Kernels of Spherical Harmonics and Spherical Frames

Daniel Potts, Gabriele Steidl and Manfred Tasche

Abstract. Our concern is with the construction of a frame in $L^2(S)$ consisting of smooth functions based on kernels of spherical harmonics. The corresponding decomposition and reconstruction algorithms utilize discrete spherical Fourier transforms. Numerical examples confirm the theoretical expectations.

§1. Introduction

Traditionally, wavelets were tailored to problems on the Euclidean space $\mathbb{R}^d$. However, in most applications one has to analyze functions defined on compact domains. In particular, in geophysics wavelets on the unit sphere S of $\mathbb{R}^3$ are of interest. There exist different approaches to the constructions of spherical wavelets. Having spherical coordinates in mind, the idea of using tensor–products of periodic wavelets and wavelets on the interval was suggested in [7]. Applying tensor–products of periodic exponential spline–wavelets and spline–wavelets on the interval, wavelets on S were constructed in [3]. Unfortunately, tensor–product wavelets can possess singularities at the poles of S. To avoid these singularities, the coefficients of scaling functions and wavelets have to satisfy a linear system on each level, whose dimension grows very fast (see [3]).

A tensor–product method with a completely different kind of wavelets, namely trigonometric wavelets [2] and polynomial wavelets [11], was considered in [10]. But these wavelets cannot be used for the detection of singularities of a given function at the poles of S.

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Another simple and powerful technique for constructing biorthogonal wavelets on the sphere applies the so–called lifting scheme. This method was developed in [12]. One example are lifted versions of generalized Haar–wavelets on spherical triangles.

Finally, spherical wavelets were constructed by using spherical harmonics. This idea was realized in a different manner in [6] (for equidistributed nodes on S) and [9] (for scattered points on S).

Our paper deals with the construction of a frame in $L^2(S)$ consisting of smooth functions arising from kernels of spherical harmonics. The corresponding decomposition and reconstruction algorithms utilize discrete spherical Fourier transforms. Numerical examples emphasize that our method is suited for the detection of singularities of a given function at the poles of S, too.

This paper is organized as follows. Section 2 briefly recalls properties of spherical harmonics. Introducing the sampling spaces of band–limited functions in Section 3, we verify in Section 4 that these spaces form a multiresolution of $L^2(S)$. In Section 5 we construct the corresponding tight frame in $L^2(S)$. We describe decomposition and reconstruction algorithms in Section 6 and conclude with numerical tests in Section 7.

§2. Spherical Harmonics

Starting with the *Legendre polynomials*

$$P_k(x) := \frac{1}{2^k\, k!}\, \frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^2-1)^k \quad (x \in [-1,1];\ k \in \mathbb{N}_0)\,,$$

we define the *associated Legendre functions* P_k^n $(n \in \mathbb{N}_0;\ k = n, n+1, \dots)$ by

$$P_k^n(x) := \left(\frac{(k-n)!}{(k+n)!}\right)^{1/2} (1-x^2)^{n/2}\, \frac{\mathrm{d}^n}{\mathrm{d}x^n} P_k(x) \quad (x \in [-1,1])\,.$$

For any fixed $n \in \mathbb{N}_0$, the functions P_k^n $(k = n, n+1, \dots)$ form a complete orthogonal system in $L^2[-1,1]$ with

$$\frac{1}{2}\int_{-1}^{1} P_k^n(x) P_l^n(x)\, \mathrm{d}x \;=\; \frac{1}{2k+1}\, \delta_{k,l} \quad (n \in \mathbb{N}_0;\ k, l = n, n+1, \dots)\,.$$

Moreover, the associated Legendre functions fulfil the three–term recurrence relation

$$P_{n-1}^n(x) := 0, \qquad P_n^n(x) := \frac{((2n)!)^{1/2}}{2^n\, n!}\, (1-x^2)^{n/2}\,,$$

$$P_{k+1}^n(x) - v_k^n\, x P_k^n(x) + w_k^n\, P_{k-1}^n(x) = 0 \quad (k = n, n+1, \dots) \tag{2.1}$$

with

$$v_k^n := \frac{2k+1}{((k-n+1)(k+n+1))^{1/2}}\,, \qquad w_k^n := \frac{((k-n)(k+n))^{1/2}}{((k-n+1)(k+n+1))^{1/2}}\,.$$

As usual, we parametrize the points of the unit sphere S in $\mathbb{R}^3$ by their spherical coordinates $(\theta,\varphi) \in [0,\pi] \times [0,2\pi)$. Then the *spherical distance* between the two points $p = (\theta,\varphi)$ and $p' = (\theta',\varphi')$ of S is determined by

$$pp' := \arccos\,(p \cdot p')$$

with the inner (cartesian) product

$$p \cdot p' = \cos\theta\ \cos\theta' \ + \ \sin\theta\ \sin\theta'\ \cos(\varphi - \varphi')\,.$$

We are interested in the Hilbert space $L^2(S)$ of all square integrable functions on S with the scalar product

$$\langle f,g\rangle := \frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} f(\theta,\varphi)\,\overline{g(\theta,\varphi)}\,\sin\theta\ \mathrm{d}\varphi\ \mathrm{d}\theta \quad (f,\, g \in L^2(S))$$

and with the corresponding norm $\|\cdot\|$. Set

$$\hat{\mathrm{I}} := \{(k,n) :\, k \in \mathbb{N}_0,\, n = -k,\ldots,k\}\,.$$

An orthogonal basis of $L^2(S)$ is given by the set $\{Y_k^n \,:\, (k,n) \in \hat{\mathrm{I}}\}$ of *spherical harmonics*

$$Y_k^n(\theta,\varphi) := P_k^{|n|}(\cos\theta)\ \mathrm{e}^{\mathrm{i}n\varphi}\,.$$

It is easy to check that

$$\begin{aligned}\langle Y_k^n, Y_l^m\rangle &= \frac{1}{2}\int_{-1}^1 P_k^{|n|}(x)P_l^{|m|}(x)\ \mathrm{d}x\ \cdot\ \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n-m)\varphi}\ \mathrm{d}\varphi \\ &= \frac{1}{2k+1}\ \delta_{k,l}\ \delta_{n,m} \quad ((k,n),\ (l,m) \in \hat{\mathrm{I}})\,. \end{aligned} \tag{2.2}$$

Furthermore, it holds the following *addition theorem for spherical harmonics*

$$\sum_{n=-k}^{k} Y_k^n(\theta,\varphi)\ Y_k^{-n}(\theta',\varphi') \ = \ P_k(\cos\theta\ \cos\theta' \ + \ \sin\theta\ \sin\theta'\ \cos(\varphi-\varphi'))$$

or shortly with $p = (\theta,\varphi)$ and $p' = (\theta',\varphi')$,

$$\sum_{n=-k}^{k} Y_k^n(p)\ Y_k^{-n}(p') \ = \ P_k(p\cdot p')\,. \tag{2.3}$$

We consider the Fourier expansion of $f \in L^2(S)$ with respect to the spherical harmonics

$$f = \sum_{(k,n)\in\hat{\mathrm{I}}} (2k+1)\, a_k^n(f)\, Y_k^n\,, \quad a_k^n(f) := \langle f, Y_k^n \rangle\,.$$

Let $l^2(\hat{\mathrm{I}})$ be the Hilbert space of all complex sequences $(a_k^n)_{(k,n)\in\hat{\mathrm{I}}}$ with

$$\sum_{(k,n)\in\hat{\mathrm{I}}} (2k+1)\, |a_k^n|^2 < \infty\,.$$

Then we refer to $F: \ L^2(S) \to l^2(\hat{\mathrm{I}})$ defined by

$$F\,f := \left(a_k^n(f)\right)_{(k,n)\in\hat{\mathrm{I}}} \quad (f \in L^2(S))$$

as *spherical Fourier transform.* For more details on spherical harmonics, see e.g. [8].

§3. Sampling Spaces

For $j \in \mathbb{N}_0$ we set

$$\begin{aligned} \mathrm{I}^j &:= \{(s,t) \ : \ s = 0,\dots,2^{j+1}; \ \ t = 0,\dots,2^{j+1}-1\}\,, \\ \hat{\mathrm{I}}^j &:= \{(k,n) \ : \ k = 0,\dots,2^j-1;\ n = -k,\dots,k\}\,. \end{aligned}$$

We consider the *sampling spaces of level* j $(j \in \mathbb{N}_0)$

$$V^j := \operatorname{span}\{Y_k^n \ : \ (k,n) \in \hat{\mathrm{I}}^j\}$$

consisting of so–called *band–limited* functions. Clearly, $\dim V^j = 2^{2j}$. Moreover, for band–limited functions, it holds the following sampling theorem.

Theorem 3.1. *Let $f \in V^j$ $(j \in \mathbb{N}_0)$ be given . Then we have for $(k,n) \in \hat{\mathrm{I}}^j$*

$$a_k^n(f) = \frac{1}{2^{j+1}} \sum_{(s,t)\in\mathrm{I}^j} \varepsilon_s^{(j+1)} w_s^{(j+1)} f(p_{s,t}^j)\, Y_k^{-n}(p_{s,t}^j)$$

with

$$p_{s,t}^j := \left(\frac{\pi s}{2^{j+1}}, \frac{\pi t}{2^j}\right),$$

$$\varepsilon_0^{(j)} = \varepsilon_{2^j}^{(j)} := 2^{-1}, \quad \varepsilon_s^{(j)} := 1 \ \ (s = 1,\dots,2^j-1)$$

and with the Clenshaw–Curtis weights

$$w_s^{(j+1)} := \frac{1}{2^{j+1}} \sum_{u=0}^{2^j} \varepsilon_u^{(j)} \frac{-2}{4u^2-1} \cos\frac{su\pi}{2^j} \quad (s = 0,\dots,2^{j+1})\,. \tag{3.1}$$

Proof: By definition of V^j, it suffices to consider the functions $f(\theta,\varphi) = Y_l^m(\theta,\varphi)$ ($(l,m) \in \hat{\mathrm{I}}^j$). Their Fourier coefficients can be written as

$$a_k^n(f) = \frac{1}{2}\int_{-1}^{1} P_l^{|m|}(x)P_k^{|n|}(x)\,\mathrm{d}x \,\cdot\, \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(m-n)\varphi}\,\mathrm{d}\varphi\,. \tag{3.2}$$

Now it holds for $m,\, n = 1-2^j, \ldots, 2^j-1$ that

$$\delta_{m,n} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(m-n)\varphi}\,\mathrm{d}\varphi = \frac{1}{2^{j+1}}\sum_{t=0}^{2^{j+1}-1} \mathrm{e}^{\pi\mathrm{i}(m-n)t/2^j}\,. \tag{3.3}$$

Hence, for $m \neq n$ we are done. For $m = n$, we verify that $P_l^{|n|}P_k^{|n|}$ is an algebraic polynomial of degree at least $2^{j+1}-2$ such that Clenshaw–Curtis quadrature gives

$$\frac{1}{2}\int_{-1}^{1} P_l^{|n|}(x)P_k^{|n|}(x)\,\mathrm{d}x = \sum_{s=0}^{2^{j+1}} \varepsilon_s^{(j+1)}\, w_s^{(j+1)} P_l^{|n|}\left(\cos\frac{\pi s}{2^{j+1}}\right)\, P_k^{|n|}\left(\cos\frac{\pi s}{2^{j+1}}\right).$$

Together with (3.2) and (3.3), this completes the proof. ∎

Note that a similar sampling theorem for band–limited function on S was proved in [4]. From Theorem 3.1 it follows a discrete orthogonality relation for spherical harmonics.

Corollary 3.2. *Let $j \in \mathbb{N}_0$ be given. Then we have for all (k,n), $(l,m) \in \hat{\mathrm{I}}^j$*

$$\frac{1}{2^{j+1}}\sum_{(s,t)\in\mathrm{I}^j} \varepsilon_s^{(j+1)} w_s^{(j+1)}\, Y_l^m(p_{s,t}^j)\, Y_k^{-n}(p_{s,t}^j) \;=\; \frac{1}{2k+1}\,\delta_{k,l}\,\delta_{m,n}\,.$$

Proof: Using Theorem 3.1 for $f = Y_l^m$, we obtain the result by (2.2). ∎

Theorem 3.1 leads to the following algorithm for the computation of the spherical Fourier transform of a band–limited function $f \in V^j$:

Algorithm 3.3 (Discrete Spherical Fourier Transform)

Input: For fixed $j \in \mathbb{N}_0$, let $f(p_{s,t}^j) \in \mathbb{C}$ ($(s,t) \in \mathrm{I}^j$) be given.

1. For every $s = 0, \ldots, 2^{j+1}$ form by fast Fourier transform

$$\hat{f}_s^n := \frac{1}{2^{j+1}}\sum_{t=0}^{2^{j+1}-1} f(p_{s,t}^j)\,\mathrm{e}^{-\mathrm{i}n\pi t/2^j} \quad (n = 1-2^j, \ldots, 2^j-1)\,.$$

2. For every $n = 1-2^j, \ldots, 2^j-1$ compute

$$\hat{\alpha}_{k,n}^j := \sum_{s=0}^{2^{j+1}} \varepsilon_s^{(j+1)} w_s^{(j+1)}\; \hat{f}_s^n\; P_k^{|n|}\left(\cos\frac{\pi s}{2^{j+1}}\right) \quad (k = |n|, \ldots, 2^j-1)$$

by the three–term recurrence relation (2.1).

Output: $\hat{a}_k^n(f) := \hat{\alpha}_{k,n}^j \in \mathbb{C}$ ($(k,n) \in \hat{\mathrm{I}}^j$).

The size of the Clenshaw–Curtis weights can be estimated as follows:

Lemma 3.4. *For $j \in \mathbb{N}_0$ and $s = 0, \ldots, 2^{j+1}$, it holds*

$$(2^{2j+2} - 1)^{-1} \le w_s^{(j+1)} < 2^{-j} - (2^{2j+2} - 1)^{-1} \, .$$

Proof: From (3.1) it follows that

$$\begin{aligned} 2^{j+1} w_s^{(j+1)} &\ge 2^{j+1} w_0^{(j+1)} \\ &= 1 - \sum_{u=1}^{2^j - 1} \left(\frac{1}{2u-1} - \frac{1}{2u+1} \right) - \frac{1}{2} \left(\frac{1}{2^{j+1}-1} - \frac{1}{2^{j+1}+1} \right) \\ &= \frac{1}{2} \left(\frac{1}{2^{j+1}-1} + \frac{1}{2^{j+1}+1} \right) = 2^{j+1} (2^{2j+2} - 1)^{-1} \, . \end{aligned}$$

On the other hand, we have

$$\begin{aligned} 2^{j+1} w_s^{(j+1)} &< 1 + \sum_{u=1}^{2^j - 1} \left(\frac{1}{2u-1} - \frac{1}{2u+1} \right) + \frac{1}{2} \left(\frac{1}{2^{j+1}-1} - \frac{1}{2^{j+1}+1} \right) \\ &= 2 - \frac{1}{2} \left(\frac{1}{2^{j+1}-1} + \frac{1}{2^{j+1}+1} \right) = 2 - 2^{j+1} (2^{2j+2} - 1)^{-1} \, . \end{aligned}$$

This completes the proof. ■

§4. Multiresolution of $L^2(S)$

The following definition of scaling functions is motivated by the outstanding properties of so–called kernel polynomials associated with orthogonal polynomials in weighted L^2–spaces (see e.g. [1, pp. 35–38]). The constructions of polynomial scaling functions and corresponding wavelets in [2,5,10,11] are based on kernel polynomials. Clearly, the concept of kernel polynomials can be transferred to orthogonal systems in other Hilbert spaces. We are concerned with the *kernels* $K^j : S \times S \to \mathbb{C}$ *of level* j $(j \in \mathbb{N}_0)$

$$K^j(p, p') := \sum_{(k,n) \in \hat{\mathrm{I}}^j} (2k+1) \, Y_k^n(p) \, Y_k^{-n}(p') \quad (p, p' \in S)$$

defined with respect to the spherical harmonics in V^j. The name "kernel" is explained by the *reproducing property*

$$\langle f, K^j(\,\cdot\,, p') \rangle = f(p') \quad (f \in V^j, \ p' \in S) \, , \tag{4.1}$$

which follows immediately from the Fourier expansion

$$f = \sum_{(k,n) \in \hat{\mathrm{I}}^j} (2k+1) \, a_k^n(f) \, Y_k^n \tag{4.2}$$

of $f \in V^j$ and from (2.2).

We introduce the *scaling functions* ϕ^j *of level* j ($j \in \mathbb{N}_0$) by

$$2^j \phi^j := \sum_{(k,n)\in \hat{\mathrm{I}}^j} (2k+1)\, Y_k^n$$

and the *"rotated" scaling functions* $\phi^j_{s,t} \in V^j$ ($(s,t) \in \mathrm{I}^j$) by

$$2^j \phi^j_{s,t} := K^j(\,\cdot\,, p^j_{s,t}) = \sum_{(k,n)\in \hat{\mathrm{I}}^j} (2k+1)\, Y_k^n\, Y_k^{-n}(p^j_{s,t})\,.$$

Then we have by (4.1) that

$$a_k^n(\phi^j_{s,t}) \;=\; Y_k^{-n}(p^j_{s,t})\, a_k^n(\phi^j) \;=\; \begin{cases} 2^{-j}\, Y_k^{-n}(p^j_{s,t}) & (k,n) \in \hat{\mathrm{I}}^j, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

The functions $\phi^j_{s,t} \in V^j$ ($(s,t) \in \mathrm{I}^j$) are smooth on the whole sphere S. Unfortunately, they are linearly dependent. By the addition theorem (2.3), the scaling functions read in terms of the spherical distance as

$$2^j \phi^j_{s,t}(p) = \sum_{k=0}^{2^j-1} (2k+1)\, P_k\left(\cos(pp^j_{s,t})\right) \;=\; \sum_{k=0}^{2^j-1} (2k+1)\, P_k(p \cdot p^j_{s,t})\,. \tag{4.4}$$

Theorem 4.1 collects further important properties of $\phi^j_{s,t}$.

Theorem 4.1. *For* $j \in \mathbb{N}_0$ *and* $(s,t) \in \mathrm{I}^j$ *it holds:*

(i) *Reproducing property:*

$$2^j \langle f, \phi^j_{s,t}\rangle \;=\; f(p^j_{s,t}) \quad (f \in V^j)\,.$$

(ii) $\|\phi^j_{s,t}\| = 1\,, \quad \phi^j_{s,t}(p^j_{s,t}) = 2^j.$

(iii) *Localization property:* $\phi^j_{s,t}$ *is localized around* $p^j_{s,t}$, *i.e.*

$$\frac{\|\phi^j_{s,t}\|}{\phi^j_{s,t}(p^j_{s,t})} \;=\; 2^{-j} = \min\left\{\|f\| : \, f \in V^j,\ f(p^j_{s,t}) = 1\right\}\,.$$

Proof: The reproducing property (i) follows immediately from (4.1). In particular, (i) yields

$$\|\phi^j_{s,t}\|^2 \;=\; \langle \phi^j_{s,t}, \phi^j_{s,t}\rangle \;=\; 2^{-j}\, \phi^j_{s,t}(p^j_{s,t})\,.$$

On the other hand, we have by (4.4) and $P_k(1) = 1$ $(k \in \mathbb{N}_0)$ that

$$\phi^j_{s,t}(p^j_{s,t}) = 2^{-j} \sum_{k=0}^{2^j-1} (2k+1)\, P_k(\cos 0) \; = 2^j\,,$$

which proves (ii).

The localization property (iii) can be seen by the following standard arguments: Let $f \in V^j$ with $f(p^j_{s,t}) = 1$, i.e.

$$1 = \sum_{(k,n)\in \hat{\mathrm{I}}} (2k+1)\, a^n_k(f)\, Y^n_k(p^j_{s,t})\,.$$

Then by applying the Cauchy–Schwarz inequality, the Parseval identity and (ii), we obtain

$$\begin{aligned} 1 &\le \left(\sum_{(k,n)\in \hat{\mathrm{I}}^j} (2k+1)\, |a^n_k(f)|^2 \right) \left(\sum_{(k,n)\in \hat{\mathrm{I}}^j} (2k+1)\, |Y^n_k(p^j_{s,t})|^2 \right) \\ &= \|f\|^2\, 2^j\, \phi^j_{s,t}(p^j_{s,t}) \; = \; 2^{2j}\, \|f\|^2\,, \end{aligned}$$

where the equality holds if and only if $a^n_k(f) = 2^{-j}\, Y^{-n}_k(p^j_{s,t})$ $((k,n) \in \hat{\mathrm{I}}^j)$, i.e. $f = 2^{-j}\, \phi^j_{s,t}$. ∎

Theorem 4.2. *The subspaces V^j $(j \in \mathbb{N}_0)$ form a multiresolution of $L^2(S)$ with the following properties:*

(i) $V^j \subset V^{j+1}$ $(j \in \mathbb{N}_0)$, $\quad \mathrm{clos} \bigcup_{j=0}^{\infty} V^j = L^2(S)\,.$

(ii) $V^j = \mathrm{span}\,\{\phi^j_{s,t} : (s,t) \in \mathrm{I}^j\}\,.$

(iii) *The set* $\{(2^{j-1}\varepsilon^{(j+1)}_s w^{(j+1)}_s)^{1/2}\phi^j_{s,t} : (s,t) \in \mathrm{I}^j\}$ *is a tight frame in* V^j, *i.e., for all* $f \in V^j$ *we have*

$$2^{j-1} \sum_{(s,t)\in \mathrm{I}^j} \varepsilon^{(j+1)}_s w^{(j+1)}_s\, |\langle f, \phi^j_{s,t}\rangle|^2 = \|f\|^2\,.$$

Proof: 1. By definition of V^j, property (i) is straightforward.

2. Any $f \in V^j$ can be represented in the form (4.2), where the Fourier coefficients $a^n_k(f)$ $((k,n) \in \hat{\mathrm{I}}^j)$ can be computed by Theorem 3.1. Together with the definition of $\phi^j_{s,t}$, this results in

$$f = \frac{1}{2} \sum_{(s,t)\in \mathrm{I}^j} \varepsilon^{(j+1)}_s w^{(j+1)}_s f(p^j_{s,t})\, \phi^j_{s,t}\,.$$

Thus (ii) is clear.

3. By the reproducing property of $\phi^j_{s,t}$ the above expression can be replaced by

$$f = 2^{j-1} \sum_{(s,t)\in \mathrm{I}^j} \varepsilon_s^{(j+1)} w_s^{(j+1)} \langle f, \phi^j_{s,t}\rangle \, \phi^j_{s,t} \, .$$

Hence we obtain the final result

$$\|f\|^2 = \langle f, f\rangle \; = \; 2^{j-1} \sum_{(s,t)\in \mathrm{I}^j} \varepsilon_s^{(j+1)} w_s^{(j+1)} \, |\langle f, \phi^j_{s,t}\rangle|^2 \, . \quad \blacksquare$$

The Fourier–transformed *two–scale relation of* ϕ^j reads as follows

$$a_k^n(\phi^j) = A^{j+1}_{k,n} \, a_k^n(\phi^{j+1}) \quad ((k,n) \in \hat{\mathrm{I}}) \tag{4.5}$$

with

$$A^{j+1}_{k,n} \; := \begin{cases} 2 & (k,n) \in \hat{\mathrm{I}}^j, \\ 0 & (k,n) \in \hat{\mathrm{I}} \setminus \hat{\mathrm{I}}^j. \end{cases}$$

§5. Wavelet Spaces

We introduce the *wavelet space* W^j *of level* j $(j \in \mathbb{N}_0)$ as the orthogonal complement of V^j in V^{j+1}, i.e. $V^{j+1} = V^j \oplus W^j$. This implies the orthogonal sum decomposition

$$L^2(S) = V^{j_0} \oplus \bigoplus_{j=j_0}^{\infty} W^j \quad (j_0 \in \mathbb{N}_0) \, .$$

Clearly,

$$\dim\, W^j \; = \; \dim\, V^{j+1} - \dim\, V^j \; = \; 3 \cdot 2^{2j} \, .$$

Let the function ψ^j $(j \in \mathbb{N}_0)$ be defined by

$$2^j \, \psi^j \; := \sum_{(k,n)\in \hat{\mathrm{I}}^{j+1}\setminus \hat{\mathrm{I}}^j} (2k+1) \, Y_k^n$$

and the "rotated" functions $\psi^j_{s,t} \in W^j$ $((s,t) \in \mathrm{I}^{j+1})$ by

$$2^j \, \psi^j_{s,t} \; := \; K^{j+1}(\,\cdot\,, p^{j+1}_{s,t}) - K^j(\,\cdot\,, p^{j+1}_{s,t}) = \sum_{(k,n)\in \hat{\mathrm{I}}^{j+1}\setminus \hat{\mathrm{I}}^j} (2k+1) \, Y_k^n \, Y_k^{-n}(p^{j+1}_{s,t}) \, .$$

By the addition theorem (2.3), the functions $\psi^j_{s,t}$ can be written alternatively as

$$2^j \psi^j_{s,t}(p) \; = \sum_{k=2^j}^{2^{j+1}-1} (2k+1) \, P_k \left(\cos(p \, p^{j+1}_{s,t}) \right) .$$

Then we have

$$a_k^n(\psi^j_{s,t}) \; = \; Y_k^{-n}(p^{j+1}_{s,t}) \, a_k^n(\psi^j) \; = \begin{cases} 2^{-j} \, Y_k^{-n}(p^{j+1}_{s,t}) & (k,n) \in \hat{\mathrm{I}}^{j+1} \setminus \hat{\mathrm{I}}^j, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

The functions $\psi^j_{s,t}$ $((s,t) \in \mathrm{I}^{j+1})$ are smooth on the whole sphere S. Obviously, they are linearly dependent. Moreover, they possess the following important properties:

Theorem 5.1. *For $j \in \mathbb{N}_0$ and $(s,t) \in \mathrm{I}^{j+1}$ it holds:*

(i) *Reproducing property:*

$$2^j \langle f, \psi^j_{s,t} \rangle = f(p^{j+1}_{s,t}) \quad (f \in W^j)\,.$$

(ii) *Orthogonality property:*

$$\langle \phi^j_{u,v}, \psi^j_{s,t} \rangle = 0 \quad ((u,v) \in \mathrm{I}^j)\,.$$

(iii) $\|\psi^j_{s,t}\| = 3^{1/2}\,, \quad \psi^j_{s,t}(p^{j+1}_{s,t}) = 3 \cdot 2^j$.

(iv) *Localization property: $\psi^j_{s,t}$ is localized around $p^{j+1}_{s,t}$, i.e.*

$$\frac{\|\psi^j_{s,t}\|}{\psi^j_{s,t}(p^{j+1}_{s,t})} = 3^{-1/2}\, 2^{-j} = \min\,\{\|f\| : \, f \in W^j,\, f(p^{j+1}_{s,t}) = 1\}\,.$$

Proof: The orthogonality property is a direct consequence of the definition of $\psi^j_{s,t}$. The proof of (i), (iii) and (iv) is omitted here, since it follows exactly the lines of the proof of Theorem 4.1. ■

Theorem 5.2. *We have*

$$W^j = \mathrm{span}\,\{\psi^j_{s,t} \, : \, (s,t) \in \mathrm{I}^{j+1}\}\,.$$

The functions $(2^{j-2}\varepsilon^{(j+2)}_s w^{(j+2)}_s)^{1/2}\psi^j_{s,t}$ $((s,t) \in \mathrm{I}^{j+1})$ constitute a tight frame in W^j, i.e.,

$$2^{j-2} \sum_{(s,t)\in\hat{\mathrm{I}}^{j+1}} \varepsilon^{(j+2)}_s w^{(j+2)}_s \, |\langle f, \psi^j_{s,t}\rangle|^2 = \|f\|^2 \tag{5.2}$$

for all $f \in W^j$.

Proof: Fourier expansion of $f \in W^j$ with Fourier coefficients $a^n_k(f)$ determined by Theorem 3.1 gives

$$\begin{aligned} f &= \sum_{(k,n)\in\hat{\mathrm{I}}^{j+1}\setminus\hat{\mathrm{I}}^j} (2k+1)\, a^n_k(f)\, Y^n_k \qquad (5.3) \\ &= \frac{1}{4} \sum_{(s,t)\in\mathrm{I}^{j+1}} \varepsilon^{(j+2)}_s w^{(j+2)}_s f(p^{j+1}_{s,t})\, \psi^j_{s,t} \end{aligned}$$

such that $W^j = \mathrm{span}\,\{\psi^j_{s,t} \, : \, (s,t) \in \mathrm{I}^{j+1}\}$. Now (5.2) follows as in the proof of Theorem 4.2. ■

The Fourier–transformed *two–scale relation of* ψ^j reads

$$a^n_k(\psi^j) = B^{j+1}_{k,n}\, a^n_k(\phi^{j+1}) \quad ((k,n) \in \hat{\mathrm{I}}) \tag{5.4}$$

with

$$B^{j+1}_{k,n} := \begin{cases} 2 & (k,n) \in \hat{\mathrm{I}}^{j+1} \setminus \hat{\mathrm{I}}^j, \\ 0 & \text{otherwise.} \end{cases}$$

§6. Decomposition and Reconstruction Algorithms

In this section, we derive efficient decomposition and reconstruction algorithms. In order to *decompose* a given function $f^{j+1} \in V^{j+1}$ $(j \in \mathbb{N}_0)$ of the form

$$f^{j+1} = \frac{1}{2} \sum_{(s,t) \in \mathrm{I}^{j+1}} \varepsilon_s^{(j+2)} w_s^{(j+2)} \alpha_{s,t}^{j+1} \, \phi_{s,t}^{j+1} \tag{6.1}$$

with $\alpha_{s,t}^{j+1} := f^{j+1}(p_{s,t}^{j+1})$ $((s,t) \in \mathrm{I}^{j+1})$, the uniquely determined functions $f^j \in V^j$ and $g^j \in W^j$ have to be found such that

$$f^{j+1} = f^j + g^j . \tag{6.2}$$

Assume that the coefficients $\alpha_{s,t}^{j+1} \in \mathbb{C}$ $((s,t) \in \mathrm{I}^{j+1})$ or their *discrete spherical Fourier transform data*

$$\hat{\alpha}_{k,n}^{j+1} := \frac{1}{2^{j+2}} \sum_{(s,t) \in \mathrm{I}^{j+1}} \varepsilon_s^{(j+2)} \, w_s^{(j+2)} \alpha_{s,t}^{j+1} \, Y_k^{-n}(p_{s,t}^{j+1}) \quad ((k,n) \in \hat{\mathrm{I}}^{j+1}) \tag{6.3}$$

are known. The functions $f^j \in V^j$ and $g^j \in W^j$ can be represented by

$$f^j = \frac{1}{2} \sum_{(s,t) \in \mathrm{I}^j} \varepsilon_s^{(j+1)} w_s^{(j+1)} \alpha_{s,t}^j \, \phi_{s,t}^j \, , \tag{6.4}$$

$$g^j = \frac{1}{4} \sum_{(u,v) \in \mathrm{I}^{j+1}} \varepsilon_u^{(j+2)} w_u^{(j+2)} \beta_{u,v}^j \, \psi_{u,v}^j \tag{6.5}$$

with unknown coefficients $\alpha_{s,t}^j$, $\beta_{u,v}^j \in \mathbb{C}$. Let $\hat{\alpha}_{k,n}^j$, $\tilde{\beta}_{l,m}^j \in \mathbb{C}$ denote the following discrete spherical Fourier transform data

$$\hat{\alpha}_{k,n}^j := \frac{1}{2^{j+1}} \sum_{(s,t) \in \mathrm{I}^j} \varepsilon_s^{(j+1)} w_s^{(j+1)} \, \alpha_{s,t}^j \, Y_k^{-n}(p_{s,t}^j) \quad ((k,n) \in \hat{\mathrm{I}}^j) \, , \tag{6.6}$$

$$\tilde{\beta}_{l,m}^j := \frac{1}{2^{j+2}} \sum_{(u,v) \in \mathrm{I}^{j+1}} \varepsilon_u^{(j+2)} w_u^{(j+2)} \, \beta_{u,v}^j \, Y_l^{-m}(p_{u,v}^{j+1}) \quad ((l,m) \in \hat{\mathrm{I}}^{j+1}) \, . \tag{6.7}$$

In order to *reconstruct* $f^{j+1} \in V^{j+1}$ $(j \in \mathbb{N}_0)$, we have to compute the coefficients $\alpha_{s,t}^{j+1}$ of the sum (6.2) with given functions $f^j \in V^j$ and $g^j \in W^j$ (see (6.1)). Assume that the coefficients $\alpha_{s,t}^j$, $\beta_{u,v}^j \in \mathbb{C}$ in (6.4) – (6.5) or their corresponding transformed data (6.6) – (6.7) are known. The decomposition and reconstruction algorithms are based on the following connection between (6.3) and (6.6) – (6.7):

Theorem 6.1. *For $j \in \mathbb{N}_0$, let $f^{j+1} \in V^{j+1}$, $f^j \in V^j$ and $g^j \in W^j$ with (6.1) – (6.7) be given. Then we have*

$$\hat{\alpha}^j_{k,n} = \hat{\alpha}^{j+1}_{k,n} \quad ((k,n) \in \hat{\mathrm{I}}^j),$$
$$\tilde{\beta}^j_{l,m} = \hat{\alpha}^{j+1}_{l,m} \quad ((l,m) \in \hat{\mathrm{I}}^{j+1} \setminus \hat{\mathrm{I}}^j).$$

Proof: From (6.1), it follows by (4.3) and (6.3) that for all $(k,n) \in \hat{\mathrm{I}}^{j+1}$

$$a^n_k(f^{j+1}) = 2^{j+1}\, \hat{\alpha}^{j+1}_{k,n}\, a^n_k(\phi^{j+1}).$$

Analogously, by (4.3), (5.1), (6.4) and (6.5) we have

$$a^n_k(f^j) = 2^j\, \hat{\alpha}^j_{k,n}\, a^n_k(\phi^j) \quad ((k,n) \in \hat{\mathrm{I}}^j), \tag{6.8}$$
$$a^m_l(g^j) = 2^j\, \tilde{\beta}^j_{l,m} a^m_l(\psi^j) \quad ((l,m) \in \hat{\mathrm{I}}^{j+1}). \tag{6.9}$$

From (6.2) it follows that for all $(k,n) \in \hat{\mathrm{I}}^{j+1}$,

$$a^n_k(f^{j+1}) = a^n_k(f^j) + a^n_k(g^j).$$

Using the Fourier–transformed two–scale relations (4.5) and (5.4), we obtain for all $(k,n) \in \hat{\mathrm{I}}^{j+1}$,

$$2\, \hat{\alpha}^{j+1}_{k,n}\, a^n_k(\phi^{j+1}) = \hat{\alpha}^j_{k,n}\, A^{j+1}_{k,n}\, a^n_k(\phi^{j+1}) + \tilde{\beta}^j_{k,n}\, B^{j+1}_{k,n}\, a^n_k(\phi^{j+1}).$$

By $a^n_k(\phi^{j+1}) \neq 0$ $((k,n) \in \hat{\mathrm{I}}^{j+1})$, this yields the assertion. ■

As consequence we obtain the following procedures:

Algorithm 6.2 (Decomposition Algorithm)

Input: For fixed $j \in \mathbb{N}_0$, let $\hat{\alpha}^{j+1}_{k,n} \in \mathbb{C}$ $((k,n) \in \hat{\mathrm{I}}^{j+1})$ be given.

Set

$$\hat{\alpha}^j_{k,n} := \hat{\alpha}^{j+1}_{k,n} \quad ((k,n) \in \hat{\mathrm{I}}^j),$$
$$\tilde{\beta}^j_{l,m} := \hat{\alpha}^{j+1}_{l,m} \quad ((l,m) \in \hat{\mathrm{I}}^{j+1} \setminus \hat{\mathrm{I}}^j).$$

Output: $\hat{\alpha}^j_{k,n} \in \mathbb{C}$ $((k,n) \in \hat{\mathrm{I}}^j)$, $\tilde{\beta}^j_{l,m} \in \mathbb{C}$ $((l,m) \in \hat{\mathrm{I}}^{j+1} \setminus \hat{\mathrm{I}}^j)$.

Algorithm 6.3 (Reconstruction Algorithm)

Input: For fixed $j \in \mathbb{N}_0$, let $\hat{\alpha}^j_{k,n} \in \mathbb{C}$ $((k,n) \in \hat{\mathrm{I}}^j)$ and $\tilde{\beta}^j_{l,m} \in \mathbb{C}$ $((l,m) \in \hat{\mathrm{I}}^{j+1} \setminus \hat{\mathrm{I}}^j)$ be given.

Set

$$\hat{\alpha}^{j+1}_{k,n} := \hat{\alpha}^j_{k,n} \qquad ((k,n) \in \hat{\mathrm{I}}^j),$$
$$\hat{\alpha}^{j+1}_{l,m} := \tilde{\beta}^j_{l,m} \qquad ((k,n) \in \hat{\mathrm{I}}^{j+1} \setminus \hat{\mathrm{I}}^j).$$

Output: $\hat{\alpha}^{j+1}_{k,n} \in \mathbb{C}$ $((k,n) \in \hat{\mathrm{I}}^{j+1})$.

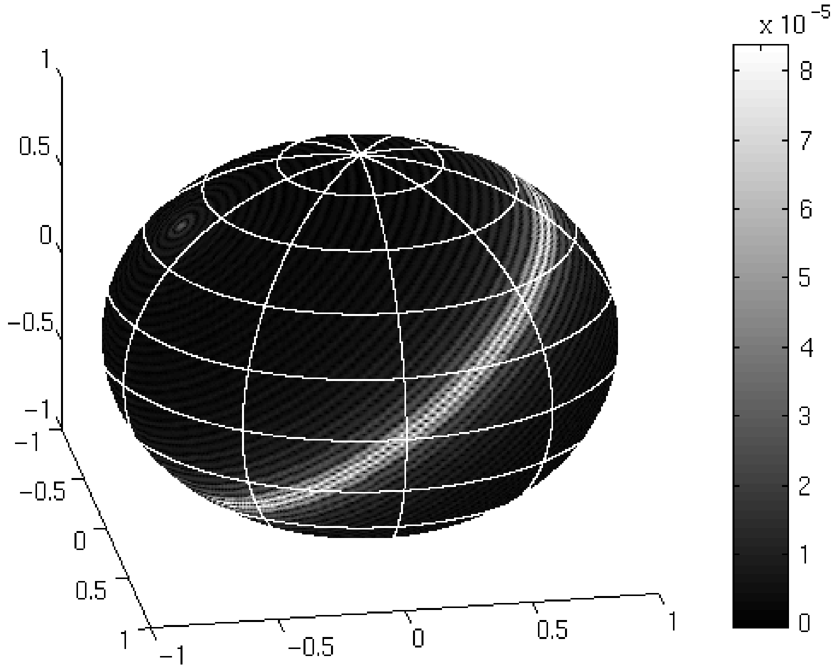

Fig. 1. Wavelet part of f_1 in W^6.

§7. Numerical Tests

Finally, we present numerical tests of the described decomposition algorithm. Consider the ellipsoid

$$x_1^2 + x_2^2 + 4\,x_3^2 = 1 \quad ((x_1, x_2, x_3) \in \mathbb{R}^3)\,.$$

Using spherical coordinates (θ, φ, r) of $\mathbb{R}^3$, we get $r^2\,(1 + 3\cos^2\theta) = 1$. Then we introduce the function $f : S \to \mathbb{R}$ defined by

$$f(\theta, \varphi) \;:=\; \begin{cases} 1 & \theta \in [0, \pi/2], \\ (1 + 3\cos^2\,\theta)^{-1/2} & \theta \in (\pi/2, \pi]. \end{cases}$$

Hence f is the union of a half–sphere and a half–ellipsoid. This function f is smooth on S without the equator $\{(\pi/2, \varphi) : \varphi \in [0, 2\pi)\}$. If we rotate the function f around the x_1–axis by the angle $\pi/4$ and $\pi/2$, we obtain the functions f_1 and f_2, respectively.

By the decomposition algorithm 6.2, we analyze the local regularity of f_1 and f_2. In order to compute values of the related wavelet parts on a grid of S, we use the representation (5.3). As shown in Figures 1 and 2, the corresponding wavelet parts in W^6 describe the smoothness of f_1 and f_2 in the right way.

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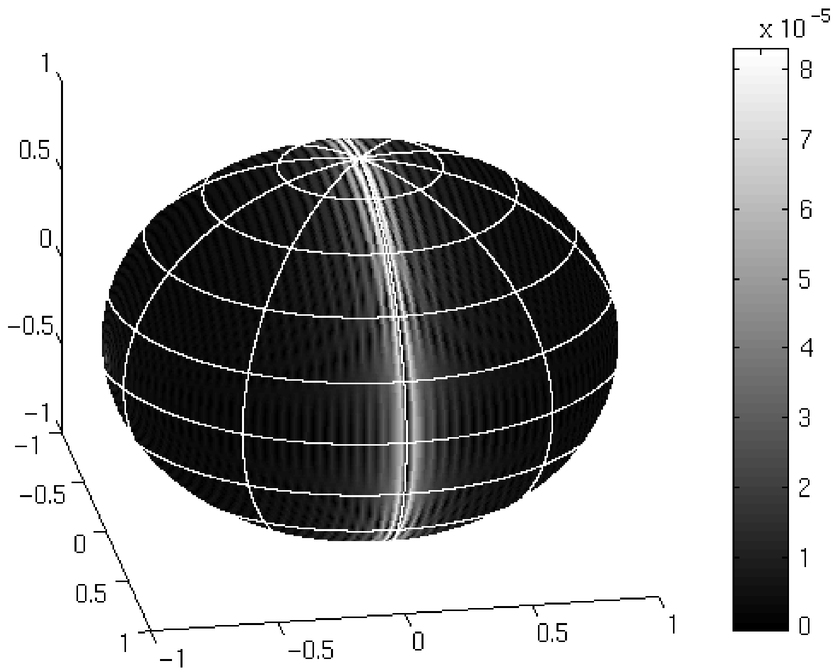

Fig. 2. Wavelet part of f_2 in W^6.

References

1. Chihara, T.S., *An Introduction to Orthogonal Polynomials*, Gordon and Breach Science Publ., New York, 1978.
2. Chui, C.K., and H.N. Mhaskar, On trigonometric wavelets, Constr. Approx. **9** (1993), 167–190.
3. Dahlke, S., W. Dahmen, E. Schmitt, and I. Weinreich, Multiresolution analysis and wavelets on S^2 and S^3, preprint, RWTH Aachen, 1994.
4. Driscoll, J.R., and D.M. Healy, Jr., Computing Fourier transforms and convolutions on the 2–sphere, Adv. in Appl. Math. **15** (1994), 202–250.
5. Fischer, B., and J. Prestin, Wavelets based on orthogonal polynomials, preprint, Medizinische Universität zu Lübeck, 1996.
6. Freeden, W., and U. Windheuser, Combined spherical harmonic and wavelet expansion, preprint, Universität Kaiserslautern, 1995.
7. Lyche, T., and L.L. Schumaker, L–spline wavelets, in *Wavelets: Theory, Algorithms and Applications*, C.K. Chui, L. Montefusco, and L. Puccio (eds.), Academic Press, San Diego, 1994, 197–212.
8. Müller, C., *Spherical Harmonics*, Springer–Verlag, Berlin, 1966.
9. Narcowich, F.J., and J.D. Ward, Nonstationary wavelets on the m–sphere for scattered data, preprint, Texas A&M University, 1995.

10. Potts, D., and M. Tasche, Interpolatory wavelets on the sphere, in *Approximation Theory* VIII, Vol. 2: *Wavelets and Multilevel Approximation*, C.K. Chui, and L.L. Schumaker (eds.), World Scientific Publ., Singapore, 1995, 335–342.
11. Tasche, M., Polynomial wavelets on $[-1, 1]$, in *Approximation Theory, Wavelets and Applications*, S.P. Singh (ed.), Kluwer Academic Publ., Dordrecht, 1995, 497–512.
12. Schröder, P., and W. Sweldens, Spherical wavelets: Efficiently representing functions on the sphere, preprint, University of South Carolina, 1995.

Daniel Potts
Fachbereich Mathematik, Universität Rostock
D – 18051 Rostock
Germany
daniel.potts@stud.uni-rostock.de

Gabriele Steidl
Fachbereich Mathematik, TH Darmstadt
Schloßgartenstr. 7
D – 64289 Darmstadt
Germany
steidl@mathematik.th-darmstadt.de

Manfred Tasche
Institut für Mathematik, Medizinische Universität zu Lübeck
Wallstr. 40
D – 23560 Lübeck
Germany
manfred.tasche@informatik.mu-luebeck.de