On numerical realizations of Shannon's sampling theorem

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In this paper, we discuss some numerical realizations of Shannon's sampling theorem. First we show the poor convergence of classical Shannon sampling sums by presenting sharp upper and lower bounds of the norm of the Shannon sampling operator. In addition, it is known that in the presence of noise in the samples of a bandlimited function, the convergence of Shannon sampling series may even break down completely. To overcome these drawbacks, one can use oversampling and regularization with a convenient window function. Such a window function can be chosen either in frequency domain or in time domain. We especially put emphasis on the comparison of these two approaches in terms of error decay rates. It turns out that the best numerical results are obtained by oversampling and regularization in time domain using a sinh-type window function or a continuous Kaiser–Bessel window function, which results in an interpolating approximation with localized sampling. Several numerical experiments illustrate the theoretical results.

Key words: Shannon sampling sums, Whittaker–Kotelnikov–Shannon sampling theorem, bandlimited function, regularization with window function, regularized Shannon sampling formulas, error estimates, numerical robustness.

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1 Introduction

The classical Whittaker–Kotelnikov–Shannon sampling theorem plays a fundamental role in signal processing, since it describes the close relation between a bandlimited function and its equidistant samples. A function $f \in L^2(\mathbb{R})$ is called bandlimited with bandwidth $\frac{N}{2}$, if the support of its Fourier transform

$$\hat{f}(v) := \int_{\mathbb{R}} f(t)\, \mathrm{e}^{-2\pi \mathrm{i} t v}\, \mathrm{d}t\,, \quad v \in \mathbb{R}\,,$$

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is contained in $\left[-\frac{N}{2}, \frac{N}{2}\right]$. Let the space of all bandlimited functions with bandwidth $\frac{N}{2}$ be denoted by

$$\mathcal{B}_{N/2}(\mathbb{R}) := \Big\{f \in L^2(\mathbb{R})\colon \operatorname{supp} \hat f \subseteq \left[-\tfrac{N}{2}, \tfrac{N}{2}\right]\Big\}\,.$$

Then the sampling theorem states that any function $f \in \mathcal{B}_{N/2}(\mathbb{R})$ can be recovered from its samples $f\big(\frac{k}{L}\big)$, $k \in \mathbb{Z}$, with $L \geq N$ as

$$f(t) = \sum_{k\in\mathbb{Z}} f\big(\tfrac{k}{L}\big)\,\mathrm{sinc}(L\pi t - \pi k)\,, \quad t \in \mathbb{R}\,, \tag{1.1}$$

where the sinc function is given by

$$\mathrm{sinc}\, x := \begin{cases} \frac{\sin x}{x} & : x \in \mathbb{R}\setminus\{0\}\,, \\ 1 & : x = 0\,. \end{cases}$$

It is well known that the series in (1.1) converges absolutely and uniformly on whole $\mathbb{R}$. Unfortunately, the practical use of this sampling theorem is limited, since it requires infinitely many samples, which is impossible in practice. Furthermore, the sinc function decays very slowly such that the *Shannon sampling series*

$$\sum_{k\in\mathbb{Z}} f\big(\tfrac{k}{L}\big)\,\mathrm{sinc}(L\pi t - k\pi)\,, \quad t \in \mathbb{R}\,, \tag{1.2}$$

with $L \geq N$ has rather poor convergence, as can be seen from the sharp upper and lower bounds of the norm of the Shannon sampling operator (see Theorem 2.2). In addition, it is known (see [7]) that in the presence of noise in the samples $f\big(\frac{\ell}{L}\big)$, $\ell \in \mathbb{Z}$, of a bandlimited function $f \in \mathcal{B}_{N/2}(\mathbb{R})$, the convergence of Shannon sampling series may even break down completely. To overcome these drawbacks, many applications employ *oversampling*, i. e., a function $f \in L^2(\mathbb{R})$ of bandwidth $\frac{N}{2}$ is sampled on a finer grid $\frac{1}{L}\mathbb{Z}$ with $L > N$, where the oversampling is measured by the *oversampling parameter* $\lambda := \frac{L-N}{N} \geq 0$. In addition, we consider various regularization techniques, where a so-called *window function* is used. Since this window function can be chosen in frequency domain or in spatial domain, we study both approaches and compare the theoretical and numerical approximation properties in terms of decay rates.

On the one hand, we investigate the regularization with a window function in frequency domain (called *frequency window function*), cf. e. g. [6, 12, 21, 14, 24]. Here we use a suitable function of the form

$$\hat\psi(v) := \begin{cases} 1 & : |v| \leq \frac{N}{2}\,, \\ \chi(|v|) & : \frac{N}{2} < |v| < \frac{L}{2}\,, \\ 0 & : |v| \geq \frac{L}{2}\,, \end{cases}$$

where $\chi\colon \left[\frac{N}{2}, \frac{L}{2}\right] \to [0, 1]$ is frequently chosen as some monotonously decreasing, continuous function with $\chi\big(\frac{N}{2}\big) = 1$ and $\chi\big(\frac{L}{2}\big) = 0$. Applying inverse Fourier transform, we determine the corresponding function ψ in time domain. Since $\hat\psi$ is compactly supported, the uncertainty principle (see [15, Lemma 2.31]) yields $\operatorname{supp}\psi = \mathbb{R}$. Then it is known that the function f can be represented in the form

$$f(t) = \sum_{k\in\mathbb{Z}} f\big(\tfrac{k}{L}\big)\,\frac{1}{L}\,\psi\big(t - \tfrac{k}{L}\big)\,, \quad t \in \mathbb{R}\,.$$

Using uniform truncation, we approximate a function $f \in \mathcal{B}_{N/2}(\mathbb{R})$ by the T-th partial sum

$$(P_{\psi,T}f)(t) := \sum_{k=-T}^{T} f\big(\tfrac{k}{L}\big)\,\frac{1}{L}\,\psi\big(t-\tfrac{k}{L}\big)\,, \quad t \in [-1,\,1]\,.$$

On the other hand, we examine the regularization with a window function in time domain (called *time window function*), cf. e. g. [18, 10, 11, 9]. Here a suitable window function $\varphi\colon \mathbb{R} \to [0,\,1]$ with compact support $\big[-\frac{m}{L},\,\frac{m}{L}\big]$ belongs to the set $\Phi_{m,L}$ (as defined in Section 4) with some $m \in \mathbb{N} \setminus \{1\}$. Then we recover a function $f \in \mathcal{B}_{N/2}(\mathbb{R})$ by the *regularized Shannon sampling formula*

$$(R_{\varphi,m}f)(t) := \sum_{k\in\mathbb{Z}} f\big(\tfrac{k}{L}\big)\,\mathrm{sinc}\big(L\pi t - \pi k\big)\,\varphi\big(t-\tfrac{k}{L}\big)\,, \quad t \in \mathbb{R}\,,$$

with $L \geq N$. By defining the set $\Phi_{m,L}$ of window functions φ, only small truncation parameters m are needed to achieve high accuracy, resulting in short sums being evaluable very fast. In other words, this approach uses localized sampling. Moreover, this method is an interpolating approximation, since for all $n,\,k \in \mathbb{Z}$ we have

$$\mathrm{sinc}\big(L\pi t - \pi k\big)\,\varphi\big(t-\tfrac{k}{L}\big)\big|_{t=\frac{n}{L}} = \delta_{n,k}\,.$$

In this paper we propose new estimates of the uniform approximation errors

$$\max_{t\in[-1,\,1]} \big|f(t) - (P_{\psi,T}f)(t)\big| \quad \text{and} \quad \max_{t\in\mathbb{R}} \big|f(t) - (R_{\varphi,m}f)(t)\big|\,,$$

where we apply several window functions $\hat{\psi}$ and φ. It is shown in the subsequent sections that the uniform approximation error decays algebraically with respect to T, if $\hat{\psi}$ is a frequency window function. Otherwise, if $\varphi \in \Phi_{m,L}$ is chosen as a time window function such as the sinh-type or continuous Kaiser-Bessel window function, then the uniform approximation error decays exponentially with respect to m.

To this end, this paper is organized as follows. First, in Section 2 we show the poor convergence of classical Shannon sampling sums and improve results on the upper and lower bounds of the norm of the Shannon sampling operator. Consequently, we study the different regularization techniques. In Section 3 we start with the regularization using a frequency window function. After recapitulating a general result in Theorem 3.3, we consider window functions of different regularity and present the corresponding algebraic decay results in Theorems 3.4 and 3.7. Subsequently, in Section 4 we proceed with the regularization using a time window function. Here we also review the known general result in Theorem 4.1 and afterwards demonstrate the exponential decay of the considered sinh-type and Kaiser-Bessel window functions in Theorems 4.2 and 4.3. Finally, in Section 5 we compare the previously considered approaches from Sections 3 and 4 to illustrate our theoretical results.

2 Poor convergence of Shannon sampling sums

Let $C_0(\mathbb{R})$ denote the Banach space of continuous functions $f\colon \mathbb{R} \to \mathbb{C}$ vanishing as $|t| \to \infty$ with norm

$$\|f\|_{C_0(\mathbb{R})} := \max_{t\in\mathbb{R}} |f(t)|\,.$$

In order to show that the Shannon sampling series (1.2) has rather poor convergence, we truncate the series (1.2) with $T \in \mathbb{N}$, and consider the *T-th Shannon sampling sum*

$$(S_T f)(t) := \sum_{k=-T}^{T} f\big(\tfrac{k}{L}\big)\, \mathrm{sinc}(L\pi t - k\pi)\,, \quad t \in \mathbb{R}\,.$$

Obviously, this operator can be formed for each $f \in C_0(\mathbb{R})$.

Lemma 2.1. *The linear operator $S_T \colon C_0(\mathbb{R}) \to C_0(\mathbb{R})$ has the norm*

$$\|S_T\| = \max_{t\in\mathbb{R}} \sum_{k=-T}^{T} \big|\mathrm{sinc}(L\pi t - k\pi)\big|\,. \tag{2.1}$$

Proof. For each $f \in C_0(\mathbb{R})$ and $t \in \mathbb{R}$ we have

$$\big|(S_T f)(t)\big| \le \sum_{k=-T}^{T} \big|f\big(\tfrac{k}{L}\big)\big|\, \big|\mathrm{sinc}(L\pi t - k\pi)\big| \le \sum_{k=-T}^{T} \big|\mathrm{sinc}(L\pi t - k\pi)\big|\, \|f\|_{C_0(\mathbb{R})}\,,$$

such that

$$\|S_T f\|_{C_0(\mathbb{R})} \le \max_{t\in\mathbb{R}} \sum_{k=-T}^{T} \big|\mathrm{sinc}(L\pi t - k\pi)\big|\, \|f\|_{C_0(\mathbb{R})}\,.$$

By defining the even nonnegative function

$$s_T(t) := \sum_{k=-T}^{T} \big|\mathrm{sinc}(L\pi t - k\pi)\big|\,, \quad t \in \mathbb{R}\,,$$

which is contained in $C_0(\mathbb{R})$, and assuming that s_T has its maximum in $t_0 \in \mathbb{R}$, this yields

$$\|S_T\| = \sup\big\{\|S_T f\|_{C_0(\mathbb{R})} \colon \|f\|_{C_0(\mathbb{R})} = 1\big\} \le s_T(t_0)\,.$$

The other way around, we consider the linear spline $g \in C_0(\mathbb{R})$ with nodes in $\frac{1}{L}\,\mathbb{Z}$, where

$$g\big(\tfrac{k}{L}\big) = \begin{cases} \mathrm{sign}\big(\mathrm{sinc}(L\pi t_0 - k\pi)\big) & : k = -T\,, \ldots, T\,, \\ 0 & : k \in \mathbb{Z} \setminus \{-T\,, \ldots, T\}\,. \end{cases}$$

Obviously, we have $\|g\|_{C_0(\mathbb{R})} = 1$ and

$$(S_T g)(t) = \sum_{k=-T}^{T} \mathrm{sign}\big(\mathrm{sinc}(L\pi t_0 - k\pi)\big)\, \mathrm{sinc}(L\pi t - k\pi)\big) \le s_T(t) \le s_T(t_0)\,.$$

Then

$$(S_T g)(t_0) = \sum_{k=-T}^{T} \big|\mathrm{sinc}(L\pi t_0 - k\pi)\big| = \max_{t\in\mathbb{R}} s_T(t) = s_T(t_0)$$

implies

$$\|S_T\| \ge \|S_T g\|_{C_0(\mathbb{R})} = \max_{t\in\mathbb{R}} \big|(S_T g)(t)\big| = s_T(t_0)$$

and hence (2.1). ■

Now we show that the norm $\|S_T\|$ is unbounded with respect to T. Here we use *Euler's constant*

$$\gamma := \lim_{T\to\infty} \left(\sum_{k=1}^{T} \frac{1}{k} - \ln T \right) = 0.57721566\ldots .$$

In the following, we improve a former result of [23, p. 142].

Theorem 2.2. *The norm of the operator $S_T\colon C_0(\mathbb{R}) \to C_0(\mathbb{R})$ can be estimated by*

$$\frac{2}{\pi}\big[\ln T + 2\ln 2 + \gamma\big] - \frac{1}{\pi T\,(2T+1)} < \|S_T\| < \frac{2}{\pi}\big[\ln T + 2\ln 2 + \gamma\big] + \frac{T+2}{\pi T\,(T+1)} . \tag{2.2}$$

Proof. As in [23, p. 142] we represent $s_T(t)$ in the form

$$s_T(t) = \sum_{k=1}^{T+1} a_k(t)\,, \quad t \in \mathbb{R}\,,$$

with

$$a_k(t) := \begin{cases} \big|\mathrm{sinc}(L\pi t - k\pi)\big| + \big|\mathrm{sinc}(L\pi t + (k-1)\pi)\big| & : k = 1\,, \ldots, T\,, \\ \big|\mathrm{sinc}(L\pi t + T\pi)\big| & : k = T+1\,. \end{cases}$$

Since $s_T(t)$ is even, we estimate the maximum of $s_T(t)$ only for $t \ge 0$. For $a_1(t)$ with $t \in \big(0,\, \frac{1}{L}\big)$ we have by trigonometric identities that

$$a_1(t) = \frac{\sin(L\pi t)}{\pi} \left(\frac{1}{Lt} + \frac{1}{1 - Lt} \right) = \frac{\sin(L\pi t)}{\pi L t\,(1 - Lt)} . \tag{2.3}$$

By Schur's expansion of $\sin(\pi x)$, see [4], we know that

$$\sin(L\pi t) = \sum_{n=1}^{\infty} \frac{1}{n!}\,\alpha_n\,(Lt)^n\,(1 - Lt)^n\,, \quad t \in \big[0,\, \tfrac{1}{L}\big]\,,$$

with positive coefficients

$$\alpha_n = \frac{\pi}{(n-1)!} \int_0^{\pi/2} t^{n-1}\,(\pi - t)^{n-1}\, \sin t \,\mathrm{d}t\,, \quad n \in \mathbb{N}\,.$$

Thus, we obtain the expansion

$$a_1(t) = 1 + \frac{1}{\pi} \sum_{n=2}^{\infty} \frac{1}{n!}\,\alpha_n\,(Lt)^{n-1}\,(1 - Lt)^{n-1}\,, \quad t \in \big[0,\, \tfrac{1}{L}\big]\,.$$

Hence, the function $a_1\colon \big[0,\, \frac{1}{L}\big] \to \mathbb{R}$ is concave and has its maximum at $t = \frac{1}{2L}$, i.e., by (2.3) we compute

$$\max_{t\in[0,\,1/L]} a_1(t) = a_1\big(\tfrac{1}{2L}\big) = \frac{4}{\pi}\,.$$

For $a_k(t)$, $k = 2, \ldots, T$, with $t \in \big[0,\, \frac{1}{L}\big]$ we have

$$a_k(t) = \frac{\sin(L\pi t)}{\pi} \left(\frac{1}{k - 1 + Lt} + \frac{1}{k - Lt} \right) = \frac{(2k-1)\,\sin(L\pi t)}{\pi\,\big[(k-1)k + Lt\,(1 - Lt)\big]}\,.$$

We define the functions $b_k\colon [0, 1] \to \mathbb{R}$, $k = 2, \ldots, T$, via

$$b_k(x) := \frac{(2k-1)}{\pi}\, a_k\big(\tfrac{x}{L}\big) = \frac{\sin(\pi x)}{(k-1)k + x(1-x)}\,, \quad x \in [0, 1]\,,$$

such that $b_k(0) = b_k(1) = 0$ and the symmetry relation $b_k(x) = b_k(1-x)$ is fulfilled, i.e., each b_k is symmetric with reference to $\frac{1}{2}$. Furthermore, by $b_k'(x) \geq 0$ for $x \in \big[0, \frac{1}{2}\big]$, the function b_k is increasing on $\big[0, \frac{1}{2}\big]$ and therefore has its maximum at $x = \frac{1}{2}$. Thus, the function $a_k\colon \big[0, \frac{1}{L}\big] \to \mathbb{R}$ has its maximum at $t = \frac{1}{2L}$, i.e.,

$$\max_{t\in[0,\,1/L]} a_k(t) = a_k\big(\tfrac{1}{2L}\big) = \frac{4}{(2k-1)\pi}\,.$$

Since $a_{T+1}(t)$ can be written as

$$a_{T+1}(t) = \frac{\sin(L\pi t)}{\pi\,(T + Lt)}$$

for $t \in \big[0, \frac{1}{L}\big]$, we obtain

$$0 < \max_{t\in[0,\,1/L]} a_{T+1}(t) < \frac{1}{\pi T}\,.$$

In the case $T \gg 1$, the function $a_{T+1}\colon \big[0, \frac{1}{L}\big] \to \mathbb{R}$ has its maximum close to $t = \frac{1}{2L}$. Hence, in summary this yields

$$\frac{4}{\pi}\sum_{k=1}^{T}\frac{1}{2k-1} < \max_{t\in[0,\,1/L]} s_T(t) < \frac{4}{\pi}\sum_{k=1}^{T}\frac{1}{2k-1} + \frac{1}{\pi T}\,.$$

For $t \in \big[\frac{n}{L}, \frac{n+1}{L}\big]$ with arbitrary $n \in \mathbb{N}$, the sum $s_T(t)$ is less than it is for $t \in \big[0, \frac{1}{L}\big]$, since for each $n \in \mathbb{N}$ and $t \in \big(0, \frac{1}{L}\big)$ we have

$$\sum_{k=-T}^{T}\frac{\sin(L\pi t)}{|L\pi t - (k-n)\pi|} < \sum_{k=-T}^{T}\frac{\sin(L\pi t)}{|L\pi t - k\pi|}$$

and therefore

$$s_T\big(\tfrac{n}{L} + t\big) < s_T(t)\,.$$

Thus, for the even function $s_T(t)$ we obtain

$$\frac{4}{\pi}\sum_{k=1}^{T}\frac{1}{2k-1} < \max_{t\in\mathbb{R}} s_T(t) < \frac{4}{\pi}\sum_{k=1}^{T}\frac{1}{2k-1} + \frac{1}{\pi T}\,.$$

By Lemma 2.1 this can also be written as

$$\frac{4}{\pi}\sum_{k=1}^{T}\frac{1}{2k-1} < \|S_T\| < \frac{4}{\pi}\sum_{k=1}^{T}\frac{1}{2k-1} + \frac{1}{\pi T}\,. \tag{2.4}$$

Note that for $T \gg 1$ the value

$$s_T\big(\tfrac{1}{2L}\big) = \frac{4}{\pi}\sum_{k=1}^{T}\frac{1}{2k-1} + \frac{2}{\pi\,(2T+1)} \tag{2.5}$$

is a good approximation of the norm $\|S_T\|$.

Now we estimate $\|S_T\|$ by $\ln T$. For this purpose we denote the *T-th harmonic number* by

$$H_T := \sum_{k=1}^{T} \frac{1}{k}\,, \quad T \in \mathbb{N}\,,$$

such that

$$\sum_{k=1}^{T} \frac{1}{2k-1} = \sum_{k=1}^{T} \left(\frac{1}{2k-1} + \frac{1}{2k}\right) - \sum_{k=1}^{T} \frac{1}{2k} = H_{2T} - \frac{1}{2}\,H_T\,. \tag{2.6}$$

Using *Euler's constant*

$$\gamma = \lim_{T\to\infty} \big(H_T - \ln T\big)\,,$$

the estimates

$$\frac{1}{2T+2} < H_T - \ln T - \gamma < \frac{1}{2T} \tag{2.7}$$

are known (see [27]). From (2.6) and (2.7) we conclude that

$$\frac{1}{2}\,\ln T + \ln 2 + \frac{1}{2}\,\gamma - \frac{1}{4T\,(2T+1)} < H_{2T} - \frac{1}{2}\,H_T < \frac{1}{2}\,\ln T + \ln 2 + \frac{1}{2}\,\gamma + \frac{1}{4T\,(T+1)}\,. \tag{2.8}$$

Therefore, applying (2.4), (2.6), and (2.8) yields the assertion (2.2). ■

We remark that Lemma 2.2 immediately implies

$$\lim_{T\to\infty} \left(\|S_T\| - \frac{2}{\pi}\,\ln T\right) = \frac{4}{\pi}\,\ln 2 + \frac{2\gamma}{\pi}\,.$$

Now let $f \in \mathcal{B}_{N/2}(\mathbb{R})$ be a given bandlimited function with bandwidth $\frac{N}{2}$. Then this function possesses a smooth representation, which we will use in the following. Let $T \in \mathbb{N}$ be sufficiently large. For given samples $f\big(\frac{k}{L}\big)$ with $k \in \mathbb{Z}$ and $L \geq N$ we consider finitely many erroneous samples

$$\tilde f_k := \begin{cases} f\big(\frac{k}{L}\big) + \varepsilon_k & : k = -T,\,\ldots,\,T\,,\\ f\big(\frac{k}{L}\big) & : k \in \mathbb{Z}\setminus\{-T,\,\ldots,\,T\}\,, \end{cases}$$

with error terms ε_k, which are bounded by $|\varepsilon_k| \leq \varepsilon$ for $k = -T,\,\ldots,\,T$. Then we reconstruct $f(t)$ by its approximation

$$\tilde f(t) := \sum_{k\in\mathbb{Z}} \tilde f_k\,\mathrm{sinc}(L\pi t - k\pi) = f(t) + \sum_{k=-T}^{T} \varepsilon_k\,\mathrm{sinc}(L\pi t - k\pi)\,, \quad t \in \mathbb{R}\,.$$

Note that by (2.3) we are given the upper error bound

$$\|\tilde f - f\|_{C_0(\mathbb{R})} \leq \varepsilon\,\max_{t\in\mathbb{R}} s_T(t) < \frac{4\varepsilon}{\pi}\sum_{k=1}^{T} \frac{1}{2k-1} + \frac{\varepsilon}{\pi T}\,. \tag{2.9}$$

Next, we present a lower error bound.

Theorem 2.3. *Let $f \in \mathcal{B}_{N/2}(\mathbb{R})$ be an arbitrary bandlimited function with bandwidth $\frac{N}{2}$. Further let $L \ge N$, $T \in \mathbb{N}$, and $\varepsilon > 0$ be given. Then for the special error terms*

$$\varepsilon_k = \varepsilon \operatorname{sign}\big(\operatorname{sinc}(\tfrac{\pi}{2} - k\pi)\big) = \varepsilon\,(-1)^{k+1} \operatorname{sign}(2k-1)\,, \quad k = -T, \ldots, T\,,$$

we have

$$\|\tilde f - f\|_{C_0(\mathbb{R})} \ge \varepsilon \left(\frac{2}{\pi}\,\ln T + \frac{4}{\pi}\,\ln 2 + \frac{2\gamma}{\pi}\right) > \varepsilon \left(\frac{2}{\pi}\,\ln T + \tfrac{5}{4}\right), \tag{2.10}$$

such that the Shannon sampling series is not numerically robust.

Proof. Due to the special choice of the error terms ε_k we obtain

$$\tilde f(t) - f(t) = \varepsilon \sum_{k=-T}^{T} \operatorname{sign}\big(\operatorname{sinc}(\tfrac{\pi}{2} - k\pi)\big)\, \operatorname{sinc}(L\pi t - k\pi)\,, \quad t \in \mathbb{R}\,. \tag{2.11}$$

By (2.5) and (2.8) we conclude that

$$\begin{aligned}
\|\tilde f - f\|_{C_0(\mathbb{R})} &\ge \big|\tilde f(\tfrac{1}{2L}) - f(\tfrac{1}{2L})\big| = \varepsilon \sum_{k=-T}^{T} \big|\operatorname{sinc}(\tfrac{\pi}{2} - k\pi)\big| = \varepsilon\, s_T\big(\tfrac{1}{2L}\big)\\
&= \varepsilon\,\frac{4}{\pi} \sum_{k=1}^{T} \frac{1}{2k-1} + \frac{2\varepsilon}{(2T+1)\pi} > \varepsilon \left(\frac{2}{\pi}\,\ln T + \frac{4}{\pi}\,\ln 2 + \frac{2\gamma}{\pi}\right) + \varepsilon\,\frac{2T-1}{(2T+1)T\,\pi}\\
&> \varepsilon \left(\frac{2}{\pi}\,\ln T + \frac{4}{\pi}\,\ln 2 + \frac{2\gamma}{\pi}\right).
\end{aligned}$$

Note that

$$\frac{4}{\pi}\,\ln 2 + \frac{2\gamma}{\pi} = 1.2500093\ldots > \tfrac{5}{4}\,.$$

This completes the proof. ∎

By Theorem 2.3 we improve a corresponding remark of [7]. Note that the norm $\|\tilde f - f\|_{C_0(\mathbb{R})}$ does not depend on the special choice of the function f. However, since $T \in \mathbb{N}$ may be large, this error behavior is not satisfactory, cf. Figure 2.1. We remark that Figure 2.1 also illustrates that the norm $\|\tilde f - f\|_{C_0(\mathbb{R})}$ seems to be independent of the oversampling parameter λ.

3 Regularization with a frequency window function

To overcome the drawbacks of poor convergence and numerical instability, one can apply regularization with a convenient window function either in the frequency domain or in the time domain. Often one employs *oversampling*, i. e., a bandlimited function $f \in \mathcal{B}_{N/2}(\mathbb{R})$ of bandwidth $\frac{N}{2}$ is sampled on a finer grid $\frac{1}{L}\,\mathbb{Z}$ with $L = N\,(1+\lambda)$, where $\lambda > 0$ is the oversampling parameter.

First, together with oversampling, we consider the *regularization with a frequency window function* of the form

$$\hat\psi(v) := \begin{cases} 1 & : |v| \le \frac{N}{2}\,, \\ \chi(|v|) & : \frac{N}{2} < |v| < \frac{L}{2}\,, \\ 0 & : |v| \ge \frac{L}{2}\,, \end{cases} \tag{3.1}$$

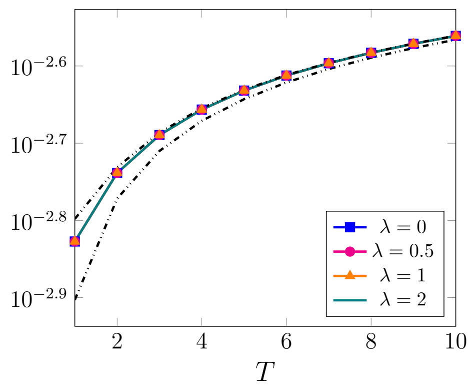


Figure 2.1: The norm $\|\tilde{f} - f\|_{C_0(\mathbb{R})}$ of (2.11) as well as its lower/upper bounds (2.10) and (2.9) for several $L = N(1+\lambda)$, $\lambda \in \{0, 0.5, 1, 2\}$, and $T \in \{1, \ldots, 10\}$, where $N = 128$ and $\varepsilon = 10^{-3}$ is chosen.

cf. [6, 14, 24], where $\chi\colon \big[\frac{N}{2}, \frac{L}{2}\big] \to [0, 1]$ is frequently chosen as some monotonously decreasing, continuous function with $\chi\big(\frac{N}{2}\big) = 1$ and $\chi\big(\frac{L}{2}\big) = 0$. Applying the inverse Fourier transform, we determine the corresponding function in time domain as

$$\psi(t) = \int_{\mathbb{R}} \hat{\psi}(v)\, \mathrm{e}^{2\pi \mathrm{i}\, vt}\, \mathrm{d}v = 2 \int_0^{L/2} \hat{\psi}(v)\, \cos(2\pi\, vt)\, \mathrm{d}v\,. \tag{3.2}$$

Example 3.1. A simple example of a frequency window function is the *linear frequency window function* (cf. [6, pp. 18–19] or [14, pp. 210–212])

$$\hat{\psi}_{\mathrm{lin}}(v) := \begin{cases} 1 & : |v| \le \frac{N}{2}\,, \\ 1 - \frac{2|v|-N}{L-N} & : \frac{N}{2} < |v| < \frac{L}{2}\,, \\ 0 & : |v| \ge \frac{L}{2}\,. \end{cases} \tag{3.3}$$

Obviously, $\hat{\psi}_{\mathrm{lin}}(v)$ is a continuous linear spline supported on $\big[-\frac{L}{2}, \frac{L}{2}\big]$, see Figure 3.1 (a). By (3.2) we receive $\psi_{\mathrm{lin}}(0) = \frac{N+L}{2}$. For $t \in \mathbb{R} \setminus \{0\}$ we obtain

$$\begin{aligned} \psi_{\mathrm{lin}}(t) &= 2 \int_0^{N/2} \cos(2\pi\, vt)\, \mathrm{d}v + 2 \int_{N/2}^{L/2} \left(1 - \frac{2v - N}{L - N}\right) \cos(2\pi\, vt)\, \mathrm{d}v \\ &= \frac{1}{(L-N)\,(\pi t)^2} \big(\cos(N\pi t) - \cos(L\pi t)\big) \\ &= \frac{2}{(L-N)\,(\pi t)^2}\, \sin\big(\tfrac{N+L}{2}\,\pi t\big)\, \sin\big(\tfrac{L-N}{2}\,\pi t\big) \\ &= \frac{N+L}{2}\, \mathrm{sinc}\big(\tfrac{N+L}{2}\,\pi t\big)\, \mathrm{sinc}\big(\tfrac{L-N}{2}\,\pi t\big)\,. \end{aligned} \tag{3.4}$$

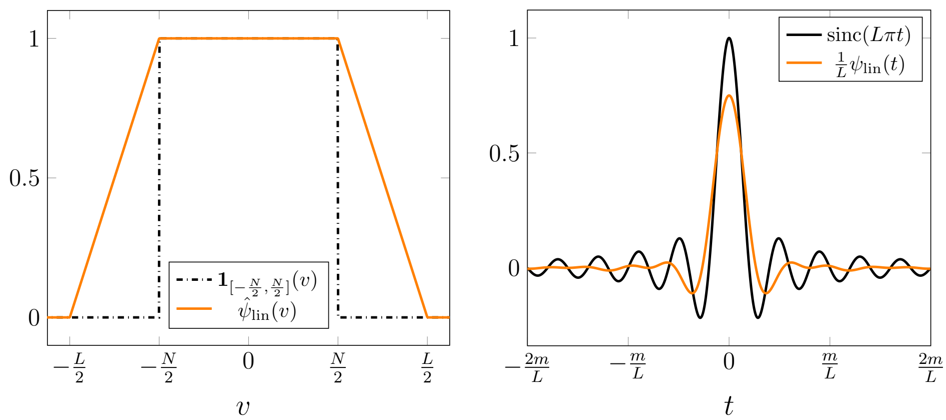
This function $\frac{1}{L}\,\psi_{\mathrm{lin}}$ is even, supported on whole $\mathbb{R}$, has its maximum at $t = 0$ such that

$$\Big\|\frac{1}{L}\,\psi_{\mathrm{lin}}\Big\|_{C_0(\mathbb{R})} = \frac{1}{L}\,\psi_{\mathrm{lin}}(0) = \frac{2+\lambda}{2+2\lambda} < 1\,.$$

In addition, $\frac{1}{L}\,\psi_{\mathrm{lin}}(t)$ has a faster decay than $\operatorname{sinc}(N\pi t)$ for $|t| \to \infty$, cf. Figure 3.1 (b). Note that if $\lambda \to 0$, we have

$$\lim_{L \to +N} \frac{1}{L}\,\psi_{\mathrm{lin}}(t) = \operatorname{sinc}(N\pi t)\,.$$

$\square$



(a) $\hat{\psi}_{\mathrm{lin}}$ in (3.3) (b) $\frac{1}{L}\,\psi_{\mathrm{lin}}$ in (3.4)

Figure 3.1: The frequency window function (3.3) and its scaled inverse Fourier transform (3.4).

Remark 3.2. Note that $\big\{\frac{1}{L}\,\psi_{\mathrm{lin}}\big(\,\cdot\, - \frac{k}{L}\big)\big\}_{k\in\mathbb{Z}}$ is a *Bessel sequence in* $L^2(\mathbb{R})$ *with the bound* $\frac{1}{L}$, i. e., for all $f \in L^2(\mathbb{R})$ we have

$$\sum_{k\in\mathbb{Z}} \big|\langle f,\, \tfrac{1}{L}\,\psi_{\mathrm{lin}}\big(\,\cdot\, - \tfrac{k}{L}\big)\rangle_{L^2(\mathbb{R})}\big|^2 \le \tfrac{1}{L}\,\|f\|^2_{L^2(\mathbb{R})}\,.$$

However, $\big\{\frac{1}{L}\,\psi_{\mathrm{lin}}\big(\,\cdot\, - \frac{k}{L}\big)\big\}_{k\in\mathbb{Z}}$ is not an orthonormal sequence and also not a Riesz sequence. To see this, we consider the 1-periodic function

$$\Psi(v) := \frac{1}{L^2} \sum_{k\in\mathbb{Z}} \big|\hat{\psi}_{\mathrm{lin}}(Lv + Lk)\big|^2\,, \quad v \in \mathbb{R}\,.$$

By (3.3) we have

$$\hat{\psi}_{\mathrm{lin}}(Lv) = \begin{cases} 1 & : |v| \le \frac{1}{2+2\lambda}\,, \\ 1 - \frac{(2+2\lambda)|v|-1}{\lambda} & : \frac{1}{2+2\lambda} < |v| < \frac{1}{2}\,, \\ 0 & : |v| \ge \frac{1}{2}\,, \end{cases}$$

i. e., $\Psi(v) \le \frac{1}{L^2}$ for all $v \in \mathbb{R}$ and $\Psi(v) = 0$ for $v \in \frac{1}{2} + \mathbb{Z}$. Then [5, Theorem 9.2.5] yields the result. $\square$

Analogous to [6, p. 19] and [14, Theorem 7.2.5], we obtain the following representation result, see also [24, p. 4].

Theorem 3.3. *Let $f \in \mathcal{B}_{N/2}(\mathbb{R})$ be a bandlimited function with bandwidth $\frac{N}{2}$. Using oversampling with $L = N\,(1+\lambda)$, $\lambda > 0$, and regularization with a frequency window function $\hat{\psi}$ of the form* (3.1)*, the function f can be represented in the form*

$$f(t) = \sum_{k\in\mathbb{Z}} f\big(\tfrac{k}{L}\big)\, \frac{1}{L}\, \psi\big(t - \tfrac{k}{L}\big)\,, \quad t \in \mathbb{R}\,. \tag{3.5}$$

For instance, using the linear frequency window function (3.3)*, one obtains the representation*

$$f(t) = \frac{N+L}{2L} \sum_{k\in\mathbb{Z}} f\big(\tfrac{k}{L}\big)\, \mathrm{sinc}\big(\tfrac{N+L}{2L}\,(\pi L t - \pi k)\big)\, \mathrm{sinc}\big(\tfrac{L-N}{2L}\,(\pi L t - \pi k)\big)\,, \quad t \in \mathbb{R}\,. \tag{3.6}$$

Proof. Since by assumption $f \in \mathcal{B}_{N/2}(\mathbb{R})$, we have $\mathrm{supp}\,\hat{f} \subseteq \big[-\frac{N}{2}, \frac{N}{2}\big] \subset \big[-\frac{L}{2}, \frac{L}{2}\big]$ and therefore the function $\hat{f}$ restricted on $\big[-\frac{L}{2}, \frac{L}{2}\big]$ can be represented by its L-periodic Fourier series

$$\hat{f}(v) = \sum_{k\in\mathbb{Z}} c_k(\hat{f})\, \mathrm{e}^{2\pi \mathrm{i}\, kv/L}\,, \quad v \in \big[-\tfrac{L}{2}, \tfrac{L}{2}\big]\,, \tag{3.7}$$

with the Fourier coefficients

$$c_k(\hat{f}) = \frac{1}{L} \int_{-L/2}^{L/2} \hat{f}(v)\, \mathrm{e}^{-2\pi \mathrm{i}\, kv/L}\, \mathrm{d}v\,.$$

Using the inverse Fourier transform, we see that

$$c_k(\hat{f}) = \frac{1}{L} \int_{\mathbb{R}} \hat{f}(v)\, \mathrm{e}^{-2\pi \mathrm{i}\, kv/L}\, \mathrm{d}v = \frac{1}{L}\, f\big(-\tfrac{k}{L}\big)\,.$$

Hence, we may denote $\hat{f}$ by (3.7) as

$$\hat{f}(v) = \frac{1}{L} \sum_{k\in\mathbb{Z}} f\big(\tfrac{k}{L}\big)\, \mathrm{e}^{-2\pi \mathrm{i}\, kv/L}\,, \quad v \in \big[-\tfrac{L}{2}, \tfrac{L}{2}\big]\,.$$

Additionally, we have $\hat{\psi}(v) = 1$ for $v \in \big[-\frac{N}{2}, \frac{N}{2}\big]$ by (3.1) as well as $\mathrm{supp}\,\hat{f} \subseteq \big[-\frac{N}{2}, \frac{N}{2}\big]$ by assumption, such that $\hat{f}(v) = \hat{f}(v)\,\hat{\psi}(v)$ for all $v \in \mathbb{R}$, and therefore

$$\begin{aligned} f(t) &= \int_{\mathbb{R}} \hat{f}(v)\, \mathrm{e}^{2\pi \mathrm{i}\, tv}\, \mathrm{d}v = \int_{-L/2}^{L/2} \hat{f}(v)\, \mathrm{e}^{2\pi \mathrm{i}\, tv}\, \mathrm{d}v = \int_{-L/2}^{L/2} \hat{f}(v)\, \hat{\psi}(v)\, \mathrm{e}^{2\pi \mathrm{i}\, tv}\, \mathrm{d}v \\ &= \sum_{k\in\mathbb{Z}} \frac{1}{L}\, f\big(\tfrac{k}{L}\big) \int_{-L/2}^{L/2} \hat{\psi}(v)\, \mathrm{e}^{2\pi \mathrm{i}\,(t-k/L)\, v}\, \mathrm{d}v = \sum_{k\in\mathbb{Z}} f\big(\tfrac{k}{L}\big)\, \frac{1}{L}\, \psi\big(t - \tfrac{k}{L}\big)\,. \end{aligned}$$

For the linear frequency window function (3.3) the representation (3.6) is obtained by inserting the inverse Fourier transform (3.4) into (3.5). This completes the proof. ■

Note that (3.5) is not an interpolating approximation, since in general we have

$$\frac{1}{L}\, \psi\big(t - \tfrac{k}{L}\big)\,\big|_{t=\frac{n}{L}} \neq \delta_{n,k}\,, \quad n, k \in \mathbb{Z}\,.$$

Moreover, since the frequency window function $\hat{\psi}$ in (3.1) is compactly supported, the uncertainty principle (see [15, Lemma 2.31]) yields $\operatorname{supp}\psi = \mathbb{R}$, such that (3.5) does not imply localized sampling for any choice of $\hat{\psi}$. In other words, the representation (3.5) still requires infinitely many samples $f\big(\frac{k}{L}\big)$. Thus, for practical realizations we need to consider a truncated version of (3.5) and hence for $T \in \mathbb{N}$ we introduce the T-th partial sum

$$(P_{\psi,T}f)(t) \coloneqq \sum_{k=-T}^{T} f\big(\tfrac{k}{L}\big)\,\frac{1}{L}\,\psi\big(t-\tfrac{k}{L}\big)\,, \quad t \in [-1,\,1]\,. \tag{3.8}$$

Then for the linear frequency window function (3.3) we show the following convergence result.

Theorem 3.4. *Let $f \in \mathcal{B}_{N/2}(\mathbb{R})$ be a bandlimited function with bandwidth $\frac{N}{2}$. Using oversampling with $L = N\,(1+\lambda)$, $\lambda > 0$, and regularization with the linear frequency window function (3.3), the T-th partial sums $P_{\mathrm{lin},T}f$ converge uniformly to f on $[-1,\,1]$ as $T \to \infty$. For $T > L$ the following estimate holds*

$$\max_{t\in[-1,\,1]} \big|f(t) - (P_{\mathrm{lin},T}f)(t)\big| \le \sqrt{\frac{2L}{3}}\,\frac{4\,(1+\lambda)}{\pi^2\lambda}\,(T-L+1)^{-3/2}\,\|f\|_{L^2(\mathbb{R})}\,. \tag{3.9}$$

Proof. By (3.5) and (3.8) we have

$$f(t) - (P_{\mathrm{lin},T}f)(t) = \sum_{|k|>T} f\big(\tfrac{k}{L}\big)\,\frac{1}{L}\,\psi_{\mathrm{lin}}\big(t-\tfrac{k}{L}\big)\,,$$

such that Cauchy–Schwarz inequality implies

$$\big|f(t) - (P_{\mathrm{lin},T}f)(t)\big| \le \bigg(\sum_{|k|>T} \big|f\big(\tfrac{k}{L}\big)\big|^2\bigg)^{1/2} \bigg(\sum_{|k|>T} \big|\tfrac{1}{L}\,\psi_{\mathrm{lin}}\big(t-\tfrac{k}{L}\big)\big|^2\bigg)^{1/2}\,. \tag{3.10}$$

Since $f \in \mathcal{B}_{N/2}(\mathbb{R})$ is bandlimited with bandwidth $\frac{N}{2}$ and $L > N$, the Parseval equation

$$\frac{1}{L}\sum_{k\in\mathbb{Z}} \big|f\big(\tfrac{k}{L}\big)\big|^2 = \|f\|_{L^2(\mathbb{R})}^2$$

holds and hence

$$\bigg(\sum_{|k|>T} \big|f\big(\tfrac{k}{L}\big)\big|^2\bigg)^{1/2} \le \sqrt{L}\,\|f\|_{L^2(\mathbb{R})}\,. \tag{3.11}$$

It can easily be seen that (3.4) satisfies the decay condition

$$\big|\tfrac{1}{L}\,\psi_{\mathrm{lin}}(x)\big| \le \frac{2}{LN\lambda\,\pi^2}\,x^{-2}\,, \quad x \in \mathbb{R}\setminus\{0\}\,,$$

and thereby

$$\big|\tfrac{1}{L}\,\psi_{\mathrm{lin}}\big(t-\tfrac{k}{L}\big)\big|^2 \le \frac{4\,(1+\lambda)^2}{\lambda^2\pi^4}\,(Lt-k)^{-4}\,.$$

Thus, for $T > L$ and $t \in [-1,\,1]$ we obtain

$$\bigg(\sum_{|k|>T} \big|\tfrac{1}{L}\,\psi_{\mathrm{lin}}\big(t-\tfrac{k}{L}\big)\big|^2\bigg)^{1/2} \le \frac{2\,(1+\lambda)}{\lambda\pi^2}\,\bigg(\sum_{|k|>T} (Lt-k)^{-4}\bigg)^{1/2}$$

$$\leq \frac{2\sqrt{2}\,(1+\lambda)}{\lambda\pi^2} \left(\sum_{k=T+1}^{\infty} (k-L)^{-4} \right)^{1/2} .$$

Using the integral test for convergence of series, we conclude

$$\sum_{k=T+1}^{\infty} (k-L)^{-4} \leq (T-L+1)^{-4} + \int_{T+1}^{\infty} (t-L)^{-4}\,\mathrm{d}t$$
$$= (T-L+1)^{-4} + \tfrac{1}{3}\,(T-L+1)^{-3} < \tfrac{4}{3}\,(T-L+1)^{-3} ,$$

which yields

$$\left(\sum_{k=T+1}^{\infty} (k-L)^{-4} \right)^{1/2} \leq \tfrac{2}{\sqrt{3}}\,(T-L+1)^{-3/2} . \tag{3.12}$$

Therefore, (3.10), (3.11), and (3.12) imply the estimate (3.9). ■

Example 3.5. Next, we visualize the error bound of Theorem 3.4, i. e., for a given function $f \in \mathcal{B}_{N/2}(\mathbb{R})$ with $L = N(1+\lambda)$, $\lambda > 0$, we show that the approximation error satisfies (3.9). For this purpose, the error

$$\max_{t\in[-1,\,1]} \big| f(t) - (P_{\mathrm{lin},T} f)(t) \big| \tag{3.13}$$

is estimated by evaluating the given function f as well as its approximation $P_{\mathrm{lin},T} f$, cf. (3.8), at equidistant points $t_s \in [-1,\,1]$, $s = 1, \ldots, S$, with $S = 10^5$. Here we study the function $f(t) = \sqrt{N}\,\mathrm{sinc}(N\pi t)$, $t \in \mathbb{R}$, such that $\|f\|_{L^2(\mathbb{R})} = 1$. We fix $N = 128$ and consider the error behavior for increasing $T \in \mathbb{N}$. More specifically, in this experiment we choose several oversampling parameters $\lambda \in \{0.5, 1, 2\}$ and truncation parameters $T \in \{10, 20, \ldots, 500\}$. The corresponding results are depicted in Figure 3.2. Note that the error bound in (3.9) is only valid for $T > L$. Therefore, we have additionally marked the point $T = L$ for each λ as a vertical dash-dotted line. It can easily be seen that also the error results are much better when $T > L$. Note, however, that increasing the oversampling parameter λ requires a much larger truncation parameter T to obtain errors of the same size. □

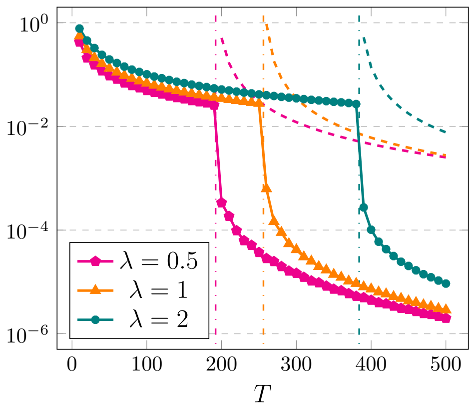


Figure 3.2: Maximum approximation error (3.13) (solid) and error constant (3.9) (dashed) using the linear frequency window ψ_{lin} from (3.4) in (3.8) for the test function $f(t) = \sqrt{N}\,\mathrm{sinc}(N\pi t)$ with $N = 128$, $T \in \{10, 20, \ldots, 500\}$, and $\lambda \in \{0.5, 1, 2\}$.

In order to obtain convergence rates better than the one in Theorem 3.4, one may consider frequency window functions (3.1) of higher smoothness.

Example 3.6. Next, we construct a continuously differentiable frequency window function by polynomial interpolation. Since the frequency window function (3.1) is even, it suffices to consider only $\chi\colon \left[\frac{N}{2},\, \frac{L}{2}\right] \to [0,\, 1]$ at the interval boundaries $\frac{N}{2}$ and $\frac{L}{2}$. Clearly, the linear frequency window function $\hat{\psi}_{\mathrm{lin}}$ in (3.3) fulfills

$$\lim_{v \to \frac{N}{2}} \chi(v) = 1\,, \quad \lim_{v \to \frac{L}{2}} \chi(v) = 0\,.$$

Thus, to obtain a smoother frequency window function, we need to additionally satisfy the first order conditions

$$\lim_{v \to \frac{N}{2}} \chi'(v) = 0\,, \quad \lim_{v \to \frac{L}{2}} \chi'(v) = 0\,.$$

Then the corresponding interpolation polynomial yields the *cubic frequency window function*

$$\hat{\psi}_{\mathrm{cub}}(v) := \begin{cases} 1 & : |v| \le \frac{N}{2}\,, \\ \frac{16}{(L-N)^3} \left(|v| - \frac{L}{2}\right)^2 \left(|v| - \frac{3N-L}{4}\right) & : \frac{N}{2} < |v| < \frac{L}{2}\,, \\ 0 & : |v| \ge \frac{L}{2}\,, \end{cases} \tag{3.14}$$

see Figure 3.3 (a). By (3.2) we see that the inverse Fourier transform of (3.14) is given by

$$\psi_{\mathrm{cub}}(t) = \frac{12(\cos(N\pi t) - \cos(L\pi t))}{\pi^4 t^4\,(L-N)^3} - \frac{6(\sin(L\pi t) + \sin(N\pi t))}{\pi^3 t^3\,(L-N)^2}\,, \quad t \in \mathbb{R} \setminus \{0\}\,, \tag{3.15}$$

and $\psi_{\mathrm{cub}}(0) = \frac{L+N}{2}$, cf. Figure 3.3 (b). □

Analogous to Theorem 3.4, the following error estimate can be derived.

Theorem 3.7. *Let $f \in \mathcal{B}_{N/2}(\mathbb{R})$ be a bandlimited function with bandwidth $\frac{N}{2}$. Using oversampling with $L = N\,(1+\lambda)$, $\lambda > 0$, and regularization with the cubic frequency window function* (3.14)*, the T-th partial sums $P_{\mathrm{cub},T}f$ converge uniformly to f on $[-1,\, 1]$ as $T \to \infty$. For $T > L$ the following estimate holds*

$$\max_{t \in [-1,\,1]} \left|f(t) - (P_{\mathrm{cub},T}f)(t)\right| \le \sqrt{\frac{12L}{5}}\, \frac{12\,(1+\lambda)^2}{\pi^3 \lambda^2}\, (T - L + 1)^{-5/2}\, \|f\|_{L^2(\mathbb{R})}\,. \tag{3.16}$$

Example 3.8. Another continuously differentiable frequency window function is given in [21] as the *raised cosine frequency window function*

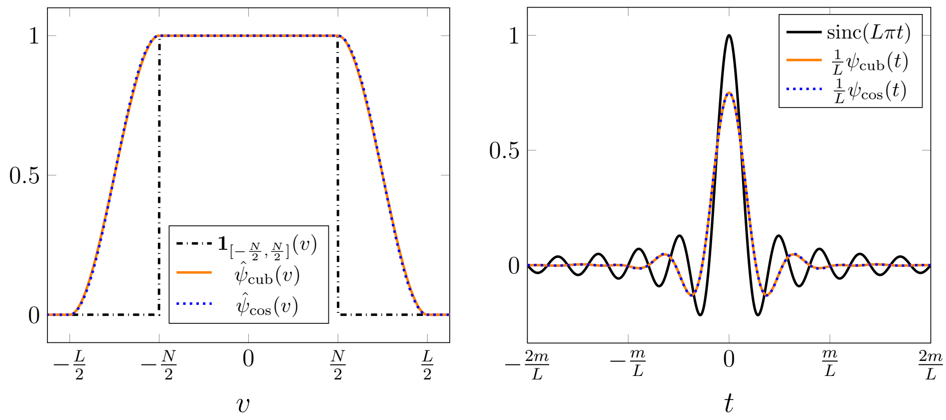
$$\hat{\psi}_{\cos}(v) := \begin{cases} 1 & : |v| \le \frac{N}{2}\,, \\ \frac{1}{2} + \frac{1}{2}\,\cos\left(\frac{2|v|-N}{L-N}\,\pi\right) & : \frac{N}{2} < |v| < \frac{L}{2}\,, \\ 0 & : |v| \ge \frac{L}{2}\,, \end{cases} \tag{3.17}$$

see Figure 3.3 (a). By (3.2) the corresponding function in time domain can be determined as

$$\psi_{\cos}(t) = \frac{L\,\mathrm{sinc}(L\pi t) + N\mathrm{sinc}(N\pi t)}{2 - 2t^2\,(L-N)^2}\,, \quad t \in \mathbb{R} \setminus \left\{\pm\, \tfrac{1}{L-N}\right\}, \tag{3.18}$$

$$\psi_{\cos}\left(\pm \tfrac{1}{L-N}\right) = \tfrac{L-N}{4}\, \cos\left(\tfrac{N\pi}{L-N}\right),$$

see Figure 3.3 (b). Note that since $\hat{\psi}_{\cos}$ in (3.17) possesses the same regularity as $\hat{\psi}_{\mathrm{cub}}$ in (3.14), both frequency window functions meet the same error bound (3.16), cf. Figure 5.1. □



(a) $\hat\psi_{\mathrm{cub}}$ and $\hat\psi_{\cos}$ (b) $\frac{1}{L}\,\psi_{\mathrm{cub}}$ and $\frac{1}{L}\,\psi_{\cos}$

Figure 3.3: The frequency window functions (3.14) and (3.17), and their scaled inverse Fourier transforms.

Note that by (3.4) and the convolution property of the Fourier transform, for $L > N$ the linear frequency window function (3.3) can be written as

$$\hat\psi_{\mathrm{lin}}(v) = \frac{2}{L-N}\,\big(\mathbf{1}_{(N+L)/4} * \mathbf{1}_{(L-N)/4}\big)(v)\,.$$

Therefore, instead of determining smooth frequency window functions of the form (3.1) by means of interpolation as in Example 3.6, they can also be constructed by convolution, cf. [12].

Lemma 3.9. *Let $L > N$ be given. Assume that $\rho\colon \mathbb{R} \to [0,\,\infty)$ is an even integrable function with* $\operatorname{supp}\rho = \big[-\frac{L-N}{4},\,\frac{L-N}{4}\big]$ *and* $\int_{\mathbb{R}} \rho(v)\,\mathrm{d}v = 1$.
Then the convolution

$$\hat\psi_{\mathrm{conv}}(v) = \big(\mathbf{1}_{(N+L)/4} * \rho\big)(v)\,,\quad v \in \mathbb{R}\,, \tag{3.19}$$

is a frequency window function of the form (3.1).

Proof. By assumptions we have

$$\hat\psi_{\mathrm{conv}}(v) = \big(\mathbf{1}_{(N+L)/4} * \rho\big)(v) = \int_{-(N+L)/4}^{(N+L)/4} \rho(v-w)\,\mathrm{d}w = \int_{v-(N+L)/4}^{v+(N+L)/4} \rho(w)\,\mathrm{d}w\,. \tag{3.20}$$

Since the convolution of two even functions is again an even function, it suffices to consider (3.20) only for $v \ge 0$. For $v \in \big[0,\,\frac{N}{2}\big]$ we have $v - \frac{L+N}{4} \le -\frac{L-N}{4} < \frac{L-N}{4} \le v + \frac{L+N}{4}$ and therefore

$$\hat\psi_{\mathrm{conv}}(v) = \int_{-(L-N)/4}^{(L-N)/4} \rho(w)\,\mathrm{d}w = 1\,.$$

For $v \in \big[\frac{N}{2},\,\frac{L}{2}\big]$ we can write

$$\hat\psi_{\mathrm{conv}}(v) = \int_{v-(L+N)/4}^{(L-N)/4} \rho(w)\,\mathrm{d}w \ge 0\,,$$

where $\hat{\psi}_{\mathrm{conv}}\big(\frac{N}{2}\big) = 1$, $\hat{\psi}_{\mathrm{conv}}\big(\frac{L}{2}\big) = 0$, and $\hat{\psi}_{\mathrm{conv}} \colon \big[\frac{N}{2},\, \frac{L}{2}\big] \to [0,\, 1]$ is monotonously non-increasing, since $\rho(w) \ge 0$ for all $w \in \mathbb{R}$ by assumption. For $v \in \big[\frac{L}{2},\, \infty\big)$ we have $v - \frac{L+N}{4} \ge \frac{L-N}{4}$, which implies by assumption $\operatorname{supp}\rho = \big[-\frac{L-N}{4},\, \frac{L-N}{4}\big]$ that

$$\hat{\psi}_{\mathrm{conv}}(v) = \int_{v-(L+N)/4}^{v+(L+N)/4} \rho(w)\,\mathrm{d}w = 0\,.$$

This completes the proof. ■

Note that the frequency window functions (3.14) and (3.17) lack such a convolutional representation. However, the convolutional approach (3.19) has substantial advantages, since the smoothness of (3.19) is determined by the smoothness of the chosen function ρ and the inverse Fourier transform is known by the convolution property as

$$\psi_{\mathrm{conv}}(t) = \frac{N+L}{2}\,\mathrm{sinc}\big(\tfrac{N+L}{2}\,\pi t\big)\,\check{\rho}(t)\,, \tag{3.21}$$

which is especially helpful if the inverse Fourier transform $\check{\rho}$ of ρ is explicitly known. Since ρ is even by assumption, we have $\check{\rho} = \hat{\rho}$ with

$$\check{\rho}(t) = \int_{\mathbb{R}} \rho(v)\,\mathrm{e}^{2\pi\mathrm{i}\,tv}\,\mathrm{d}v = 2\int_0^{\infty} \rho(v)\,\cos(2\pi\,vt)\,\mathrm{d}v\,.$$

Example 3.10. For the special choice of $\rho(v) = \frac{2n}{L-N}\,M_n\big(\frac{2n}{L-N}\,v\big)$ with $n \in \mathbb{N}$, where M_n is the centered cardinal B-spline of order n, we have

$$\check{\rho}(t) = \Big(\mathrm{sinc}\big(\tfrac{L-N}{2n}\,\pi t\big)\Big)^n\,.$$

Using $n = 1$ this again yields (3.4), whereas for $n = 2$ we obtain

$$\psi_{\mathrm{conv},2}(t) = \frac{N+L}{2}\,\mathrm{sinc}\big(\tfrac{N+L}{2}\,\pi t\big)\,\Big(\mathrm{sinc}\big(\tfrac{L-N}{4}\,\pi t\big)\Big)^2\,. \tag{3.22}$$

Note that the frequency window function $\hat{\psi}_{\mathrm{conv},2}$, cf. (3.22), possesses the same regularity as $\hat{\psi}_{\mathrm{cub}}$ in (3.14) and $\hat{\psi}_{\cos}$ in (3.17), and therefore they all meet the same error bound (3.16), cf. Figure 5.1. □

Example 3.11. In [12] the infinitely differentiable function

$$\rho_\infty(v) = \begin{cases} c\,\exp\Big(\big[\big(\tfrac{4v}{L-N}\big)^2 - 1\big]^{-1}\Big) & : |v| < \frac{L-N}{4}\,, \\ 0 & : \text{otherwise}\,, \end{cases}$$

with the scaling factor

$$c = \frac{1}{2}\left(\int_0^{(L-N)/4} \exp\Big(\big[\big(\tfrac{4v}{L-N}\big)^2 - 1\big]^{-1}\Big)\,\mathrm{d}v\right)^{-1}.$$

is considered. The corresponding frequency window function (3.19) is denoted by $\hat{\psi}_\infty$. However, since for this function ρ_∞ the inverse Fourier transform $\check{\rho}_\infty$ cannot explicitly be stated, the function (3.21) in time domain can only be approximated, which was done by a piecewise rational approximation $\check{\rho}_{\mathrm{rat}}$ in [12]. We remark that because of this additional approximation a numerical decay of the expected rate is doubtful, since the issue of robustness of the corresponding regularized Shannon series remained unclear. This effect can also be seen in Figure 5.1, where the corresponding frequency window function (3.19), denoted by $\hat{\psi}_{\mathrm{rat}}$, shows similar behavior as the linear frequency window function $\hat{\psi}_{\mathrm{lin}}$ in (3.3).

A similar comment applies to [24], where an infinitely differentiable window function $\hat{\psi}$ is aimed for as well. Since no such $\hat{\psi}$ with explicit inverse Fourier transform (3.2) is known, in [24] the function ψ in time domain is estimated with some Gabor approximation. Although an efficient computation scheme via fast Fourier transform (FFT) was introduced in [25], the numerical nonrobustness by this approximation seems to be neglected in this work. $\square$

Finally, we remark that already in [6, p. 19] it was stated that a faster decay than for $\hat{\psi}_{\mathrm{lin}}$ from (3.3) can be obtained by choosing $\hat{\psi}$ in (3.1) smoother, but at the price of a very large constant. This can also be seen in Figure 5.1, where the results for the window functions $\hat{\psi}_{\mathrm{cub}}$ in (3.14), $\hat{\psi}_{\mathrm{cos}}$ in (3.17), $\hat{\psi}_{\mathrm{conv},2}$ in (3.22), and $\hat{\psi}_{\mathrm{rat}}$ from Example 3.11 are plotted as well. For this reason many authors restricted themselves to the linear frequency window function $\hat{\psi}_{\mathrm{lin}}$ in (3.3). Furthermore, the numerical results in Figure 5.1 encourage the suggestion that in practice only algebraic decay rates are achievable for the regularization with a frequency window function.

4 Regularization with a time window function

To preferably obtain better decay rates, we now consider a second regularization technique, namely regularization with a convenient window function in the time domain. To this end, for $L > N$ and any $m \in \mathbb{N} \setminus \{1\}$ with $2m \ll L$, we introduce the set $\Phi_{m,L}$ of all window functions $\varphi\colon \mathbb{R} \to [0,\, 1]$ with the following properties:

- Each $\varphi \in \Phi_{m,L}$ is supported on $\big[-\frac{m}{L},\, \frac{m}{L}\big]$. Further, φ is even and continuous on $\big[-\frac{m}{L},\, \frac{m}{L}\big]$.
- Each $\varphi \in \Phi_{m,L}$ restricted on $\big[0,\, \frac{m}{L}\big]$ is monotonously non-increasing with $\varphi(0) = 1$.
- For each $\varphi \in \Phi_{m,L}$, the Fourier transform

  $$\hat{\varphi}(v) = \int_{\mathbb{R}} \varphi(t)\, \mathrm{e}^{-2\pi \mathrm{i}\, vt}\, \mathrm{d}t = 2 \int_0^{m/L} \varphi(t)\, \cos(2\pi vt)\, \mathrm{d}t\,, \quad v \in \mathbb{R}\,,$$

  is explicitly known.

As examples of such window functions we consider the sinh-*type window function*

$$\varphi_{\mathrm{sinh}}(t) := \begin{cases} \frac{1}{\sinh \beta}\, \sinh\Big(\beta \sqrt{1 - \big(\frac{Lt}{m}\big)^2}\Big) & : t \in \big[-\frac{m}{L},\, \frac{m}{L}\big]\,, \\ 0 & : t \in \mathbb{R} \setminus \big[-\frac{m}{L},\, \frac{m}{L}\big]\,, \end{cases} \tag{4.1}$$

with $\beta = \frac{\pi m\,(L-N)}{L}$, and the *continuous Kaiser–Bessel window function*

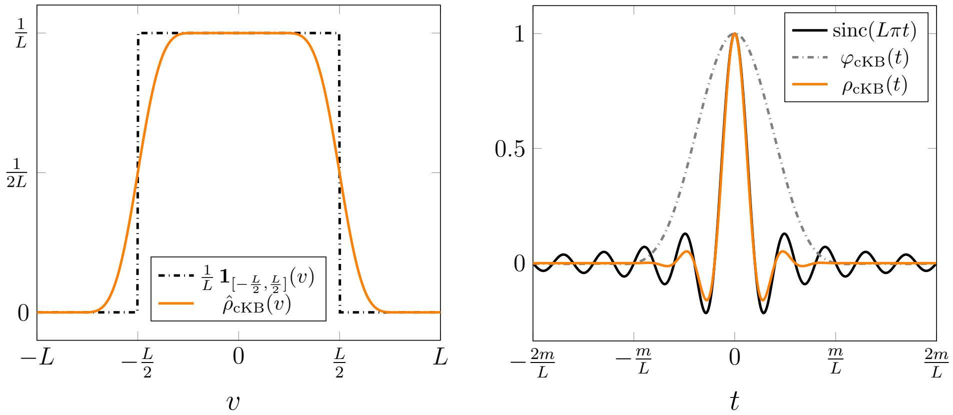
$$\varphi_{\mathrm{cKB}}(t) := \begin{cases} \frac{1}{I_0(\beta)-1}\, \Big(I_0\Big(\beta \sqrt{1 - \big(\frac{Lt}{m}\big)^2}\Big) - 1\Big) & : t \in \big[-\frac{m}{L},\, \frac{m}{L}\big]\,, \\ 0 & : t \in \mathbb{R} \setminus \big[-\frac{m}{L},\, \frac{m}{L}\big]\,, \end{cases} \tag{4.2}$$

with $\beta = \frac{\pi m\,(L-N)}{L}$, where $I_0(x)$ denotes the *modified Bessel function of first kind* given by

$$I_0(x) := \sum_{k=0}^{\infty} \frac{1}{\big((2k)!!\big)^2}\, x^{2k}\,, \quad x \in \mathbb{R}\,.$$

Both window functions are well-studied in the context of the nonequispaced fast Fourier transform (NFFT), see e. g. [16] and references therein.

A visualization of the continuous Kaiser–Bessel window function (4.2) as well as the corresponding regularization $\rho_{\mathrm{cKB}}(t) := \mathrm{sinc}(L\pi t)\,\varphi_{\mathrm{cKB}}(t)$ of the sinc function can be found in Figure 4.1. We remark that in contrast to Figure 3.1 here the function ρ_{cKB} in time domain is compactly supported and its Fourier transform $\hat\rho_{\mathrm{cKB}}$ is supported on whole $\mathbb{R}$, where for the frequency window function (3.3) it is vice versa (see [15, Lemma 2.31]). Note that a visualization for the sinh-type window (4.1) would look the same as Figure 4.1.



(a) $\hat\rho_{\mathrm{cKB}}$ (b) window function (4.2)

Figure 4.1: The regularized sinc function $\rho_{\mathrm{cKB}}(t) := \mathrm{sinc}(L\pi t)\,\varphi_{\mathrm{cKB}}(t)$ using the continuous Kaiser–Bessel window function (4.2) and its Fourier transform $\hat\rho_{\mathrm{cKB}}$.

Then we recover a bandlimited function $f \in \mathcal{B}_{N/2}(\mathbb{R})$ by the *regularized Shannon sampling formula*

$$(R_{\varphi,m}f)(t) := \sum_{k\in\mathbb{Z}} f\big(\tfrac{k}{L}\big)\,\mathrm{sinc}(L\pi t - \pi k)\,\varphi\big(t - \tfrac{k}{L}\big)\,, \quad t \in \mathbb{R}\,, \tag{4.3}$$

with $L \ge N$. Since by assumption $\mathrm{sinc}(\pi(n-k)) = \delta_{n,k}$ for all $n,\,k \in \mathbb{Z}$ with the Kronecker delta $\delta_{n,k}$ and $\varphi(0) = 1$, this procedure is an *interpolating approximation* of f, because

$$\mathrm{sinc}(L\pi t - \pi k)\,\varphi\big(t - \tfrac{k}{L}\big)\,\big|_{t=\frac{n}{L}} = \delta_{n,k}\,.$$

Furthermore, the use of the compactly supported window function $\varphi \in \Phi_{m,L}$ leads to *localized sampling* of the bandlimited function $f \in \mathcal{B}_{N/2}(\mathbb{R})$, i. e., the computation of $(R_{\varphi,m}f)(t)$ for $t \in \mathbb{R} \setminus \frac{1}{L}\,\mathbb{Z}$ requires only $2m+1$ samples $f\big(\frac{k}{L}\big)$, where $k \in \mathbb{Z}$ fulfills the conditions $|k - Lt| \le m$. Consequently, for given $f \in \mathcal{B}_{N/2}(\mathbb{R})$ and $L \ge N$, the reconstruction of f on the interval $[-1,\,1]$ requires $2m + 2L + 1$ samples $f\big(\frac{k}{L}\big)$ with $k = -m - L,\,\dots,\,m + L$. In addition, we again employ oversampling of the bandlimited function $f \in \mathcal{B}_{N/2}(\mathbb{R})$, i. e., f is sampled on a finer grid $\frac{1}{L}\,\mathbb{Z}$ with $L > N$.

This concept of regularized Shannon sampling formulas with localized sampling and oversampling has already been studied by various authors. A survey of different approaches for window functions can be found in [19], while the prominent Gaussian window function was e. g. considered in [18, 20, 22, 26, 10]. Since this Gaussian window function has also been studied in [9], where superiority of the sinh-type window function (4.1) was shown, we now focus on time window functions $\varphi \in \Phi_{m,L}$, such as (4.1) and (4.2).

Similar as in [9], for given $f \in \mathcal{B}_{N/2}(\mathbb{R})$ and $\varphi \in \Phi_{m,L}$ the uniform approximation error $\|f - R_{\varphi,m}f\|_{C_0(\mathbb{R})}$ of the regularized Shannon sampling formula can be estimated as follows.

Theorem 4.1. *Let $f \in \mathcal{B}_{N/2}(\mathbb{R})$ with $N \in \mathbb{N}$, $L > N$, and $m \in \mathbb{N} \setminus \{1\}$ be given. Further let $\varphi \in \Phi_{m,L}$.*
Then the regularized Shannon sampling formula (4.3) *satisfies*

$$\|f - R_{\varphi,m}f\|_{C_0(\mathbb{R})} \le \big(E_1(m,N,L) + E_2(m,L)\big)\, \|f\|_{L^2(\mathbb{R})}\,,$$

with the corresponding error constants

$$E_1(m,N,L) := \sqrt{N} \max_{v \in [-N/2,\, N/2]} \left| 1 - \int_{v-L/2}^{v+L/2} \hat{\varphi}(u)\, \mathrm{d}u \right|, \tag{4.4}$$

$$E_2(m,L) := \frac{\sqrt{2L}}{\pi\, m}\, \varphi\big(\tfrac{m}{L}\big)\,. \tag{4.5}$$

Proof. For a proof of Theorem 4.1 see [9, Thm. 3.2]. ■

Now we specify the result of Theorem 4.1 for certain window functions. To this end, assume that $f \in \mathcal{B}_{N/2}(\mathbb{R})$ with $N \in \mathbb{N}$ and $L = (1+\lambda)\,N$, $\lambda > 0$. Additionally, we choose $m \in \mathbb{N} \setminus \{1\}$. We demonstrate that for the window functions (4.1) and (4.2) the approximation error of the regularized Shannon sampling formula (4.3) decreases exponentially with respect to m. We start with the sinh-type window function (4.1).

Theorem 4.2. *Let $f \in \mathcal{B}_{N/2}(\mathbb{R})$ with $N \in \mathbb{N}$ be given. Assume that $L = (1+\lambda)\,N$ with $\lambda > 0$ and $m \in \mathbb{N} \setminus \{1\}$. Let φ_{sinh} be the* sinh-*type window function* (4.1) *with parameter $\beta = \frac{\pi m \lambda}{1+\lambda}$. Then the regularized Shannon sampling formula* (4.3) *with the* sinh-*type window function* (4.1) *satisfies the error estimate*

$$\|f - R_{\mathrm{sinh},m}f\|_{C_0(\mathbb{R})} \le \sqrt{N}\, \mathrm{e}^{-m\pi\lambda/(1+\lambda)}\, \|f\|_{L^2(\mathbb{R})}\,. \tag{4.6}$$

Proof. For a proof of Theorem 4.2 see [9, Theorem 6.1]. ■

Next, we continue with the continuous Kaiser–Bessel window function (4.2).

Theorem 4.3. *Let $f \in \mathcal{B}_{N/2}(\mathbb{R})$ with $N \in \mathbb{N}$ be given. Assume that $L = (1+\lambda)\,N$ with $\lambda > 0$ and $m \in \mathbb{N} \setminus \{1\}$. Let φ_{cKB} be the continuous Kaiser–Bessel window function* (4.2) *with parameter $\beta = \frac{\pi m \lambda}{1+\lambda}$.*
Then the regularized Shannon formula (4.3) *with the continuous Kaiser–Bessel window function* (4.2) *satisfies the error estimate*

$$\|f - R_{\mathrm{cKB},m}f\|_{C_0(\mathbb{R})} \le \sqrt{N}\, \frac{1}{I_0(\beta) - 1} \left(1 + \frac{4m\lambda}{1+\lambda}\right) \|f\|_{L^2(\mathbb{R})}\,. \tag{4.7}$$

Proof. By means of Theorem 4.1 we only have to compute the error constants (4.4) and (4.5). Note that (4.2) implies $\varphi_{\mathrm{cKB}}\big(\frac{m}{L}\big) = 0$, such that the error constant (4.5) vanishes. For computing the error constant (4.4) we introduce the function $\eta\colon \big[-\frac{N}{2},\, \frac{N}{2}\big] \to \mathbb{R}$ given by

$$\eta(v) = 1 - \int_{L/2-v}^{L/2+v} \hat{\varphi}_{\mathrm{cKB}}(u)\,\mathrm{d}u\,. \tag{4.8}$$

As known by [13, p. 3, 1.1, and p. 95, 18.31], the Fourier transform of (4.2) has the form

$$\hat{\varphi}_{\mathrm{cKB}}(v) = \frac{2m}{\big(I_0(\beta)-1\big)\,L} \cdot \begin{cases} \Big(\frac{\sinh\big(\beta\sqrt{1-w^2}\big)}{\beta\sqrt{1-w^2}} - \mathrm{sinc}(\beta w)\Big) & : |w| < 1\,, \\ \Big(\mathrm{sinc}\big(\beta\sqrt{w^2-1}\,\big) - \mathrm{sinc}(\beta w)\Big) & : |w| \ge 1\,, \end{cases} \tag{4.9}$$

with the scaled frequency $w = \frac{2\pi m}{\beta L}\, v$. Thus, substituting $w = \frac{2\pi m}{\beta L}\, u$ in (4.8) yields

$$\eta(v) = 1 - \frac{\beta L}{2\pi m} \int_{-a(-v)}^{a(v)} \hat{\varphi}_{\mathrm{cKB}}\big(\tfrac{\beta L}{2m\pi}\, w\big)\,\mathrm{d}w$$

with the increasing linear function $a(v) := \frac{2m\pi}{\beta L}\,\big(v + \frac{L}{2}\big)$. By the choice of the parameter $\beta = \frac{m\pi\lambda}{1+\lambda}$ with $\lambda > 0$ we have $a\big(-\frac{N}{2}\big) = 1$ and $a(v) \ge 1$ for all $v \in \big[-\frac{N}{2},\, \frac{N}{2}\big]$. Using (4.9), we decompose $\eta(v)$ in the form

$$\eta(v) = \eta_1(v) - \eta_2(v)\,, \quad v \in \big[-\tfrac{N}{2},\, \tfrac{N}{2}\big]\,,$$

with

$$\eta_1(v) = 1 - \frac{\beta}{\pi\,\big(I_0(\beta)-1\big)} \int_{-1}^{1} \bigg(\frac{\sinh\big(\beta\sqrt{1-w^2}\,\big)}{\beta\sqrt{1-w^2}} - \mathrm{sinc}(\beta w)\bigg)\,\mathrm{d}w\,,$$

$$\eta_2(v) = \frac{\beta}{\pi\,\big(I_0(\beta)-1\big)} \bigg(\int_{-a(-v)}^{-1} + \int_1^{a(v)}\bigg)\Big(\mathrm{sinc}\big(\beta\sqrt{w^2-1}\,\big) - \mathrm{sinc}(\beta w)\Big)\,\mathrm{d}w\,.$$

By [8, 3.997–1] we have

$$\begin{aligned}\int_{-1}^{1} \frac{\sinh\big(\beta\sqrt{1-w^2}\,\big)}{\beta\sqrt{1-w^2}}\,\mathrm{d}w &= \frac{2}{\beta}\int_0^1 \frac{\sinh\big(\beta\sqrt{1-w^2}\,\big)}{\sqrt{1-w^2}}\,\mathrm{d}w \\ &= \frac{2}{\beta}\int_0^{\pi/2} \sinh(\beta\cos s)\,\mathrm{d}s = \frac{\pi}{\beta}\,\mathbf{L}_0(\beta)\,,\end{aligned}$$

where $\mathbf{L}_0(x)$ denotes the *modified Struve function* given by (see [1, 12.2.1])

$$\mathbf{L}_0(x) := \sum_{k=0}^{\infty} \frac{(x/2)^{2k+1}}{\big(\Gamma\big(k+\frac{3}{2}\big)\big)^2} = \frac{2x}{\pi}\sum_{k=0}^{\infty} \frac{x^{2k}}{\big((2k+1)!!\big)^2}\,.$$

Note that the function $I_0(x) - \mathbf{L}_0(x)$ is completely monotonic on $[0,\,\infty)$ (see [3, Theorem 1]) and tends to zero as $x \to \infty$. Applying the *sine integral function*

$$\mathrm{Si}(x) := \int_0^x \frac{\sin w}{w}\,\mathrm{d}w = \int_0^x \mathrm{sinc}\, w\,\mathrm{d}w\,, \quad x \in \mathbb{R}\,,$$

implies

$$\int_{-1}^{1} \operatorname{sinc}(\beta w)\,\mathrm{d}w = 2\int_{0}^{1} \operatorname{sinc}(\beta w)\,\mathrm{d}w = \frac{2}{\beta}\,\mathrm{Si}(\beta)\,.$$

Hence, we obtain

$$\eta_1(v) = 1 - \frac{1}{I_0(\beta)-1}\left(\mathbf{L}_0(\beta) - \frac{2}{\pi}\,\mathrm{Si}(\beta)\right) = \frac{1}{I_0(\beta)-1}\left(I_0(\beta) - \mathbf{L}_0(\beta) - 1 + \frac{2}{\pi}\,\mathrm{Si}(\beta)\right).$$

Note that for suitable $\beta = \frac{m\pi\lambda}{1+\lambda}$ we find $\big[I_0(\beta) - \mathbf{L}_0(\beta) - 1 + \frac{2}{\pi}\,\mathrm{Si}(\beta)\big] \in (0,\,1)$, cf. Figure 4.2. In addition, it is known that $I_0(x) \ge 1$, $x \in \mathbb{R}$, such that $\eta_1(v) > 0$.

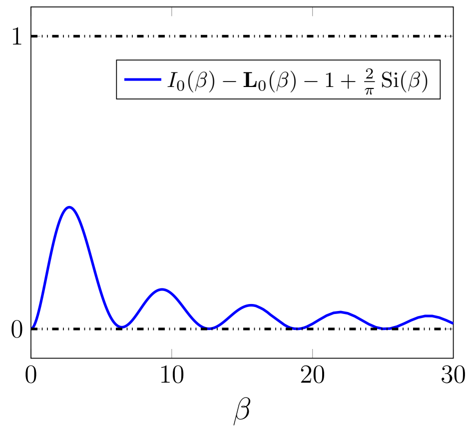


Figure 4.2: Visualization of $\big[I_0(\beta) - \mathbf{L}_0(\beta) - 1 + \frac{2}{\pi}\,\mathrm{Si}(\beta)\big] \in (0,\,1)$ for suitable $\beta = \frac{m\pi\lambda}{1+\lambda}$.

Now we estimate $\eta_2(v)$ for $v \in \big[-\frac{N}{2},\,\frac{N}{2}\big]$ by the triangle inequality as

$$|\eta_2(v)| \le \frac{\beta}{\pi\,\big(I_0(\beta)-1\big)}\left(\int_{-a(-v)}^{-1} + \int_{1}^{a(v)}\right)\Big|\operatorname{sinc}\big(\beta\sqrt{w^2-1}\,\big) - \operatorname{sinc}(\beta w)\Big|\,\mathrm{d}w\,.$$

By [17, Lemma 4] we have for $|w| \ge 1$ that

$$\Big|\operatorname{sinc}\big(\beta\sqrt{w^2-1}\,\big) - \operatorname{sinc}(\beta w)\Big| \le \frac{2}{w^2}\,.$$

Thus, we conclude

$$|\eta_2(v)| \le \frac{4\beta}{\pi\,\big(I_0(\beta)-1\big)}\int_{1}^{\infty} \frac{1}{w^2}\,\mathrm{d}w = \frac{4\beta}{\pi\,\big(I_0(\beta)-1\big)}$$

and using Figure 4.2 and $\beta = \frac{m\pi\lambda}{1+\lambda}$ we therefore obtain

$$\begin{aligned}|\eta(v)| \le \eta_1(v) + |\eta_2(v)| &\le \frac{1}{I_0(\beta)-1}\left(I_0(\beta) - \mathbf{L}_0(\beta) - 1 + \frac{2}{\pi}\,\mathrm{Si}(\beta) + \frac{4\beta}{\pi}\right)\\ &\le \frac{1}{I_0(\beta)-1}\left(1 + \frac{4m\lambda}{1+\lambda}\right).\end{aligned}$$

Since the function $\mathrm{e}^{-x} I_0(x)$ is strictly decreasing on $[0, \infty)$ and tends to zero as $x \to \infty$ (see [2]), we have

$$\frac{1}{I_0(\beta) - 1} = \frac{\mathrm{e}^{-\beta}}{\mathrm{e}^{-\beta} I_0(\beta) - \mathrm{e}^{-\beta}} = \frac{1}{\mathrm{e}^{-\beta} I_0(\beta) - \mathrm{e}^{-\beta}} \, \mathrm{e}^{-m\pi\lambda/(1+\lambda)} \, .$$

Thus, the approximation error of the regularized Shannon formula (4.3) with the continuous Kaiser–Bessel window function (4.2) decreases exponentially with respect to m. ■

5 Comparison of the two regularization methods

Finally, we compare the behavior of the regularization methods presented in Sections 3 and 4. For a given test function $f \in \mathcal{B}_{N/2}(\mathbb{R})$ with $L = N(1+\lambda)$, $\lambda > 0$, we consider the approximation error

$$\max_{t \in [-1,\,1]} \big| f(t) - (P_{\psi,T} f)(t) \big| \tag{5.1}$$

for $\psi \in \{\psi_{\mathrm{lin}}, \psi_{\mathrm{cub}}, \psi_{\mathrm{cos}}, \psi_{\mathrm{conv},2}, \psi_{\mathrm{rat}}\}$, cf. (3.4), (3.15), (3.18), (3.22), and Example 3.11, as well as the corresponding error constants (3.9) and (3.16). In addition, we study the approximation error

$$\max_{t \in [-1,\,1]} | f(t) - (R_{\varphi,m} f)(t) | \tag{5.2}$$

with $\varphi \in \{\varphi_{\mathrm{sinh}}, \varphi_{\mathrm{cKB}}\}$, cf. (4.1) and (4.2), and the corresponding error constants (4.6) and (4.7). By the definition of the regularized Shannon sampling formula in (4.3) we have

$$(R_{\varphi,m} f)(t) = \sum_{\ell=-L-m}^{L+m} f\big(\tfrac{\ell}{L}\big)\, \rho\big(t - \tfrac{\ell}{L}\big)\,, \quad t \in [-1, 1]\,, \tag{5.3}$$

with the regularized sinc function

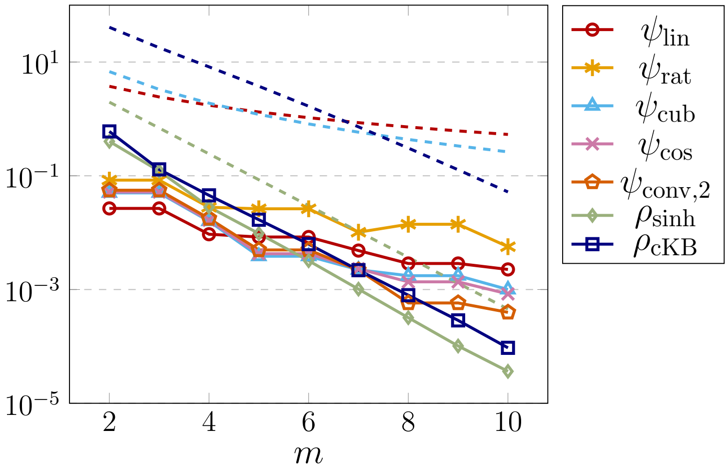
$$\rho(t) := \mathrm{sinc}(L\pi t)\, \varphi(t)\,. \tag{5.4}$$

Thus, to compare (5.3) to $P_{\psi,T} f$ from (3.8), we set $T = L + m$, such that both approximations use the same number of samples. As in Example 3.5 the errors (5.1) and (5.2) shall be estimated by evaluating a given function f and its approximation at equidistant points $t_s \in [-1,\, 1]$, $s = 1, \ldots, S$, with $S = 10^5$. Analogous to [13, Section IV, C] we choose the test function

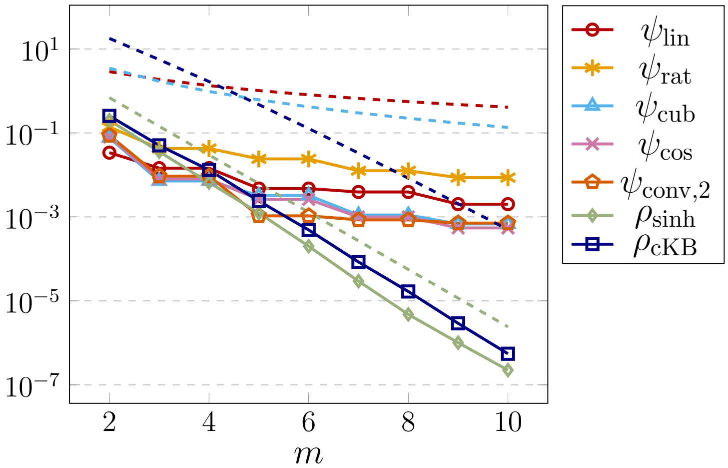
$$f(t) = \sqrt{\tfrac{4N}{5}}\, \big[\mathrm{sinc}(N\pi t) + \tfrac{1}{2}\, \mathrm{sinc}(N\pi(t-1))\big]\,, \quad t \in \mathbb{R}\,, \tag{5.5}$$

with $\|f\|_2 = 1$. We fix $N = 256$ and consider several values of $m \in \mathbb{N} \setminus \{1\}$ and $\lambda \in \{0.5, 1, 2\}$.

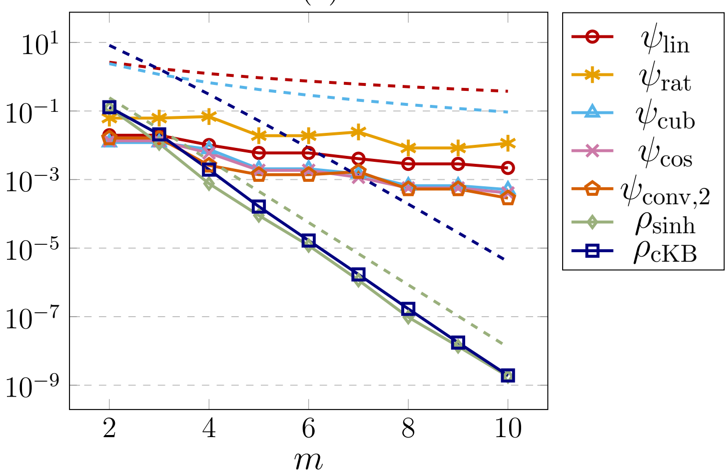
The associated results are displayed in Figure 5.1. Note that for all tested functions the theoretical error behavior perfectly coincides with the numerical outcomes. Moreover, it can clearly be seen that for higher oversampling parameter λ and higher cut-off parameter m, the error results using (4.3) get much better than the ones using (3.5), due to the exponential error decay rate shown for (4.3). This is to say, our numerical results show that regularization with a time window function performs much better than regularization with a frequency window function, since an exponential decay can (up to now) only be realized using a time window function. Furthermore, the great importance of an explicit representation of the regularizing window function can be seen, cf. Example 3.11.



(a) $\lambda = 0.5$



(b) $\lambda = 1$



(c) $\lambda = 2$

Figure 5.1: Maximum approximation error (solid) and error constants (dashed) using regularizations (3.5) with several frequency window functions compared to regularizations (4.3) with time window functions φ_{sinh} and φ_{cKB}, cf. (5.4), for the test function (5.5) with $N = 256$, $m \in \{2, 3, \ldots, 10\}$, and $\lambda \in \{0.5, 1, 2\}$.

In summary, we found that the regularized Shannon sampling formula with the sinh-type time window function is the best, since this approach is the most accurate, easy to compute, and requires less data (for comparable accuracy) than the regularization with a frequency window function.

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References

[1] M. Abramowitz and I.A. Stegun, editors. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Dover, New York, 1972.

[2] Á. Baricz. Bounds for modified Bessel functions of the first and second kinds. *Proc. Edinb. Math. Soc. (2)*, 53:575–599, 2010.

[3] Á. Baricz and T.K. Pogány. Functional inequalities for modified Struve functions II. *Math. Inequal. Appl.*, 17:1387–1398, 2014.

[4] G. Berger and M. Tasche. Hermite-Lagrange interpolation and Schur's expansion of $\sin \pi x$. *J. Approx. Theory*, 53:17–25, 1988.

[5] O. Christensen. *An Introduction to Frames and Riesz Bases.* 2nd edition, Birkhäuser/Springer, Basel, 2016.

[6] I. Daubechies. *Ten Lectures on Wavelets.* SIAM, Philadelphia, 1992.

[7] I. Daubechies and R. DeVore. Approximating a bandlimited function using very coarsely quantized data: A family of stable sigma-delta modulators of arbitrary order. *Ann. of Math. (2)*, 158:679–710, 2003.

[8] I.S. Gradshteyn and I.M. Ryzhik. *Table of Integrals, Series, and Products.* Academic Press, New York, 1980.

[9] M. Kircheis, D. Potts, and M. Tasche. On regularized Shannon sampling formulas with localized sampling. *Sampl. Theory Signal Process. Data Anal.*, 20 (20), 2022, 34 pp.

[10] R. Lin and H. Zhang. Convergence analysis of the Gaussian regularized Shannon sampling formula. *Numer. Funct. Anal. Optim.*, 38(2):224–247, 2017.

[11] C.A. Micchelli, Y. Xu, and H. Zhang. Optimal learning of bandlimited functions from localized sampling. *J. Complexity*, 25(2):85–114, 2009.

[12] F. Natterer. Efficient evaluation of oversampled functions. *J. Comput. Appl. Math.*, 14(3):303–309, 1986.

[13] F. Oberhettinger. *Tables of Fourier Transforms and Fourier Transforms of Distributions.* Springer, Berlin, 1990.

[14] J. R. Partington. *Interpolation, Identification, and Sampling.* Clarendon Press, London Mathematical Society Monographs New Series, 1997.

[15] G. Plonka, D. Potts, G. Steidl, and M. Tasche. *Numerical Fourier Analysis.* Birkhäuser/Springer, Cham, 2018.

[16] D. Potts and M. Tasche. Uniform error estimates for nonequispaced fast Fourier transforms. *Sampl. Theory Signal Process. Data Anal. 19 (17): 1–42*, 2021.

[17] D. Potts and M. Tasche. Continuous window functions for NFFT. *Adv. Comput. Math. 47 (53): 1–34*, 2021.

[18] L. Qian. On the regularized Whittaker–Kotelnikov–Shannon sampling formula. *Proc. Amer. Math. Soc.*, 131(4):1169–1176, 2003.

[19] L. Qian. The regularized Whittaker-Kotelnikov-Shannon sampling theorem and its application to the numerical solutions of partial differential equations. *PhD thesis, National Univ. Singapore*, 2004.

[20] L. Qian and D.B. Creamer. Localized sampling in the presence of noise. *Appl. Math. Letter*, 19:351–355, 2006.

[21] T. S. Rappaport. *Wireless Communications: Principles and Practice.* Prentice Hall, New Jersey, 1996.

[22] G. Schmeisser and F. Stenger. Sinc approximation with a Gaussian multiplier. *Sampl. Theory Signal Image Process.*, 6(2):199–221, 2007.

[23] F. Stenger. *Numerical Methods Based on Sinc and Analytic Functions.* Springer, New York, 1993.

[24] T. Strohmer and J. Tanner. Implementations of Shannon's sampling theorem, a time–frequency approach. *Sampl. Theory Signal Image Process.*, 4(1):1–17, 2005.

[25] T. Strohmer and J. Tanner. Fast reconstruction methods for bandlimited functions from periodic nonuniform sampling. *SIAM J. Numer. Anal.*, 44(3):1071-1094, 2006.

[26] K. Tanaka, M. Sugihara, and K. Murota. Complex analytic approach to the sinc-Gauss sampling formula. *Japan J. Ind. Appl. Math.*, 25:209–231, 2008.

[27] R.M. Young. Euler's constant. *Math. Gaz.*, 75:187–190, 1991.