimensional transformed rank-1 lattices $\Lambda_{\psi(\circ,\boldsymbol{\eta})}(\mathbf{z}, M)$ as defined in (3.26), generated by $\mathbf{z} = (1,7)^\top$ and $M = 150$. We compare the lattices transformed by the sine transformation (3.11) with our previously introduced torus-to-cube transformations (3.9) with $\boldsymbol{\eta} = (\eta_1, \eta_2)^\top = (3,3)^\top$.

4.1 Evaluation of transformed multivariate trigonometric polynomials

Given a frequency set $I \subset \mathbb{Z}^d$ of finite cardinality $|I| < \infty$ we consider the multivariate trigonometric polynomial $h \in \Pi_{I,\psi(\circ,\boldsymbol{\eta})}$ as in (3.28) with Fourier coefficients $\hat{h}_{\mathbf{k}}$. The evaluation of h

Algorithm 4.1 Evaluation at rank-1 lattice

Input:	$M \in \mathbb{N}$	lattice size of $\Lambda_{\psi(\circ,\boldsymbol{\eta})}(\mathbf{z}, M)$
	$\mathbf{z} \in \mathbb{Z}^d$	generating vector of $\Lambda_{\psi(\circ,\boldsymbol{\eta})}(\mathbf{z}, M)$
	$I \subset \mathbb{Z}^d$	frequency set of finite cardinality
	$\hat{\mathbf{h}} = \left(\hat{h}_{\mathbf{k}}\right)_{\mathbf{k}\in I}$	Fourier coefficients of $h \in \Pi_{I,\psi(\circ,\boldsymbol{\eta})}$

$\hat{\mathbf{g}} = (0)_{l=0}^{M-1}$
for each $\mathbf{k} \in I$ **do**
  $\hat{g}_{\mathbf{k}\cdot\mathbf{z} \bmod M} = \hat{g}_{\mathbf{k}\cdot\mathbf{z} \bmod M} + \hat{h}_{\mathbf{k}}$
end for
$\mathbf{h} = \text{iFFT\_1D}(\hat{\mathbf{g}})$
$\mathbf{h} = M\mathbf{h}$

Output:	$\mathbf{h} = \mathbf{A}\hat{\mathbf{h}} = \left(h(\mathbf{y}_j)\sqrt{\frac{\omega(\mathbf{y}_j,\boldsymbol{\mu})}{\varrho(\mathbf{y}_j,\boldsymbol{\eta})}}\right)_{j=0}^{M-1}$	function values of $h \in \Pi_{I,\psi(\circ,\boldsymbol{\eta})}$

at lattice points $\mathbf{y}_j \in \Lambda_{\psi(\circ,\boldsymbol{\eta})}(\mathbf{z}, M)$ simplifies to

$$h(\mathbf{y}_j)\sqrt{\frac{\omega(\mathbf{y}_j,\boldsymbol{\mu})}{\varrho(\mathbf{y}_j,\boldsymbol{\eta})}} = \sum_{\mathbf{k}\in I} \hat{h}_{\mathbf{k}}\, \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\psi^{-1}(\mathbf{y}_j,\boldsymbol{\eta})}$$
$$= \sum_{\ell=0}^{M-1} \left(\sum_{\substack{\mathbf{k}\in I,\\ \mathbf{k}\cdot\mathbf{z}\equiv\ell\,(\bmod M)}} \hat{h}_{\mathbf{k}} \right) \mathrm{e}^{2\pi\mathrm{i}\ell\frac{j}{M}} = \sum_{\ell=0}^{M-1} \hat{g}_\ell\, \mathrm{e}^{2\pi\mathrm{i}\ell\frac{j}{M}},$$

with

$$\hat{g}_\ell = \sum_{\substack{\mathbf{k}\in I,\\ \mathbf{k}\cdot\mathbf{z}\equiv\ell\,(\bmod M)}} \hat{h}_{\mathbf{k}}.$$

In total, the evaluation of such a function is realized by simply pre-computing $(\hat{g}_\ell)_{\ell=0}^{M-1}$ and applying a one-dimensional inverse fast Fourier transform, see Algorithm 4.1.

4.2 Reconstruction of transformed multivariate trigonometric polynomials

For the reconstruction of a multivariate trigonometric polynomial $h \in \Pi_{I,\psi(\circ,\boldsymbol{\eta})}$ as in (3.28) from lattice points $\mathbf{y}_j \in \Lambda_{\psi(\circ,\boldsymbol{\eta})}(\mathbf{z}, M, I)$ we utilize the exact integration property (3.29) and the fact that we have

$$\sum_{j=0}^{M-1} \left(\mathrm{e}^{2\pi\mathrm{i}\frac{(\mathbf{k}-\mathbf{h})\cdot\mathbf{z}}{M}}\right)^j = \begin{cases} M & \text{for } \mathbf{k}\cdot\mathbf{z} \equiv \mathbf{k}\cdot\mathbf{h}\,(\bmod M),\\ 0 & \text{otherwise,}\end{cases} \tag{4.2}$$

and thus $\mathbf{A}^*\mathbf{A} = M\mathbf{I}$ with $\mathbf{I} \in \mathbb{C}^{|I|\times|I|}$ being the identity matrix. For fixed parameters $\boldsymbol{\eta}, \boldsymbol{\mu} \in \mathbb{R}_+^d$ and $\mathbf{x}_j = (x_1^j, \dots, x_d^j)$ we have input sample points as in (4.1) of the form

$$h(\mathbf{y}_j)\sqrt{\frac{\omega(\mathbf{y}_j,\boldsymbol{\mu})}{\varrho(\mathbf{y}_j,\boldsymbol{\eta})}} = h(\psi(\mathbf{x}_j,\boldsymbol{\eta}))\sqrt{\omega(\psi(\mathbf{x}_j,\boldsymbol{\eta}),\boldsymbol{\mu})\prod_{k=1}^d \psi_k'(x_k^j,\eta_k)} = f(\mathbf{x}_j,\boldsymbol{\eta},\boldsymbol{\mu}) = f(\mathbf{x}_j).$$

Algorithm 4.2 Reconstruction from sampling values along a transformed reconstructing rank-1 lattice

Input:	$I \subset \mathbb{Z}^d$	frequency set of finite cardinality
	$M \in \mathbb{N}$	lattice size of $\Lambda_{\psi(\circ,\boldsymbol{\eta})}(\mathbf{z}, M, I)$
	$\mathbf{z} \in \mathbb{Z}^d$	generating vector of $\Lambda_{\psi(\circ,\boldsymbol{\eta})}(\mathbf{z}, M, I)$
	$\mathbf{h} = \left(h(\mathbf{y}_j)\sqrt{\frac{\omega(\mathbf{y}_j,\boldsymbol{\mu})}{\varrho(\mathbf{y}_j,\boldsymbol{\eta})}}\right)_{j=0}^{M-1}$	function values of $h \in \Pi_{I,\psi(\circ,\boldsymbol{\eta})}$

$\hat{\mathbf{g}} = \text{FFT\_1D}(\mathbf{h})$
for each $\mathbf{k} \in I$ **do**
  $\hat{h}_{\mathbf{k}} = \frac{1}{M}\hat{g}_{\mathbf{k}\cdot\mathbf{z} \bmod M}$
end for

Output:	$\hat{\mathbf{h}} = M^{-1}\mathbf{A}^*\mathbf{h} = \left(\hat{h}_{\mathbf{k}}\right)_{\mathbf{k}\in I}$	Fourier coefficients supported on I

For the reconstruction of the Fourier coefficients $\hat{h}_{\mathbf{k}}$ we use a single one-dimensional fast Fourier transform. The entries of the resulting vector $(\hat{g}_\ell)_{\ell=0}^{M-1}$ are renumbered by means of the unique inverse mapping $\mathbf{k} \mapsto \mathbf{k}\cdot\mathbf{z} \bmod M$, see Algorithm 4.2.

4.3 Discrete approximation error

In order to use Algorithms 4.1 and 4.2 to illustrate the proposed error bounds of Theorems 3.4 and 3.5 we sample both the test function h and the approximated Fourier partial sum $S_I^\Lambda h$ in order to discretize and thus approximate the error $\|h - S_I^\Lambda h\|_{L_\infty\left([-\frac{1}{2},\frac{1}{2}]^d, \sqrt{\frac{\omega(\circ,\boldsymbol{\mu})}{\varrho(\circ,\boldsymbol{\eta})}}\right)}$ that is equal to $\left\|f - S_I^\Lambda f\right\|_{L_\infty(\mathbb{T}^d)}$ as shown in the proof of Theorem 3.4. At first we use the sample data vector $\mathbf{h} = \left(h(\mathbf{y}_j)\sqrt{\frac{\omega(\mathbf{y}_j,\boldsymbol{\mu})}{\varrho(\mathbf{y}_j,\boldsymbol{\eta})}}\right)_{j=0}^{M-1}$ with lattice points $\mathbf{y}_j \in \Lambda_{\psi(\circ,\boldsymbol{\eta})}(\mathbf{z}, M, I)$ and apply Algorithm 4.2 yielding a vector of approximated Fourier coefficients via $\hat{\mathbf{h}} = M^{-1}\mathbf{A}^*\mathbf{h}$. If we immediately put this vector $\hat{\mathbf{h}}$ into Algorithm 4.1 then we have computed the vector $\mathbf{h}_{\text{approx}} := M^{-1}\mathbf{A}\mathbf{A}^*\mathbf{h} = \left(\sqrt{\frac{\omega(\mathbf{y}_j,\boldsymbol{\mu})}{\varrho(\mathbf{y}_j,\boldsymbol{\eta})}} S_{I_N^d}^\Lambda h(\mathbf{y}_j)\right)_{j=0}^{M-1}$. In [18, Corollary 1] and [21, Theorem 2.1] it was shown under mild assumptions that for each frequency set $I \subset \mathbb{Z}^d$ that induces a reconstructing rank-1 lattice, there is an $M \in \mathbb{N}$ such that $|I| \leq M \lesssim |I|^2$. The upper bound can be improved to $M \leq C|I|\log|I|$ with high probability by using multiple rank-1 lattices as shown in [19, 22]. Furthermore, in (4.2) we already observed that for a reconstructing rank-1 lattice $\Lambda_{\psi(\circ,\boldsymbol{\eta})}(\mathbf{z}, M, I)$ we have $\mathbf{A}^*\mathbf{A} = M\mathbf{I}$ with $\mathbf{I} \in \mathbb{C}^{|I|\times|I|}$ being the identity matrix. However, $\mathbf{A}\mathbf{A}^* \in \mathbb{C}^{M\times M}$ is generally not an identity matrix. Hence, there is a gap between the initially given values $\mathbf{h}$ and the resulting vector $\mathbf{h}_{\text{approx}}$ that we quantify with the *relative discrete approximation error*

$$\varepsilon_{\text{lattice}} := \frac{\|\mathbf{h} - \mathbf{h}_{\text{approx}}\|_{\ell_\infty}}{\|\mathbf{h}\|_{\ell_\infty}} = \frac{\max_{j=0,\dots,M-1}\left|\sqrt{\frac{\omega(\mathbf{y}_j,\boldsymbol{\mu})}{\varrho(\mathbf{y}_j,\boldsymbol{\eta})}}\left(h(\mathbf{y}_j) - S_{I_N^d}^\Lambda h(\mathbf{y}_j)\right)\right|}{\max_{j=0,\dots,M-1}\left|\sqrt{\frac{\omega(\mathbf{y}_j,\boldsymbol{\mu})}{\varrho(\mathbf{y}_j,\boldsymbol{\eta})}}\, h(\mathbf{y}_j)\right|}. \tag{4.3}$$

Thus, we have a discretization of the particular weighted L_∞-norm appearing in Theorem 3.4 and for hyperbolic crosses I_N^d we have the upper bound

$$\begin{aligned}\|\mathbf{h} - \mathbf{h}_{\text{approx}}\|_{\ell_\infty} &\leq \left\| h - S_{I_N^d}^{\Lambda} h \right\|_{L_\infty\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d, \sqrt{\frac{\omega(\circ,\boldsymbol{\mu})}{\varrho(\circ,\boldsymbol{\eta})}}\right)} \\ &= \left\| f - S_{I_N^d}^{\Lambda} f \right\|_{L_\infty(\mathbb{T}^d)} \leq 2N^{-m} \|h\|_{\mathcal{H}^m\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)}\end{aligned} \tag{4.4}$$

for appropriately chosen parameters $\boldsymbol{\eta}, \boldsymbol{\mu} \in \mathbb{R}_+^d$. Hence, the theoretical results predict a certain decay rate of the discretized approximation error for increasing $N \in \mathbb{N}$ with fixed $m \in \mathbb{N}$ and suitably chosen parameter $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$.

It's important to note, that the particular discretization (4.3) was exclusively sampled at the rank-1 lattice nodes, so that we don't measure the quality of the approximation at any point outside the rank-1 lattice. This limitation is overcome by oversampling.

On the other hand, for the L_2-approximation error we lack a similar discretization approach. However, by Theorem 3.5 we know that for fixed $m \in \mathbb{N}$ and suitably chosen parameters $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ the error $\left\| h - S_{I_N^d}^{\Lambda} h \right\|_{L_2\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d, \omega\right)} = \left\| f - S_{I_N^d}^{\Lambda} f \right\|_{L_2(\mathbb{T}^d)}$ is bounded above by $N^{-m} (\log N)^{(d-1)/2} \|f\|_{\mathcal{H}^m(\mathbb{T}^d)}$. By Parseval's equation we have

$$\begin{aligned}\left\| f - S_{I_N^d}^{\Lambda} f \right\|_{L_2(\mathbb{T}^d)}^2 = \sum_{\mathbf{k} \in \mathbb{Z}^d} |\hat{f}_{\mathbf{k}} - \hat{f}_{\mathbf{k}}^{\Lambda}|^2 &= \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus I_N^d} |\hat{f}_{\mathbf{k}}|^2 + \sum_{\mathbf{k} \in I_N^d} |\hat{f}_{\mathbf{k}} - \hat{f}_{\mathbf{k}}^{\Lambda}|^2 \\ &= \|f\|_{L_2(\mathbb{T}^d)}^2 + \sum_{\mathbf{k} \in I_N^d} \left(|\hat{f}_{\mathbf{k}} - \hat{f}_{\mathbf{k}}^{\Lambda}|^2 - |\hat{f}_{\mathbf{k}}|^2 \right).\end{aligned}$$

Hence, we can evaluate the L_2-approximation error if we used Algorithm 4.2 to reconstruct the approximated Fourier coefficients $\hat{f}_{\mathbf{k}}^{\Lambda}$ and if it is possible to calculate the Fourier coefficients $\hat{f}_{\mathbf{k}}$ for all $\mathbf{k} \in I_N^d$.

5 Examples

We always assume a constant weight function $\omega \equiv 1$. In dimensions $d \in \{1, 2, 5\}$ we consider certain choices for test functions $h \in \mathcal{C}_{\text{mix}}^m \left(\left[-\frac{1}{2}, \frac{1}{2} \right]^d \right)$ in combination with the logarithmic transformation (3.9) and the sine transformation (3.11). For both transformations we discuss the proposed smoothness conditions (3.22) in Theorem 3.3 and when they are fulfilled. These smoothness conditions lead to ranges of the multivariate parameter $\boldsymbol{\eta} \in \mathbb{R}_+^d$ appearing in the logarithmic transformation (3.9) for which the transformed functions f of the form (3.21) have a guaranteed Sobolev smoothness degree $m \in \mathbb{N}$, i.e. $f \in \mathcal{H}^m(\mathbb{T}^d)$. For such functions we have proven L_∞-approximation error bounds in Theorem 3.4. In the end we compare the corresponding relative discrete approximation errors $\varepsilon_{\text{lattice}}$ given in (4.3) for both the logarithmic transformation (3.9) and the sine transformation (3.11) with various values of $\boldsymbol{\eta} \in \mathbb{R}_+^d$.

Throughout this section we repeatedly specify parameter vectors $\boldsymbol{\eta} = (\eta, \ldots, \eta)^\top$ that have the same number in each entry, for which we recall the short notation of just using a single bold number, that appeared earlier in the definition (2.6) of rank-1 lattices $\Lambda(\mathbf{z}, M)$ in form of $\mathbf{1} = (1, \ldots, 1)^\top$.

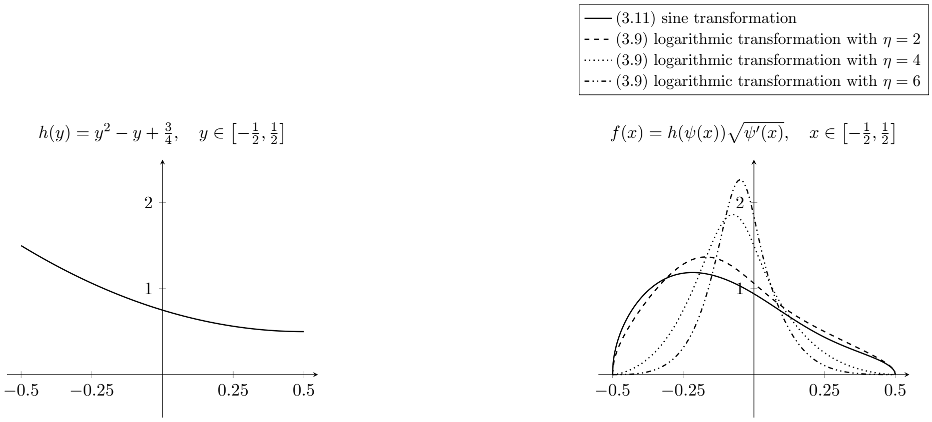

Figure 5.1: Plot of the univariate test function h given in (5.1) on the left. A comparison of the transformed functions f after applying the sine transformation as in (5.4) and after applying the logarithmic transformation (5.2) as in (5.3) with $\eta \in \{2, 4, 6\}$ on the right.

5.1 Univariate approximation

The univariate test function is

$$h(y) = y^2 - y + \frac{3}{4}, \quad y \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right], \tag{5.1}$$

and shown on the left of Figure 5.1. We choose a constant weight function $\omega(y, \mu) \equiv 1$ for all $\mu \in \mathbb{R}_+$ and the logarithmic transformation $\psi(x, \eta)$ with $x \in [-\frac{1}{2}, \frac{1}{2}]$, the parameter $\eta \in \mathbb{R}_+$, being in the form of (3.1).

Due to the choice of a constant weight function $\omega \equiv 1$ the test function h is simply in $L_2\left([-\frac{1}{2}, \frac{1}{2}], \omega(\circ, \mu)\right) = L_2\left([-\frac{1}{2}, \frac{1}{2}]\right)$. The test function h in (5.1) combined with the constant weight function $\omega \equiv 1$ and the logarithmic transformations given in (3.9) by

$$\psi(x, \eta) = \frac{1}{2}\frac{(1+2x)^\eta - (1-2x)^\eta}{(1+2x)^\eta + (1-2x)^\eta}, \tag{5.2}$$

lead to transformed functions $f(\circ, \eta, 1) =: f(\circ, \eta)$ in the sense of (3.21) of the form

$$f(x, \eta) = h(\psi(x, \eta))\sqrt{\psi'(x, \eta)} = \left(\psi(x, \eta)^2 - \psi(x, \eta) + \frac{3}{4}\right)\sqrt{\psi'(x, \eta)}. \tag{5.3}$$

In Figure 5.1 we have a side-by-side comparison of the graphs of these transformed functions $f(x, \eta)$ with the parameter $\eta \in \{1, 2, 4, 6\}$.

We proceed to determine the values $\eta \in \mathbb{R}_+$ for which $f(\circ, \eta)$ as in (5.3) is an element of $\mathcal{H}^m(\mathbb{T})$ by investigating conditions (3.18) in Theorem 3.2. First of all, we observe that for $\eta > 1$ the transformations $\psi(\circ, \eta)$ in (5.2) are transformations of the form (3.3). As the test function (5.1) is obviously in $\mathcal{C}^m\left([-\frac{1}{2}, \frac{1}{2}]\right)$ for any $m \in \mathbb{N}_0$, we proceed to check conditions (3.18) for a given $m \in \mathbb{N}_0$. For a constant weight function $\omega \equiv 1$ these conditions simplify to the task of determining the values $\eta \in \mathbb{R}_+$ for which we have

$$\left\| \left(\sqrt{\psi'(\circ, \eta)}\right)^{(j)}(\circ) \right\|_{L_\infty\left([-\frac{1}{2}, \frac{1}{2}]\right)} < \infty,$$

as well as

$$\left(\sqrt{\psi'(x,\eta)}\right)^{(j)}(x) \to 0 \quad \text{for} \quad |x| \to \frac{1}{2}$$

for all $j = 0, \ldots, m$. We obtain the following:

- For $m = 0$ we already mentioned in (3.9) that the functions $\psi'(\circ, \eta)$ are finite for $\eta \geq 1$ but converge to 0 at the boundary points $\pm\frac{1}{2}$ only for $\eta > 1$.
- For natural degrees of smoothness m the m-th derivative of $\sqrt{\psi'(\circ,\eta)}$ is in $\mathcal{C}_0\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$ if $\eta > 2m + 1$.
- For values $2m + 1 < \eta < 2m + 3$ the $(m + 1)$-th and all higher derivatives of $\sqrt{\psi'(\circ,\eta)}$ are unbounded and in case of $\eta = 2m + 3$ they are bounded but not $\mathcal{C}_0\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$.

Hence, according to the conditions in Theorem 3.2 the transformed function f in (5.3) is in $\mathcal{H}^m(\mathbb{T})$ for $\eta > 2m + 1$.

Switching to the sine transformation (3.11) leads to a transformed function f as given in (3.21) of the form

$$f(x) = h(\psi(x))\sqrt{\psi'(x)} = \left(\frac{1}{4}\sin^2(\pi x) - \frac{1}{2}\sin(\pi x) + \frac{3}{4}\right)\sqrt{\frac{\pi}{2}\cos(\pi x)}, \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right], \tag{5.4}$$

that according to Theorem 3.2 is only guaranteed to be in $\mathcal{H}^0(\mathbb{T}) = L_2(\mathbb{T})$ because all derivatives of $\sqrt{\psi'}$ are unbounded.

Finally we compare the relative discrete approximation errors $\varepsilon_{\text{lattice}}$ as in (4.3) of the sine transformed function in (5.4) and of the logarithmically transformed functions in (5.3) with $\eta = 2$ and $\eta = 4$. For this matter we consider hyperbolic crosses I_N^d, which in $d = 1$ is simply the integer set $I_N^1 = \{-N, \ldots, N\}$ and let $N \in \{4, 5, \ldots, 80\}$. In Figure 5.2 we showcase that the approximation errors of both the sine transformed and the logarithmically transformed functions for $\eta = 2$ behave similarly because they are both $\mathcal{H}^0(\mathbb{T})$-functions and are thus not guaranteed to have any upper bound as in (4.4). By increasing the parameter to $\eta = 4$ it smoothed the logarithmically transformed function by one Sobolev smoothness degree, so that $f \in \mathcal{H}^1(\mathbb{T})$, causing a faster decaying upper bound (4.4) and thus a faster decay of the relative approximation error $\varepsilon_{\text{lattice}}$ as in (4.3). Another parameter increase to $\eta = 6$ increases the Sobolev smoothness by another degree so that $f \in \mathcal{H}^2(\mathbb{T})$ and for $\eta = 8$ we have $f \in \mathcal{H}^3(\mathbb{T})$, resulting in even faster decays of the respective relative approximation errors $\varepsilon_{\text{lattice}}$ for large enough $N \in \mathbb{N}$.

Remark 5.1. *For the error function transformation* (3.10) *we obtained a very similar result regarding the parameter bound and the resulting approximation error decay. In fact, for non-negative integer degrees of smoothness* $m \in \mathbb{N}_0$ *the* m*-th derivative of* $\sqrt{\psi'(\circ,\eta)}$ *is in* $\mathcal{C}_0\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$ *if* $\eta > 2m + 1$, *too. Due to the now exponential density functions* $\varrho(\circ, \eta)$ *we obtain an overall faster decay of the discretized approximation error. However, as with the logarithmic transformation* (3.9) *the rate of decay at first is only as fast as the decay obtained with the sine transformation* (3.11) *and increases rapidly once we increase the parameter values to* $\eta \geq 3$. □

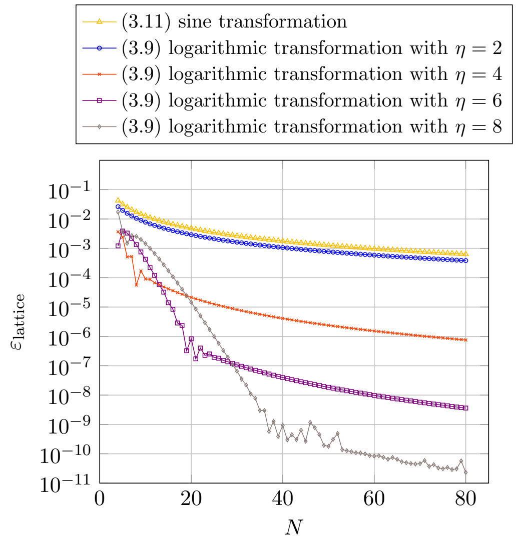

Figure 5.2: Comparison of discrete ℓ_∞-approximation error $\varepsilon_{\text{lattice}}$ as given in (4.3) of the one-dimensional test function (5.1) in combination with the sine transformation (3.11) and the logarithmic transformation (3.9) with $\eta \in \{2, 4, 6, 8\}$.

5.2 High-dimensional approximation

Before discussing a particular multivariate approximation setup we again stress the fact that we have the fast Algorithm 4.1 and Algorithm 4.2 that are based on a single one-dimensional inverse FFT and an one-dimensional FFT, respectively.

We consider the test function

$$h(\mathbf{y}) = h(y_1, \dots, y_d) = \sum_{j=1}^{d} y_j, \quad \mathbf{y} \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^d. \tag{5.5}$$

We choose a constant weight function $\omega(\circ, \boldsymbol{\mu}) \equiv 1$ for all $\boldsymbol{\mu} \in \mathbb{R}_+^d$ and the logarithmic transformation $\psi(\mathbf{x}, \boldsymbol{\eta}) = ((\psi_j(x_j, \eta_j))_{j=1}^d)^\top$ with $\mathbf{x} \in [-\frac{1}{2}, \frac{1}{2}]^d$, the parameter $\boldsymbol{\eta} \in \mathbb{R}_+^d$ and its univariate components $\psi_j(x_j, \eta_j)$ in the form of (5.2).

Due to the choice of a constant weight function $\omega \equiv 1$ the test function h is simply in $L_2\left([-\frac{1}{2}, \frac{1}{2}]^d, \omega(\circ, \boldsymbol{\mu})\right) = L_2\left([-\frac{1}{2}, \frac{1}{2}]^d\right)$. The test function h in (5.5) combined with the constant weight function $\omega \equiv 1$ and the logarithmic transformations (5.2) lead to transformed functions $f(\circ, \boldsymbol{\eta}, 1) =: f(\circ, \boldsymbol{\eta})$ of the form (3.21), reading as

$$\begin{aligned} f(\mathbf{x}, \boldsymbol{\eta}) &= h(\psi_1(x_1, \eta_1), \dots, \psi_d(x_d, \eta_d)) \prod_{j=1}^{d} \sqrt{\psi_j'(x_j, \eta_j)} \\ &= \frac{1}{2} \sum_{j=1}^{d} \left(\frac{(1+2x_j)^{\eta_j} - (1-2x_j)^{\eta_j}}{(1+2x_j)^{\eta_j} + (1-2x_j)^{\eta_j}} \right) \prod_{j=1}^{d} \sqrt{4\eta_j \frac{(1-4x_j^2)^{\eta_j - 1}}{((1+2x_j)^{\eta_j} + (1-2x_j)^{\eta_j})^2}}. \end{aligned} \tag{5.6}$$

In Figure 5.3 we have a side-by-side comparison of the graphs of these transformed functions $f(\mathbf{x}, \boldsymbol{\eta})$ for $d = 2$ with the parameter $\boldsymbol{\eta} \in \{\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{6}\}$.

We proceed to determine the values $\boldsymbol{\eta} \in \mathbb{R}_+^d$ for which $f(\circ, \boldsymbol{\eta})$ as in (5.6) is element of $\mathcal{H}^m(\mathbb{T}^d)$ by investigating conditions (3.22) in Theorem 3.3 for the derivatives of ψ'. First

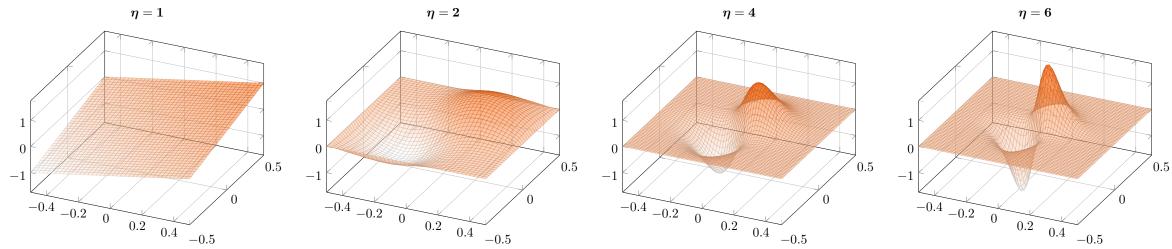

Figure 5.3: Plots of the two-dimensional transformed function $f(\circ, \boldsymbol{\eta})$ in (5.6) for $\boldsymbol{\eta} \in \{\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{6}\}$ with the logarithmic transformation $\psi(\circ, \boldsymbol{\eta})$ as in (5.2).

of all, we observe that for $\eta_1, \ldots, \eta_d > 1$ the components $\psi_1, \ldots, \psi_d$ of the transformations $\psi(\circ, \boldsymbol{\eta})$ in (5.2) are transformations as defined in (3.3). As the test function (5.5) is obviously in $\mathcal{C}^m_{\text{mix}}\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right)$ for any $m \in \mathbb{N}_0$, we proceed to check conditions (3.22) for a given $m \in \mathbb{N}_0$. For a constant weight function these conditions simplify to the task of determining the values $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_d)^\top \in \mathbb{R}^d_+$ for which we have

$$\left\| \left(\sqrt{\psi'_\ell(\circ, \eta_\ell)}\right)^{(j_\ell)} (\circ) \right\|_{L_\infty([-\frac{1}{2}, \frac{1}{2}])} < \infty,$$

as well as

$$\left(\sqrt{\psi'_\ell}\right)^{(j_\ell)} (x_\ell) \to 0 \quad \text{for} \quad |x_\ell| \to \frac{1}{2}$$

for all $\ell = 1, \ldots, d$ and $j_\ell = 0, \ldots, m$. For each dimension $\ell = 1, \ldots, d$ we observe the following:

- For $m = 0$ we already mentioned in (3.9) that the functions $\psi'_\ell(\circ, \eta_\ell)$ are finite for $\eta_\ell \geq 1$ but converge to 0 at the boundary points $\pm\frac{1}{2}$ only for $\eta_\ell > 1$.
- For natural degrees of smoothness $m \geq 1$ the m-th derivative of $\sqrt{\psi'_\ell(\circ, \eta_\ell)}$ is in $\mathcal{C}_0\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$ if $\eta_\ell > 2m + 1$.
- For values $2m+1 < \eta_\ell < 2m+3$ the $(m+1)$-th and all higher derivatives of $\sqrt{\psi'_\ell(\circ, \eta_\ell)}$ are unbounded and in case of $\eta_\ell = 2m + 3$ they are bounded but not $\mathcal{C}_0\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$.

Hence, according to the conditions in Theorem 3.2 the transformed function f in (5.6) is in $\mathcal{H}^m(\mathbb{T}^d)$ if $\eta_\ell > 2m + 1$ for all $\ell = 1, \ldots, d$.

Switching to the sine transformation (3.11) leads to a transformed function f as given in (3.21) of the form

$$f(\mathbf{x}) = h(\psi(\mathbf{x})) \prod_{j=1}^{d} \sqrt{\psi'_j(x_j)} = \frac{1}{2^d} \sum_{j=1}^{d} \sin(\pi x_j) \prod_{j=1}^{d} \sqrt{\frac{\pi}{2}\cos(\pi x_j)}, \quad \mathbf{x} \in \left[-\frac{1}{2}, \frac{1}{2}\right]^d,$$

that according to Theorem 3.2 is only in $\mathcal{H}^0(\mathbb{T}^d)$ because all derivatives of all $\sqrt{\psi'_j(\circ)}$ are unbounded.

Finally, for dimensions $d = 2$ and $d = 5$ we compare the relative discrete approximation errors $\varepsilon_{\text{lattice}}$ as in (4.3) of the sine transformed function in (5.4) and of the logarithmically

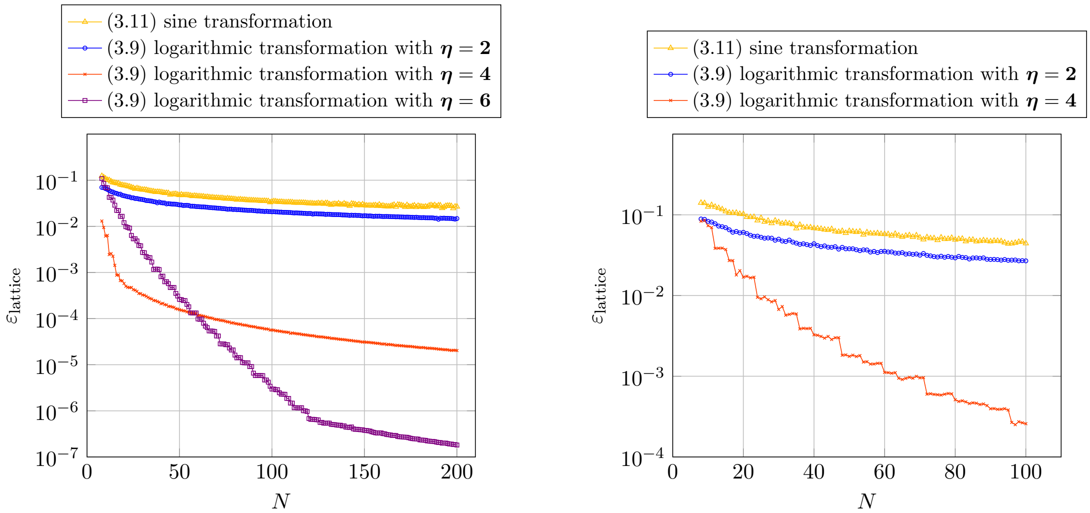

Figure 5.4: Comparison of discrete ℓ_∞-approximation error $\varepsilon_{\text{lattice}}$ as given in (4.3) of the d-dimensional test function (5.5). On the left, in dimension $d = 2$ we compare the sine transformation (3.11) and the logarithmic transformation (3.9) with $\boldsymbol{\eta} \in \{\mathbf{2}, \mathbf{4}, \mathbf{6}\}$. On the right, in dimension $d = 5$ we compare the sine transformation (3.11) and the logarithmic transformation (3.9) with $\boldsymbol{\eta} \in \{\mathbf{2}, \mathbf{4}\}$

transformed functions in (5.6) with $\boldsymbol{\eta} = \mathbf{2}$, $\boldsymbol{\eta} = \mathbf{4}$ and in case of $d = 2$ also with $\boldsymbol{\eta} = \mathbf{6}$. For this matter we consider hyperbolic crosses I_N^d as defined in (2.2) for $N \in \{8, 9, \ldots, 200\}$ in $d = 2$ and for $N \in \{8, 9, \ldots, 100\}$ in $d = 5$. We again emphasize the major advantage of Algorithms 4.1 and 4.2 in having a complexity of just $\mathcal{O}(M \log M + d|I_N^d|)$ due to being based on a single univariate inverse FFT and univariate FFT, respectly. Thus, their computation time is rather quick considering the fact that we are dealing with up to $|I_{100}^5| = 665.145$ frequencies. In Figure 5.4 we showcase that the approximation errors of both the sine transformed and the logarithmically transformed functions for $\boldsymbol{\eta} = \mathbf{2}$ behave similarly because they are both $\mathcal{H}^0(\mathbb{T}^d)$-functions and are thus not guaranteed to have any upper bound as in (4.4). By increasing the parameter to $\boldsymbol{\eta} = \mathbf{4}$ it smoothed the logarithmically transformed function by one Sobolev smoothness degree, so that $f \in \mathcal{H}^1(\mathbb{T}^d)$, causing a faster decaying upper bound (4.4) and thus a faster decay of the relative approximation error $\varepsilon_{\text{lattice}}$ as in (4.3). Another parameter increase to $\boldsymbol{\eta} = \mathbf{6}$ for $d = 2$ increases the Sobolev smoothness by another degree so that $f \in \mathcal{H}^2(\mathbb{T}^2)$ and the relative approximation error $\varepsilon_{\text{lattice}}$ decays even faster for large enough $N \in \mathbb{N}$.

6 Conclusion

In this paper we considered functions $h \in L_2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d, \omega(\circ, \boldsymbol{\mu})\right) \cap \mathcal{C}^m\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right)$ with a parameterized weight function $\omega(\circ, \boldsymbol{\mu}) : \left[-\frac{1}{2}, \frac{1}{2}\right]^d \to [0, \infty), \boldsymbol{\mu} \in \mathbb{R}_+^d$ and discussed a particular periodization strategy such that they are transformed into functions f that are continuously extendable on the torus $\mathbb{T}^d$. The applied multivariate torus-to-cube transformations $\psi : \left[-\frac{1}{2}, \frac{1}{2}\right]^d \to \left[-\frac{1}{2}, \frac{1}{2}\right]^d$, in particular parameterized torus-to-cube transformations $\psi(\circ, \boldsymbol{\eta}) : \left[-\frac{1}{2}, \frac{1}{2}\right]^d \to \left[-\frac{1}{2}, \frac{1}{2}\right]^d$ with $\boldsymbol{\eta} \in \mathbb{R}_+^d$, in combination with the weight function $\omega(\circ, \boldsymbol{\mu})$

let us furthermore control in which Sobolev space $\mathcal{H}^m(\mathbb{T}^d)$ with $m \in \mathbb{N}$ a function h defined on the cube $\left[-\frac{1}{2}, \frac{1}{2}\right]^d$ is transformed into. Due to the embedding of the Sobolev spaces $\mathcal{H}^m(\mathbb{T}^d)$ into the Wiener algebra $\mathcal{A}(\mathbb{T}^d)$ of functions with absolutely summable Fourier coefficients, we have information on the rate of decay of the Fourier coefficients $\hat{f}_{\mathbf{k}}$ and $\hat{h}_{\mathbf{k}}$ without having to calculate them – which in a lot of cases is not possible in the first place. Thus, the L_2- and L_∞-approximation error bounds for smooth functions on the torus $\mathbb{T}^d$ proposed in [41, Theorem 2.30] and [21, Theorem 3.3] can be transfered to classes of non-periodic functions defined on $\left[-\frac{1}{2}, \frac{1}{2}\right]^d$ by means of the inverse transformation $\psi^{-1}(\circ, \boldsymbol{\eta}) : \left[-\frac{1}{2}, \frac{1}{2}\right]^d \to \left[-\frac{1}{2}, \frac{1}{2}\right]^d$. Furthermore, only slight modifications are necessary to incorporate such transformations into the algorithms based on single reconstructing rank-1 lattices for the evaluation and the reconstruction of transformed multivariate trigonometric polynomials presented in [17, Algorithm 3.1 and 3.2]. Algorithms based on multiple reconstructing rank-1 lattices [19] and sparse fast Fourier transformations [33] can be adjusted, too, but weren't discussed in depth in this work.

Our numerical tests in up to dimension $d = 5$ show that these algorithms are still working within the proposed upper bounds for the approximation error. These tests also highlight the limited smoothness effect of static torus-to-cube transformations ψ as opposed to certain parameterized torus-to-cube transformations $\{\psi(\circ, \boldsymbol{\eta})\}_{\boldsymbol{\eta} \in \mathbb{R}^d_+}$, with which we can control the eventual smoothening effect.

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References

[1] J. P. Boyd. *Chebyshev and Fourier Spectral Methods.* Dover Press, New York, NY, USA, second edition, 2000.

[2] G. Byrenheid, D. Dũng, W. Sickel, and T. Ullrich. Sampling on energy-norm based sparse grids for the optimal recovery of Sobolev type functions in H^γ. *J. Approx. Theory*, 207:207–231, 2016.

[3] G. Byrenheid, L. Kämmerer, T. Ullrich, and T. Volkmer. Tight error bounds for rank-1 lattice sampling in spaces of hybrid mixed smoothness. *Numer. Math.*, 136:993–1034, 2017.

[4] B. Choi, M. Iwen, and F. Krahmer. Sparse harmonic transforms: A new class of sublinear-time algorithms for learning functions of many variables. *Found. Comput. Math.*, 2020.

[5] R. Cools, F. Y. Kuo, and D. Nuyens. Constructing lattice rules based on weighted degree of exactness and worst case error. *Computing*, 87:63–89, 2010.

[6] R. Cools, F. Y. Kuo, D. Nuyens, and G. Suryanarayana. Tent-transformed lattice rules for integration and approximation of multivariate non-periodic functions. *J. Complexity*, 36:166–181, 2016.

[7] R. Cools and D. Nuyens. Fast algorithms for component-by-component construction of rank-1 lattice rules in shift-invariant reproducing kernel Hilbert spaces. *Math. Comp.*, 75:903–920, 2004.

[8] J. Dick, T. Goda, K. Suzuki, and T. Yoshiki. Construction of interlaced polynomial lattice rules for infinitely differentiable functions. *Numer. Math.*, 137(2):257–288, 2017.

[9] J. Dick, F. Y. Kuo, and I. H. Sloan. High-dimensional integration: The quasi-Monte Carlo way. *Acta Numer.*, 22:133–288, 2013.

[10] J. Dick, D. Nuyens, and F. Pillichshammer. Lattice rules for nonperiodic smooth integrands. *Numer. Math.*, 126:259–291, 2014.

[11] D. Dung, V. N. Temlyakov, and T. Ullrich. Hyperbolic Cross Approximation. *Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser/Springer*, 2018.

[12] T. Goda and J. Dick. Construction of interlaced scrambled polynomial lattice rules of arbitrary high order. *Found. Comput. Math.*, 15(5):1245–1278, Oct. 2015.

[13] T. Goda, K. Suzuki, and T. Yoshiki. Lattice rules in non-periodic subspaces of Sobolev spaces. *Numer. Math.*, 141(2):399–427, 2019.

[14] M. Griebel and J. Hamaekers. Fast discrete Fourier transform on generalized sparse grids. In J. Garcke and D. Pflüger, editors, *Sparse Grids and Applications - Munich 2012*, volume 97 of *Lect. Notes Comput. Sci. Eng.*, pages 75–107. Springer International Publishing, 2014.

[15] M. Griebel and J. Hamaekers. Generalized sparse grid interpolation based on the fast discrete Fourier transform. 2019. Submitted to Proceedings of SGA 2018. Available as INS Preprint No. 1902.

[16] L. Kaemmerer, T. Ullrich, and T. Volkmer. Worst case recovery guarantees for least squares approximation using random samples. *ArXiv e-prints*, 2019. arXiv:1911.10111.

[17] L. Kämmerer. *High Dimensional Fast Fourier Transform Based on Rank-1 Lattice Sampling.* Dissertation. Universitätsverlag Chemnitz, 2014.

[18] L. Kämmerer. Reconstructing multivariate trigonometric polynomials from samples along rank-1 lattices. In G. E. Fasshauer and L. L. Schumaker, editors, *Approximation Theory XIV: San Antonio 2013*, pages 255–271. Springer International Publishing, 2014.

[19] L. Kämmerer. Multiple rank-1 lattices as sampling schemes for multivariate trigonometric polynomials. *J. Fourier Anal. Appl.*, 24:17–44, 2018.

[20] L. Kämmerer. Constructing spatial discretizations for sparse multivariate trigonometric polynomials that allow for a fast discrete Fourier transform. *Appl. Comput. Harmon. Anal.*, 47:702–729, 2019.

[21] L. Kämmerer, D. Potts, and T. Volkmer. Approximation of multivariate periodic functions by trigonometric polynomials based on rank-1 lattice sampling. *J. Complexity*, 31:543–576, 2015.

[22] L. Kämmerer, D. Potts, and T. Volkmer. High-dimensional sparse FFT based on sampling along multiple rank-1 lattices. *ArXiv e-prints*, 2017. arXiv:1711.05152.

[23] N. M. Korobov. On the approximate computation of multiple integrals. *Dokl. Akad. Nauk*, 124:1207–1210, 1959. In Russian.

[24] D. Krieg and M. Ullrich. Function values are enough for L_2-approximation. *ArXiv e-prints*, 2019. arXiv:1905.02516.

[25] P. Kritzer, F. Pillichshammer, L. Plaskota, and G. W. Wasilkowski. On efficient weighted integration via a change of variables. *Numer. Math.*, 2020.

[26] T. Kühn, W. Sickel, and T. Ullrich. Approximation of mixed order Sobolev functions on the d-torus: asymptotics, preasymptotics, and d-dependence. *Constr. Approx.*, 42(3):353–398, 2015.

[27] F. Y. Kuo, G. Migliorati, F. Nobile, and D. Nuyens. Function integration, reconstruction and approximation using rank-1 lattices. *Math. of Comp.*, 2020. to appear.

[28] F. Y. Kuo, I. H. Sloan, and H. Woźniakowski. Periodization strategy may fail in high dimensions. *Numer. Algorithms*, 46(4):369–391, 2007.

[29] F. Y. Kuo, G. W. Wasilkowski, and B. J. Waterhouse. Randomly shifted lattice rules for unbounded integrands. *J. Complexity*, 22(5):630–651, 2006.

[30] R. Nasdala and D. Potts. Transformed rank-1 lattices for high-dimensional approximation. *Electron. Trans. Numer. Anal.*, 53:239–282, 2020.

[31] H. Niederreiter. Quasi-Monte Carlo methods and pseudo-random numbers. *B. Am. Math. Soc.*, 84:957–1041, 1978.

[32] D. Potts and T. Volkmer. Fast and exact reconstruction of arbitrary multivariate algebraic polynomials in Chebyshev form. In *11th international conference on Sampling Theory and Applications (SampTA 2015)*, pages 392–396, 2015.

[33] D. Potts and T. Volkmer. Sparse high-dimensional FFT based on rank-1 lattice sampling. *Appl. Comput. Harmon. Anal.*, 41:713–748, 2016.

[34] H.-J. Schmeisser and H. Triebel. *Topics in Fourier analysis and function spaces*, volume 42 of *Mathematik und ihre Anwendungen in Physik und Technik.* Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1987.

[35] J. Shen, T. Tang, and L.-L. Wang. *Spectral Methods*, volume 41 of *Springer Ser. Comput. Math.* Springer-Verlag Berlin Heidelberg, Berlin, 2011.

[36] I. H. Sloan and S. Joe. *Lattice methods for multiple integration.* Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1994.

[37] I. H. Sloan and P. J. Kachoyan. Lattice methods for multiple integration: Theory, error analysis and examples. *SIAM J. Numer. Anal.*, 24:116–128, 1987.

[38] V. N. Temlyakov. Reconstruction of periodic functions of several variables from the values at the nodes of number-theoretic nets. *Anal. Math.*, 12:287–305, 1986. In Russian.

[39] V. N. Temlyakov. *Approximation of periodic functions.* Computational Mathematics and Analysis Series. Nova Science Publishers Inc., Commack, NY, 1993.

[40] T. Ullrich. Smolyak's algorithm, sparse grid approximation and periodic function spaces with dominating mixed smoothness. Dissertation, Friedrich-Schiller-Universität Jena, 2007.

[41] T. Volkmer. *Multivariate Approximation and High-Dimensional Sparse FFT Based on Rank-1 Lattice Sampling.* Dissertation. Universitätsverlag Chemnitz, 2017.

[42] J. Vybiral. Function spaces with dominating mixed smoothness. Dissertation, Friedrich-Schiller-Universität Jena, 2005.

[43] F. Weisz. Summability of multi-dimensional trigonometric Fourier series. *Surv. Approx. Theory*, 7:1–179, 2012.