

# Multiple Rank-1 Lattices as Sampling Schemes for Multivariate Trigonometric Polynomials

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We present a new sampling method that allows for the unique reconstruction of (sparse) multivariate trigonometric polynomials. The crucial idea is to use several rank-1 lattices as spatial discretization in order to overcome limitations of a single rank-1 lattice sampling method. The structure of the corresponding sampling scheme allows for the fast computation of the evaluation and the reconstruction of multivariate trigonometric polynomials, i.e., a fast Fourier transform. Moreover, we present a first algorithm that constructs a reconstructing sampling scheme consisting of several rank-1 lattices for arbitrary, given frequency index sets. Various numerical tests indicate the advantages of the constructed sampling schemes.

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## 1 Introduction

Already in the 1950's high-dimensional integration problems were treated using so called rank-1 lattices as quasi Monte-Carlo rules, cf. [23] for an overview on the early work on lattice rules. Many integration problems were tackled using the concept of rank-1 lattices even in current papers. The main reason for most of the recent activities is the seminal result of A. V. Reztsov and I. H. Sloan in [27]. They proved that a dimension incremental, so called component-by-component, construction of a rank-1 lattice guarantees optimal errors for the numerical integration of specific integrands.

However, in the 80's V. N. Temlyakov [28] investigated the approximation properties of so called Korobov lattices, that are in fact rank-1 lattices of a specific type, cf. [16]. The result was unsatisfactory, since the upper bound of the approximation error was not nearly as good as the optimal ones in relation to the number of used sampling values.

In the late 80's and 90's, the concept of sparse grids becomes very popular since the approximation of multivariate functions of dominating mixed smoothness from their function

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values at sparse grids is almost optimal, cf. e.g. [2, 29, 25, 7]. In addition to these considerations there were developed fast algorithms for the evaluation of hyperbolic cross trigonometric polynomials and their reconstruction from the sampling values at sparse grids, cf. [10, 8, 9]. Later on, the condition number of the corresponding linear mapping was investigated. It was proved that the condition number increases significantly with growing problem sizes, cf. [14]. Hence, the computation of approximations based on noisy sampling values at sparse grids may cause substantial errors.

In the years after the seminal results for integration lattices [27], several papers picked up the idea of approximating functions using the sampling values along rank-1 lattices, cf. [21, 30, 18, 19, 20, 22]. The error bounds for the approximation of functions of dominating mixed smoothness are almost similar to those of V. N. Temlyakov [28]. The crucial innovation of the new papers about rank-1 lattices was that the used sampling schemes are component-by-component constructed, i.e., the results offers a constructive method to built up the sampling schemes that allow for the error estimates. However, the results in [13] allowed for the considerations in [4], where a lower bound for the error of each sampling operator that approximates functions of dominating mixed smoothness using sampling values along rank-1 lattices is determined. The lower bounds are very close to the already known upper bounds in terms of the number of used sampling values. Thus, there is no chance to get nearly optimal worst case errors for sampling operators that uses all sampling values along a single lattice.

The sensitivity against noise of the sparse grid sampling method as well as the necessarily disproportional number of required sampling values for lattice methods call for new concepts for the spatial discretization of functions of dominating mixed smoothness. The new ideas should avoid the disadvantages and may combine the advantages of sparse grids and lattices, i.e., we prefer low oversampling, robustness against noise, available fast algorithms.

In this paper we introduce such a new concept for the discretization of trigonometric polynomials. Our new sampling method is motivated by the structure of sparse grids. A sparse grid is a composition of specific rank-1 up to rank-d lattices, cf. [2, 31, 26], that are well-suited in order to sample trigonometric polynomials with frequencies supported on hyperbolic crosses. In the present paper, we introduce sampling schemes that are compositions of multiple rank-1 lattices. We investigate necessary and sufficient conditions on the sampling scheme in order to guarantee a unique reconstruction of multivariate trigonometric polynomials. It should be noted at this point that we are interested in arbitrary multivariate trigonometric polynomials, i.e., we do not restrict the frequency support to any structure.

Furthermore, we present fast algorithms, i.e., fast Fourier transforms (FFTs), that computes

- the evaluation of multivariate trigonometric polynomials at all sampling nodes, cf. Algorithm 3, and
- the reconstruction of multivariate trigonometric polynomials from the function values at the sampling nodes, cf. Section 4.

As a matter of course, we are interested in sampling sets that allow for the unique reconstruction of multivariate trigonometric polynomials supported on specific frequency index set. We present a method, cf. Algorithm 5, that determines a set of rank-1 lattices that guarantees the unique reconstruction of multivariate trigonometric polynomials supported on a given frequency index set. The constructed sampling sets allow for a specific fast algorithm that computes the reconstruction in a direct way, cf. Algorithm 6.

The present paper deals only with multivariate trigonometric polynomials. We stress on the fact that one may use the new sampling method in an adaptive way similar to the ideas from [24]. Therein D. Potts and T. Volkmer presented a dimension incremental approach in order to identify trigonometric polynomials, i.e., they determine the frequency support as well as the frequencies of multivariate trigonometric polynomials. Certainly, one can also use this algorithm to determine the significant frequencies of multivariate periodic functions. Resulting trigonometric polynomials can be used as approximations of the treated functions. The main ingredient of this approach is the usage of reconstructing sampling schemes for trigonometric polynomials supported on arbitrary, multi-dimensional frequency index sets. In [24], the authors made use of single rank-1 lattices as sampling schemes. Anyway, the strategy introduced in the present paper may offer more suitable spatial discretizations, in particular related to the number of used sampling values.

This paper is organized as follows. Section 2 describes the framework and reveals some basic facts about sampling sets that are unions of single rank-1 lattices. The evaluation of a trigonometric polynomial at these spatial discretizations is illustrated in Section 3. In Section 4 we examine the Fourier matrix and determine necessary conditions for a unique reconstruction of trigonometric polynomials from sampling values along a set of rank-1 lattices. Moreover, we present a deterministic approach in order to construct spatial discretizations for hyperbolic cross trigonometric polynomials that allow for the unique reconstruction of these polynomials. However, this approach does not promise the desired advantages. Subsequently, we state an additional algorithm that determines reconstructing multiple rank-1 lattices for arbitrary frequency index sets, i.e., spatial discretizations that consist of sets of rank-1 lattices and allow for the unique reconstruction of trigonometric polynomials supported on the given frequency index set. This algorithm constitutes the crucial idea of this paper. In Section 5 several numerical tests indicate preferable properties of the sampling method. Section 6 specifies the most important open questions on the presented approach and suggest some potential solution approaches.

## 2 Prerequisites

In this paper we deal with multivariate trigonometric polynomials

$$p: \mathbb{T}^d \rightarrow \mathbb{C}, \quad \mathbf{x} \mapsto \sum_{\mathbf{k} \in I} \hat{p}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}},$$

where  $\mathbb{T}^d \cong [0, 1)^d$  is the  $d$ -dimensional torus, the complex numbers  $(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in I} \in \mathbb{C}^{|I|}$  are called Fourier coefficients of  $p$ , and the frequency index set  $I \subset \mathbb{Z}^d$  is of finite cardinality. The term  $\mathbf{k} \cdot \mathbf{x} = \sum_{j=1}^d k_j x_j$  is the usual scalar product of two  $d$ -dimensional vectors. Furthermore, the space of all trigonometric polynomials supported on the frequency index set  $I$  is denoted by  $\Pi_I := \text{span}\{e^{2\pi i \mathbf{k} \cdot \mathbf{x}} : \mathbf{k} \in I\}$ . The evaluation of the multivariate trigonometric polynomial  $p$  at a finite set  $\mathcal{X} \subset \mathbb{T}^d$  of sampling nodes is specified by the matrix-vector product

$$\mathbf{A}(\mathcal{X}, I) \hat{\mathbf{p}} = \mathbf{p},$$

where  $\hat{\mathbf{p}} = (\hat{p}_{\mathbf{k}})_{\mathbf{k} \in I} \in \mathbb{C}^{|I|}$  is the vector of the Fourier coefficients of the trigonometric polynomial  $p$ , the right hand side  $\mathbf{p} = (p(\mathbf{x}))_{\mathbf{x} \in \mathcal{X}}$  contains the function values of  $p$  at all nodes that belong to the sampling set  $\mathcal{X}$ , and the Fourier matrix  $\mathbf{A}(\mathcal{X}, I)$  is given by

$$\mathbf{A}(\mathcal{X}, I) = \left( e^{2\pi i \mathbf{k} \cdot \mathbf{x}} \right)_{\mathbf{x} \in \mathcal{X}, \mathbf{k} \in I}.$$

As usual, we assume a fixed order of the elements of the sampling set  $\mathcal{X}$  and the frequency index set  $I$  in order to use the matrix-vector notation.

A very specific kind of sampling schemes are so-called rank-1 lattices

$$\mathcal{X} = \Lambda(\mathbf{z}, M) := \left\{ \frac{j}{M} \mathbf{z} \bmod \mathbf{1} : j = 0, \dots, M-1 \right\} \subset \mathbb{T}^d,$$

which are well-investigated in the field of numerical integration, cf. [26, 5]. The  $d$ -dimensional integer vector  $\mathbf{z} \in \mathbb{Z}^d$  is called generating vector and the positive integer  $M \in \mathbb{N}$  the lattice size of the rank-1 lattice  $\Lambda(\mathbf{z}, M)$ . The author studied such sampling schemes for the reconstruction of trigonometric polynomials in [13]. Various specific examples illustrate the advantages of rank-1 lattices for the unique reconstruction of trigonometric polynomials. The main disadvantage of a single rank-1 lattice sampling is the necessarily growing oversampling, i.e.,  $M/|I|$  necessarily increases for growing  $|I|$  in order to ensure a unique reconstruction of trigonometric polynomials supported on specifically structured frequency index sets  $I$ , e.g., hyperbolic cross frequency index sets  $I$ , cf. also [15, 12] for more details on this specific topic.

In order to overcome this problem, we would like to extend the method of rank-1 lattice sampling in a more or less usual manner by sampling along multiple rank-1 lattices. The corresponding sampling sets are joined rank-1 lattices

$$\mathcal{X} = \Lambda(\mathbf{z}_1, M_1, \mathbf{z}_2, M_2, \dots, \mathbf{z}_s, M_s) := \bigcup_{r=1}^s \Lambda(\mathbf{z}_r, M_r)$$

and we call such a construction multiple rank-1 lattice.

The  $\mathbf{0}$  is contained in each of the rank-1 lattices  $\Lambda(\mathbf{z}_r, M_r), r = 1, \dots, s$ . Consequently, the number of distinct elements in  $\Lambda(\mathbf{z}_1, M_1, \mathbf{z}_2, M_2, \dots, \mathbf{z}_s, M_s)$  is bounded from above by  $|\Lambda(\mathbf{z}_1, M_1, \mathbf{z}_2, M_2, \dots, \mathbf{z}_s, M_s)| \leq 1 - s + \sum_{r=1}^s M_r$ . Due to the fact that there may be  $1 \leq r_1 < r_2 \leq s$  such that  $|\Lambda(\mathbf{z}_{r_1}, M_{r_1}) \cap \Lambda(\mathbf{z}_{r_2}, M_{r_2})| > 1$  or  $1 \leq r_3 \leq s$  such that  $|\Lambda(\mathbf{z}_{r_3}, M_{r_3})| < M_{r_3}$ , we have to expect a lower number of elements within  $\Lambda(\mathbf{z}_1, M_1, \mathbf{z}_2, M_2, \dots, \mathbf{z}_s, M_s)$  than the upper bound promises. However, we do not care about duplicate rows within the Fourier matrix with the exception of the duplicates that arises from  $\mathbf{x} = \mathbf{0}$ . We define the Fourier matrix

$$\mathbf{A} := \mathbf{A}(\Lambda(\mathbf{z}_1, M_1, \mathbf{z}_2, M_2, \dots, \mathbf{z}_s, M_s), I) := \begin{pmatrix} \left( e^{2\pi i \frac{j}{M_1} \mathbf{k} \cdot \mathbf{z}_1} \right)_{j=0, \dots, M_1-1, \mathbf{k} \in I} \\ \left( e^{2\pi i \frac{j}{M_2} \mathbf{k} \cdot \mathbf{z}_2} \right)_{j=1, \dots, M_2-1, \mathbf{k} \in I} \\ \vdots \\ \left( e^{2\pi i \frac{j}{M_s} \mathbf{k} \cdot \mathbf{z}_s} \right)_{j=1, \dots, M_s-1, \mathbf{k} \in I} \end{pmatrix},$$

where we assume that the frequency indices  $\mathbf{k} \in I$  are in a fixed order.

In the following, we prove some basics about multiple rank-1 lattices.

**Lemma 2.1.** *Let  $\Lambda(\mathbf{z}_1, M_1)$  and  $\Lambda(\mathbf{z}_2, M_2)$  be two rank-1 lattices with relatively prime lattice sizes  $M_1$  and  $M_2$ . Then, the rank-1 lattice  $\Lambda(M_2 \mathbf{z}_1 + M_1 \mathbf{z}_2, M_1 M_2)$  is a supset of  $\Lambda(\mathbf{z}_1, M_1)$  and  $\Lambda(\mathbf{z}_2, M_2)$ , respectively. Thus, the multiple rank-1 lattice  $\Lambda(\mathbf{z}_1, M_1, \mathbf{z}_2, M_2) \subset \Lambda(M_2 \mathbf{z}_1 + M_1 \mathbf{z}_2, M_1 M_2)$  is a subset of the rank-1 lattice  $\Lambda(M_2 \mathbf{z}_1 + M_1 \mathbf{z}_2, M_1 M_2)$ . Furthermore, the cardinality of  $\Lambda(\mathbf{z}_1, M_1, \mathbf{z}_2, M_2)$  is given by  $|\Lambda(\mathbf{z}_1, M_1)| + |\Lambda(\mathbf{z}_2, M_2)| - 1$ .*

*Proof.* Due to the coprimality of the numbers  $M_1$  and  $M_2$ , the Chinese remainder theorem implies that there exists one  $\ell \in \{0, M_1M_2 - 1\}$  such that

$$\ell \equiv j_1 \pmod{M_1} \quad \text{and} \quad \ell \equiv j_2 \pmod{M_2}.$$

Consequently, we obtain

$$\frac{\ell(M_2\mathbf{z}_1 + M_1\mathbf{z}_2)}{M_1M_2} \pmod{\mathbf{1}} = \left( \frac{\ell\mathbf{z}_1}{M_1} + \frac{\ell\mathbf{z}_2}{M_2} \right) \pmod{\mathbf{1}} = \left( \frac{j_1\mathbf{z}_1}{M_1} \pmod{\mathbf{1}} + \frac{j_2\mathbf{z}_2}{M_2} \pmod{\mathbf{1}} \right) \pmod{\mathbf{1}}.$$

This yields

$$\frac{\ell(M_2\mathbf{z}_1 + M_1\mathbf{z}_2)}{M_1M_2} \pmod{\mathbf{1}} = \frac{j_1\mathbf{z}_1}{M_1} \pmod{\mathbf{1}} \quad \text{and} \quad \frac{\ell(M_2\mathbf{z}_1 + M_1\mathbf{z}_2)}{M_1M_2} \pmod{\mathbf{1}} = \frac{j_2\mathbf{z}_2}{M_2} \pmod{\mathbf{1}}$$

for  $j_2 = 0$  and  $j_1 = 0$ , respectively.

The coprimality of  $M_1$  and  $M_2$  directly implies that  $\Lambda(\mathbf{z}_1, M_1) \cap \Lambda(\mathbf{z}_2, M_2) = \{\mathbf{0}\}$  and the assertion follows.  $\blacksquare$

**Corollary 2.2.** *Let the multiple rank-1 lattice  $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$  with pairwise coprime lattice sizes  $M_1, \dots, M_s$  be given. Then, the cardinality of  $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$  is given by*

$$|\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)| = 1 - s + \sum_{r=1}^s |\Lambda(\mathbf{z}_r, M_r)|$$

and the embedding

$$\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s) \subset \Lambda(\mathbf{z}, M)$$

holds, where the generating vector  $\mathbf{z} = \sum_{r=1}^s (\prod_{\substack{l=1 \\ l \neq r}}^s M_l) \mathbf{z}_r$  and the lattice size  $M = \prod_{r=1}^s M_r$  are explicitly given.

*Proof.* An iterative application of Lemma 2.1 yields the assertion.  $\blacksquare$

As a consequence of Corollary 2.2, a multiple rank-1 lattice  $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$  with pairwise coprime  $M_1, \dots, M_s$  and a full rank matrix  $\mathbf{A}(\Lambda(\mathbf{z}_1, M_1, \mathbf{z}_2, M_2, \dots, \mathbf{z}_s, M_s), I)$  is a subset of the rank-1 lattice  $\Lambda(\mathbf{z}, M)$ ,  $\mathbf{z} = \sum_{r=1}^s (\prod_{\substack{l=1 \\ l \neq r}}^s M_l) \mathbf{z}_r$  and  $M = \prod_{r=1}^s M_r$ . The matrix  $\mathbf{A}(\Lambda(\mathbf{z}, M), I)$  has full column rank and, in particular, pairwise orthogonal columns, cf. [13, Lemma 3.1]. Further restrictions on the rank-1 lattices  $\Lambda(\mathbf{z}_1, M_1), \dots, \Lambda(\mathbf{z}_s, M_s)$  allow for an easy determination of the number of distinct sampling values that are contained in  $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$ .

**Corollary 2.3.** *Under the assumptions of Corollary 2.2 and the additional requirements*

- $M_r$  is prime for all  $r = 1, \dots, s$  and
- $\mathbf{0} \neq \mathbf{z}_r \in [0, M_r - 1]^d \cap \mathbb{Z}^d$ ,

we conclude

$$|\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)| = 1 - s + \sum_{r=1}^s M_r.$$

*Proof.* The additional requirements of the corollary guarantee  $|\Lambda(\mathbf{z}_r, M_r)| = M_r$ ,  $r = 1, \dots, s$ . Hence, the statement follows directly from Corollary 2.2.  $\blacksquare$



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**Algorithm 3** Evaluation at multiple rank-1 lattices
 

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Input:	$M_1, \dots, M_s \in \mathbb{N}$ $\mathbf{z}_1, \dots, \mathbf{z}_s \in \mathbb{Z}^d$ $I \subset \mathbb{Z}^d$ $\hat{\mathbf{p}} = (\hat{p}_{\mathbf{k}})_{\mathbf{k} \in I}$	lattice sizes of rank-1 lattices $\Lambda(\mathbf{z}_l, M_l)$ generating vectors of $\Lambda(\mathbf{z}_l, M_l)$ frequency index set Fourier coefficients of $p \in \Pi_I$
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1: **for**  $l = 1, \dots, s$  **do**

2:    $\mathbf{p}_l = \text{LFFT}(M_l, \mathbf{z}_l, I, \hat{\mathbf{p}})$

3: **end for**

4:  $\mathbf{p} = (\mathbf{p}_1[1], \dots, \mathbf{p}_1[M_1], \mathbf{p}_2[2], \dots, \mathbf{p}_2[M_2], \dots, \mathbf{p}_s[2], \dots, \mathbf{p}_s[M_s])^\top$

Output:	$\mathbf{p} = \mathbf{A}\hat{\mathbf{p}}$	function values of $p \in \Pi_I$
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Complexity:  $\mathcal{O}(\sum_{l=1}^s M_l \log M_l + sd|I|)$

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unique reconstruction of each trigonometric polynomial  $p \in \Pi_I$  from its values along the sampling set  $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$ .

## 4 Reconstruction Properties of Multiple Rank-1 Lattices

In order to investigate the reconstruction properties of a sampling set  $\mathcal{X} \subset \mathbb{T}^d$ ,  $|\mathcal{X}| < \infty$ , with respect to a given frequency index set  $I$ , we have to consider the corresponding Fourier matrix  $\mathbf{A}(\mathcal{X}, I)$ . A unique reconstruction of all trigonometric polynomials  $p \in \Pi_I$  from the sampling values  $(p(\mathbf{x}))_{\mathbf{x} \in \mathcal{X}}$  necessarily implies a full column rank of the matrix  $\mathbf{A}(\mathcal{X}, I)$ . We may use single rank-1 lattices  $\Lambda(\mathbf{z}, M)$  as sampling set  $\mathcal{X}$ , i.e.,  $\mathcal{X} = \Lambda(\mathbf{z}, M)$ . In [13], we investigated necessary and sufficient conditions on the reconstruction property of single rank-1 lattices  $\Lambda(\mathbf{z}, M)$  as spatial discretizations. Roughly speaking, we may have to expect oversampling that increase with growing cardinality of the frequency index set  $I$ , due to the group structure of the lattice nodes.

Motivated by the construction idea of sparse grids, which are in general a union of different lattices of several ranks, we will join a few rank-1 lattices as spatial discretization in order to construct sampling schemes  $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$  that guarantee full column ranks of the Fourier matrices  $\mathbf{A}(\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s), I)$  for given frequency index sets  $I$ . In general, we are interested in practically suitable construction strategies of such multiple rank-1 lattices  $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$ . We call a sampling set  $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$  with

$$\det(\mathbf{A}^*(\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s), I)\mathbf{A}(\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s), I)) > 0$$

a reconstructing multiple rank-1 lattice for the frequency index set  $I$ .

For a given frequency index set  $I$  and given sampling set  $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$ ,  $|\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)| \geq |I|$ , we can check the reconstruction property in different ways. For instance, one can compute the echelon form or (lower bounds on) the smallest singular value of the matrices  $\mathbf{A}$  or  $\mathbf{A}^*\mathbf{A}$  in order to check whether the rank of the matrix is full or not. We emphasize, that the complexity of the test methods is at least  $\Omega(|I|^2)$  and in the case of the computation of lower bounds on the smallest singular values, cf. [32], the test may fail.

However, if the matrix  $\mathbf{A}$  is of full column rank, one can reconstruct the Fourier coefficients

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**Algorithm 4** Adjoint evaluation at multiple rank-1 lattices
 

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Input: $M_1, \dots, M_s \in \mathbb{N}$ $\mathbf{z}_1, \dots, \mathbf{z}_s \in \mathbb{Z}^d$ $I \subset \mathbb{Z}^d$ $\mathbf{p} = \begin{pmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_s \end{pmatrix}$	lattice sizes of rank-1 lattices $\Lambda(\mathbf{z}_l, M_l)$ generating vectors of $\Lambda(\mathbf{z}_l, M_l)$ frequency index set sampling values of $p \in \Pi_I$ , $\mathbf{p}_l = \left( p\left(\frac{j}{M_l} \mathbf{z}_l\right) \right)_{j=1-\delta_{1,l}, \dots, M_l}$
1: $\hat{\mathbf{a}} = (0)_{\mathbf{k} \in I}$ 2: <b>for</b> $l = 1, \dots, s$ <b>do</b> 3: $\mathbf{g} = (\delta_{1,l} \mathbf{p}[1], \mathbf{p}[M_1 + \dots + M_{l-1} - l + 3], \dots, \mathbf{p}[M_1 + \dots + M_l - l + 1])^\top$ 4: $\hat{\mathbf{a}} = \hat{\mathbf{a}} + \text{aLFFT}(M_l, \mathbf{z}_l, I, \mathbf{g})$ 5: <b>end for</b>	
Output: $\hat{\mathbf{a}} = \mathbf{A}^* \mathbf{f}$	result of adjoint matrix times vector product
Complexity: $\mathcal{O}(\sum_{l=1}^s M_l \log M_l + sd I )$	

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of a multivariate trigonometric polynomial  $p \in \Pi_I$  by solving the normal equation

$$\mathbf{A}^* \mathbf{A} \hat{\mathbf{p}} = \mathbf{A}^* \mathbf{p}.$$

Usually, one approximates the inverse of the matrix  $\mathbf{A}^* \mathbf{A}$  using a conjugate gradient method and fast algorithms that compute the matrix-vector products associated with  $\mathbf{A}$  and  $\mathbf{A}^*$ , cf. Algorithm 3 and Algorithm 4 associated with Algorithm 2. Thus, the fast reconstruction of a trigonometric polynomial  $p \in \Pi_I$  using the samples along a reconstructing multiple rank-1 lattice  $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$  for  $I$  is guaranteed. Anyway, we are still interested in a practically suitable construction strategy in order to determine reconstructing multiple rank-1 lattices  $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$  for given frequency index sets  $I$ . In the following subsection, we develop a deterministic construction of reconstructing multiple rank-1 lattices for hyperbolic cross trigonometric polynomials. Subsequently, we present an approach that even works on completely arbitrary frequency index sets.

#### 4.1 A deterministic approach for hyperbolic crosses

In approximation considerations, trigonometric polynomials with frequencies supported on specific index sets that consists of sets of anisotropic full grids has proved very beneficial. In particular, the estimates of the errors of the approximation of periodic functions of dominating mixed smoothness and hybrid mixed smoothness by (energy-norm based) hyperbolic cross trigonometric polynomials are (nearly) optimal. For that reason, we treat a simple, obvious, deterministic construction of multiple rank-1 lattices as sampling scheme for such polynomials in this subsection. We restrict ourselves to dyadic hyperbolic crosses

$$I_{\text{dhc}, N}^d := \bigcup_{\substack{\mathbf{l} \in \mathbb{N}_0^d \\ \|\mathbf{l}\|_1 = n}} \hat{G}_{\mathbf{l}}, \quad \hat{G}_{\mathbf{l}} = \mathbb{Z}^d \cap \prod_{s=1}^d (-2^{l_s-1}, 2^{l_s-1}], \quad (4.1)$$

$N = 2^n$ ,  $n \in \mathbb{N}_0$ , and explain the idea, that generally succeeds for frequency index sets that consists of sets of anisotropic full grids. The natural spatial discretization of trigonometric



polynomials with frequencies supported on dyadic hyperbolic crosses  $I_{\text{dhc},N}^d$  are given by dyadic sparse grids

$$S_{\text{dhc},N}^d := \bigcup_{\substack{\mathbf{l} \in \mathbb{N}_0^d \\ \|\mathbf{l}\|_1 = n}} G_{\mathbf{l}}, \quad G_{\mathbf{l}} = \times_{s=1}^d \left\{ 0, \frac{1}{2^{l_s}}, \frac{3}{2^{l_s}}, \frac{3}{2^{l_s}} \cdots, \frac{2^{l_s} - 1}{2^{l_s}} \right\}.$$

The components of this construction are the full grids  $G_{\mathbf{l}}$ ,  $\|\mathbf{l}\|_1 = n$ , in spatial domain. In general, these full grids are not rank-1 lattices but rank- $s$  lattices with  $s \in \mathbb{N}$  up to  $d$ . In order to determine a similar spatial discretization consisting of rank-1 lattices, we determine the following frequency index set

$$I_{\text{pdhc},N}^d := \bigcup_{\substack{\mathbf{l} \in \mathbb{N}_0^d \\ \|\mathbf{l}\|_1 = n}} \hat{P}_{\mathbf{l}},$$

where the sets  $\hat{P}_{\mathbf{l}}$  are given by

$$\hat{P}_{\mathbf{l}} := \mathbb{Z}^d \cap \times_{s=1}^d \left( \frac{-p_{\mathbf{l},s}}{2}, \frac{p_{\mathbf{l},s}}{2} \right)$$

and the numbers  $p_{\mathbf{l},s}$  are determined by

$$p_{\mathbf{l},s} := \begin{cases} 1 & l_s = 0 \\ \min_p \text{ prime} \{ p \in \mathbb{N} : p > 2^{l_s} \text{ and } p \notin \{p_{\mathbf{l},t} : t = 1, \dots, s-1\} \} & l_s > 0. \end{cases}$$

Moreover, proper spatial full grid discretizations for  $\hat{P}_{\mathbf{l}}$  are given by

$$P_{\mathbf{l}} := \times_{s=1}^d \left\{ 0, \frac{1}{p_{\mathbf{l},s}}, \frac{3}{p_{\mathbf{l},s}}, \frac{3}{p_{\mathbf{l},s}} \cdots, \frac{p_{\mathbf{l},s} - 1}{p_{\mathbf{l},s}} \right\} = \Lambda(\mathbf{z}_{\mathbf{l}}, M_{\mathbf{l}}),$$

where the lattice size  $M_{\mathbf{l}} = \prod_{s=1}^d p_{\mathbf{l},s}$  and the components  $\mathbf{z}_{\mathbf{l},s}$  of the generating vector  $\mathbf{z}_{\mathbf{l}}$  are determined by  $\mathbf{z}_{\mathbf{l},s} = M_{\mathbf{l}}/p_{\mathbf{l},s}$ . Thus, a suitable spatial discretization for trigonometric polynomials supported on the frequency index set  $I_{\text{pdhc},N}^d$  that consists of rank-1 lattices is the union of all  $P_{\mathbf{l}}$

$$S_{\text{pdhc},N}^d := \bigcup_{\substack{\mathbf{l} \in \mathbb{N}_0^d \\ \|\mathbf{l}\|_1 = n}} P_{\mathbf{l}}.$$

Due to Bertrand's postulate the numbers  $p_{\mathbf{l},s}$  are bounded from above by  $p_{\mathbf{l},s} < 2^{l_s+s}$ , and thus  $|\hat{P}_{\mathbf{l}}| = |P_{\mathbf{l}}| \leq 2^{|\mathbf{l}|_1 + d(d+1)/2}$  holds. Accordingly, the cardinality of the spatial discretization  $S_{\text{pdhc},N}^d$  is bounded from above by  $C_d n^{d-1} 2^n$ , which has the same dependence on  $N = 2^n$  as known for  $S_{\text{dhc},N}^d$ .

We assume that the matrix  $\mathbf{A}(S_{\text{pdhc},N}^d, I_{\text{pdhc},N}^d)$  is of full column rank. Accordingly, the sampling values of a trigonometric polynomial with frequencies supported on the dyadic hyperbolic cross  $I_{\text{dhc},N}^d$  at the sparse grid  $S_{\text{pdhc},N}^d$  completely determines the frequency values of this trigonometric polynomial since the embeddings  $I_{\text{dhc},N}^d \subset I_{\text{pdhc},N}^d$  hold.

**Remark 4.1.** Clearly, the full grids  $\hat{P}_l$  may be more close to the grids  $\hat{G}_l$  if we choose the edge lengths  $p_{l,s}$  relatively prime for fixed  $l$  and  $s = 1, \dots, d$ . The advantage of this approach may be lower cardinalities of the full grids  $P_l$  in spatial domain. The lattices  $P_l$  would be rank-1 lattices, nevertheless.  $\square$

**Remark 4.2.** Since we are interested in sampling sets that avoid the disadvantages of rank-1 lattices (increasing oversampling) and sparse grids (growing condition numbers of the Fourier matrices), we considered these characteristics in more detail. Similar to the matrices  $\mathbf{A} \left( S_{\text{dhc},N}^d, I_{\text{dhc},N}^d \right)$ , the Fourier matrices  $\mathbf{A} \left( S_{\text{pdhc},N}^d, I_{\text{dhc},N}^d \right)$  seem to suffer from growing condition numbers in general, cf. Section 5.2 and Figure 5.3.  $\square$

Due to the fact that the numerical tests suggest that there is no significant advantage of the sampling sets  $S_{\text{pdhc},N}^d$  compared to the sparse grids  $S_{\text{dhc},N}^d$ , we do not consider this approach in more detail. Instead, we present the crucial idea of this paper that allows for the construction of suitable spatial discretizations even for arbitrary multivariate trigonometric polynomials.

## 4.2 Construction of reconstructing multiple rank-1 lattices for arbitrary frequency index sets

In this subsection we characterize an algorithm that determines reconstructing multiple rank-1 lattices for given frequency index sets  $I \subset \mathbb{Z}^d$ ,  $|I| < \infty$ . Accordingly, we assume the frequency index set  $I$  of finite cardinality being given and fixed. We suggest to reconstruct a trigonometric polynomial  $p \in \Pi_I$  step by step. Independent of the structure of the frequency index set  $I$ , we fix a lattice size  $M_1 \sim |I|$  and a generating vector  $\mathbf{z}_1 \in [1, M_1 - 1]^d$  and reconstruct only these frequencies  $\hat{p}_{\mathbf{k}}$  that can be uniquely reconstructed by means of the sampling values along the used rank-1 lattice. The indices  $\mathbf{k}$  of these frequencies  $\hat{p}_{\mathbf{k}}$  are simply given by

$$I_1 = \{\mathbf{k} \in I : \mathbf{k} \cdot \mathbf{z}_1 \not\equiv \mathbf{h} \cdot \mathbf{z}_1 \pmod{M_1} \text{ for all } \mathbf{h} \in I \setminus \{\mathbf{k}\}\}.$$

Assuming that the frequencies  $(\hat{p}_{\mathbf{k}})_{\mathbf{k} \in I_1}$  are already determined, we have to reconstruct a trigonometric polynomial  $p_1 \in \Pi_{I \setminus I_1}$  supported by the frequency index set  $I \setminus I_1$  now. We determine the sampling values of this trigonometric polynomial by

$$p_1(\mathbf{x}) = p(\mathbf{x}) - \sum_{\mathbf{k} \in I_1} \hat{p}_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}.$$

We apply this strategy successively as long as there are frequencies that need to be reconstructed.

Algorithm 5 indicates one possibility of an algorithm that determines a set of rank-1 lattices that allows for the application of the mentioned reconstruction strategy. Some basic points are given by

- considering each prime number as lattice size not more than once [Line 5 to 9],
- applying a component-by-component (CBC) strategy, if possible [Lines 10 to 13],
- applying the strategy indicated above [Lines 14 to 24].

---

**Algorithm 5** Determining reconstructing multiple rank-1 lattices
 

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Input:	$I \subset \mathbb{Z}^d$ $\sigma \in \mathbb{R}$ $n \in \mathbb{N}$	frequency index set oversampling factor $\sigma \geq 1$ number of random test vectors
1: $l = 1$ 2: $M_{\text{candidates}} = \emptyset$ 3: <b>while</b> $ I  > 1$ <b>do</b> 4: $s = l$ 5: $M = \text{nextprime}(\sigma I )$ 6: <b>while</b> $M \in M_{\text{candidates}}$ <b>do</b> 7: $M = \text{nextprime}(M)$ 8: <b>end while</b> 9: $M_{\text{candidates}} = M_{\text{candidates}} \cup \{M\}$ 10: <b>if</b> $M > \max \left\{ \frac{ I ^2}{2} + 2, \max \{2\ \mathbf{k}\ _\infty : \mathbf{k} \in I\} \right\}$ <b>then</b> 11:     determine $\mathbf{z}_l$ using the CBC method from [13] such that $\Lambda(\mathbf{z}_l, M)$ is a reconstructing rank-1 lattice for $I$ 12: $M_l = M$ 13: $I = \emptyset$ 14: <b>else</b> 15: <b>for</b> $j = 1, \dots, n$ <b>do</b> 16:       choose random integer vector $\mathbf{v}_j \in [1, M - 1]^d$ 17:       compute $K_j =  I \setminus \{\mathbf{k} \in I : \exists \mathbf{h} \in I \setminus \{\mathbf{k}\} \text{ with } \mathbf{k} \cdot \mathbf{v}_j \equiv \mathbf{h} \cdot \mathbf{v}_j \pmod{M}\} $ 18: <b>end for</b> 19: <b>if</b> $\max_{j \in \{1, \dots, n\}} K_j > 0$ <b>then</b> 20:       determine a $\mathbf{z}_l = \mathbf{v}_{j_0}$ such that $K_{j_0} = \max_{j \in \{1, \dots, n\}} K_j$ 21: $M_l = M$ 22: $I = \{\mathbf{k} \in I : \exists \mathbf{h} \in I \setminus \{\mathbf{k}\} \text{ with } \mathbf{k} \cdot \mathbf{z}_l \equiv \mathbf{h} \cdot \mathbf{z}_l \pmod{M_l}\}$ 23: $l = l + 1$ 24: <b>end if</b> 25: <b>end if</b> 26: <b>end while</b>		
Output:	$M_1, \dots, M_s$ $\mathbf{z}_1, \dots, \mathbf{z}_s$ $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$	lattice sizes of rank-1 lattices and generating vectors of rank-1 lattices such that is a reconstructing multiple rank-1 lattice

---

Obviously, if the branch consisting of the CBC strategy is involved, the algorithm immediately terminates. Since we deal with each prime number only once and the condition in Line 10 is bounded from above for each subset of a given frequency index set  $I$  of finite cardinality, the CBC strategy branch will sooner or later be involved in the case that the algorithm does not terminate using only the branch depending on randomness.

In each step that does not determine the generating vector  $\mathbf{z}_l$  using the CBC strategy, we are interested in a rank-1 lattice that allows for the reconstruction of as many as possible frequencies of a subset of  $I$ . For that reason, we check a few generating vectors for their number of reconstructible frequency indices in Algorithm 5 and choose one of the most successful one, cf. Line 15 to 23.

We show that Algorithm 5 determines a sampling set such that  $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$  implies a full column rank matrix  $\mathbf{A}(\Lambda(\mathbf{z}_1, M_1, \mathbf{z}_2, M_2, \dots, \mathbf{z}_s, M_s), I)$ .

**Lemma 4.3.** *Let the matrix  $\mathbf{B} \in \mathbb{C}^{n \times m}$  of the following form*

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{pmatrix}$$

be given. The matrices  $\mathbf{B}_1 \in \mathbb{C}^{n_1 \times m_1}$ ,  $\dots$ ,  $\mathbf{B}_4 \in \mathbb{C}^{n_2 \times m_2}$  are submatrices of  $\mathbf{B}$ , i.e.,  $n = n_1 + n_2$  and  $m = m_1 + m_2$ . In addition, we assume that

- $\mathbf{B}_1$  has full column rank, i.e., the columns of  $\mathbf{B}_1$  are linear independent,
- $\mathbf{B}_4$  has full column rank, i.e., the columns of  $\mathbf{B}_4$  are linear independent, and
- the columns of  $\mathbf{B}_2$  are not in the span of the columns of  $\mathbf{B}_1$ .

Then the matrix  $\mathbf{B}$  has full column rank.

*Proof.* The matrix  $\mathbf{B} \in \mathbb{C}^{n \times m}$ ,  $n \geq m$ , has full column rank iff the columns of the matrix  $\mathbf{B}$  are all linear independent, i.e. the formula

$$\sum_{j=1}^m \lambda_j \mathbf{b}_j = \mathbf{0} \tag{4.2}$$

has the unique solution  $\boldsymbol{\lambda} = \mathbf{0}$ . We will exploit the full column rank of the matrix  $\mathbf{B}_1 \in \mathbb{C}^{n_1 \times m_1}$ ,  $n_1 \geq m_1$ , and, thus, we consider the sum

$$\sum_{j=1}^m \lambda_j \mathbf{b}'_j = \mathbf{0},$$

where  $\mathbf{b}'_j = (b_{j,l})_{l=1}^{n_1}$  are vectors consisting of the first  $n_1$  elements of the vectors  $\mathbf{b}_j$ . Due to the fact that the columns of  $\mathbf{B}_2 \in \mathbb{C}^{n_1 \times m_2}$  are not in the span of the columns of  $\mathbf{B}_1$  and the columns of  $\mathbf{B}_1$  are linear independent, we achieve  $\lambda_j = 0$  for all  $j = 1, \dots, m_1$ .

Accordingly, (4.2) simplifies to

$$\sum_{j=m_1+1}^m \lambda_j \mathbf{b}_j = \mathbf{0}.$$

For the remaining vectors  $\mathbf{b}_j$ ,  $j = m_1 + 1, \dots, m$ , we know that the vectors of the last  $m_2$  components of  $\mathbf{b}_j$  are linear independent and, consequently, we obtain  $\lambda_j = 0$  for all  $j = 1, \dots, m$ . ■

**Theorem 4.4.** *Algorithm 5 determines sampling sets such that the corresponding Fourier matrix  $\mathbf{A}(\Lambda(\mathbf{z}_1, M_1, \mathbf{z}_2, M_2, \dots, \mathbf{z}_s, M_s), I)$  is a full column rank matrix.*

*Proof.* In order to exploit Lemma 4.3 we will need to rearrange the order of the columns of  $\mathbf{A}(\Lambda(\mathbf{z}_1, M_1, \mathbf{z}_2, M_2, \dots, \mathbf{z}_s, M_s), I)$  in a suitable way. We assume that  $M_1, \dots, M_s$  and

$z_1, \dots, z_s$  is the output of Algorithm 5. Consequently, we can determine the following frequency index sets

$$\begin{aligned} I_1^{\mathbb{C}} &= \{\mathbf{k} \in I: \exists \mathbf{h} \in I \setminus \{\mathbf{k}\} \text{ with } \mathbf{k} \cdot \mathbf{z}_1 \equiv \mathbf{h} \cdot \mathbf{z}_1 \pmod{M_1}\} & \text{and } I_1 &= I \setminus I_1^{\mathbb{C}}, \\ I_2^{\mathbb{C}} &= \{\mathbf{k} \in I_1^{\mathbb{C}}: \exists \mathbf{h} \in I_1^{\mathbb{C}} \setminus \{\mathbf{k}\} \text{ with } \mathbf{k} \cdot \mathbf{z}_2 \equiv \mathbf{h} \cdot \mathbf{z}_2 \pmod{M_2}\} & \text{and } I_2 &= I_1^{\mathbb{C}} \setminus I_2^{\mathbb{C}}, \\ &\vdots & & \vdots \\ I_s^{\mathbb{C}} &= \{\mathbf{k} \in I_{s-1}^{\mathbb{C}}: \exists \mathbf{h} \in I_{s-1}^{\mathbb{C}} \setminus \{\mathbf{k}\} \text{ with } \mathbf{k} \cdot \mathbf{z}_s \equiv \mathbf{h} \cdot \mathbf{z}_s \pmod{M_s}\} = \emptyset & \text{and } I_s &= I_{s-1}^{\mathbb{C}} \setminus I_s^{\mathbb{C}}. \end{aligned}$$

The resulting frequency index sets  $I_j$ ,  $j = 1, \dots, s$ , are a disjoint partition of  $I$ . We rearrange the columns of the matrix  $\mathbf{A}(\Lambda(\mathbf{z}_1, M_1, \mathbf{z}_2, M_2, \dots, \mathbf{z}_s, M_s), I)$  and we achieve a matrix

$$\tilde{\mathbf{A}} := \begin{pmatrix} \mathbf{K}^{1,1} & \dots & \mathbf{K}^{1,s} \\ \vdots & \ddots & \vdots \\ \mathbf{K}^{s,1} & \dots & \mathbf{K}^{s,s} \end{pmatrix},$$

where the submatrices  $\mathbf{K}^{r,l}$  are given by  $\mathbf{K}^{r,l} := \left( e^{2\pi i \frac{j}{M_r} \mathbf{k} \cdot \mathbf{z}_r} \right)_{j=1-\delta_{1,r}, \dots, M_r-1, \mathbf{k} \in I_l}$ ,  $(r, l) \in [1, s]^2 \cap \mathbb{N}^2$ .

We define the matrices

$$\tilde{\mathbf{A}}_l := (\mathbf{K}^{r,t})_{r=1, \dots, l, t=1, \dots, l} = \begin{pmatrix} & & & \mathbf{K}^{1,l} \\ & \tilde{\mathbf{A}}_{l-1} & & \vdots \\ & & & \mathbf{K}^{l-1,l} \\ \mathbf{K}^{l,1} & \dots & \mathbf{K}^{l,l-1} & \mathbf{K}^{l,l} \end{pmatrix},$$

which are in fact submatrices of  $\tilde{\mathbf{A}}$ . In particular, we obtain  $\tilde{\mathbf{A}}_1 = \mathbf{K}^{1,1}$  and  $\tilde{\mathbf{A}}_s = \tilde{\mathbf{A}}$ .

In the following, we conclude the full column rank of  $\tilde{\mathbf{A}}_l$  from the full column rank of  $\tilde{\mathbf{A}}_{l-1}$ , the full column rank of  $\mathbf{K}^{l,l}$ , and the linear independence of each column of the matrix

$$\begin{pmatrix} \mathbf{K}^{1,l} \\ \vdots \\ \mathbf{K}^{l-1,l} \end{pmatrix}$$

and all columns of the matrix  $\tilde{\mathbf{A}}_{l-1}$ , cf. Lemma 4.3.

We start with  $l = 1$ , i.e.,  $\tilde{\mathbf{A}}_1 = \mathbf{K}^{1,1}$ . Since  $\Lambda(\mathbf{z}_1, M_1)$  is a reconstructing rank-1 lattice for  $I_1$ , the matrix  $\tilde{\mathbf{A}}_1$  has linear independent columns. Due to the construction of  $I_1$  and  $I_1^{\mathbb{C}}$ , each column of one of the matrices  $\mathbf{K}^{1,r}$ ,  $r = 2, \dots, s$ , is not in the span of the columns of  $\mathbf{K}^{1,1}$ .

We prove the full rank of  $\tilde{\mathbf{A}}_l$ ,  $l = 2, \dots, s$ , inductively. For that, we assume  $\tilde{\mathbf{A}}_{l-1}$  to be of full column rank. Additionally, we know that  $\mathbf{K}^{l,l}$  is of full column rank, since  $\Lambda(\mathbf{z}_l, M_l)$  is a reconstructing rank-1 lattice for  $I_l$ . In order to apply Lemma 4.3, we have to show, that no column of the matrix

$$\begin{pmatrix} \mathbf{K}^{1,l} \\ \vdots \\ \mathbf{K}^{l-1,l} \end{pmatrix} \tag{4.3}$$

is a linear combination of the columns of  $\tilde{\mathbf{A}}_{l-1}$ . We assume the contrary, i.e., let  $\mathbf{k} \in I_l$  be a frequency index,  $\mathbf{a}_{\mathbf{k}}$  the corresponding column of the matrix in (4.3), such that

$$\mathbf{a}_{\mathbf{k}} = \tilde{\mathbf{A}}_{l-1} \boldsymbol{\lambda} = \begin{pmatrix} \mathbf{K}^{1,1} & \dots & \mathbf{K}^{1,l-1} \\ \vdots & \dots & \vdots \\ \mathbf{K}^{l-1,1} & \dots & \mathbf{K}^{l-1,l-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda}_1 \\ \vdots \\ \boldsymbol{\lambda}_{l-1} \end{pmatrix}, \quad (4.4)$$

where we look for the vectors  $\boldsymbol{\lambda}_r \in \mathbb{C}^{|I_r|}$ ,  $r = 1, \dots, l-1$ . We solve this linear equation recursively. The first  $M_1$  rows of  $\mathbf{a}_{\mathbf{k}}$  are not in the span of the pairwise orthogonal columns of  $\mathbf{K}^{1,1}$  and the columns of the first  $M_1$  rows of  $\mathbf{K}^{1,r}$ ,  $r = 2, \dots, l-1$  are also not in the span of the columns of  $\mathbf{K}^{1,1}$ . Consequently, we observe that  $\boldsymbol{\lambda}_1 = \mathbf{0} \in \mathbb{C}^{|I_1|}$  holds. Accordingly, the solution of (4.4) is given by

$$\boldsymbol{\lambda} := (\boldsymbol{\lambda}_1^\top, \dots, \boldsymbol{\lambda}_{l-1}^\top)^\top = \underbrace{(0, \dots, 0)}_{|I_1| \text{ times}}, \boldsymbol{\lambda}_2^\top, \dots, \boldsymbol{\lambda}_{l-1}^\top)^\top$$

and we search for the solution of

$$\mathbf{a}_{\mathbf{k}} = \begin{pmatrix} \mathbf{K}^{1,2} & \dots & \mathbf{K}^{1,l-1} \\ \vdots & \dots & \vdots \\ \mathbf{K}^{l-1,2} & \dots & \mathbf{K}^{l-1,l-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda}_2 \\ \vdots \\ \boldsymbol{\lambda}_{l-1} \end{pmatrix}.$$

Next, we consider the rows numbered by  $M_1 + 1, \dots, M_1 + M_2 - 1$  and obtain

$$(a_{\mathbf{k},j})_{j=1+M_1}^{-1+M_1+M_2} = \left( \mathbf{K}^{2,2}, \dots, \mathbf{K}^{2,l-1} \right) \begin{pmatrix} \boldsymbol{\lambda}_2 \\ \vdots \\ \boldsymbol{\lambda}_{l-1} \end{pmatrix}.$$

Since  $\Lambda(\mathbf{z}_2, M_2)$  is a reconstructing rank-1 lattice for  $I_2$  and there is no  $\mathbf{k}' \in \bigcup_{j=3}^l I_j$  that aliases to a frequency index  $\mathbf{k}'' \in I_2$  with respect to  $\Lambda(\mathbf{z}_2, M_2)$ , we obtain the result  $\boldsymbol{\lambda}_2 = (0, \dots, 0)^\top \in \mathbb{C}^{|I_2|}$ . These considerations lead inductively to the formulas

$$(a_{\mathbf{k},j})_{j=3-t+\sum_{r=1}^{t-1} M_r}^{1-t+\sum_{r=1}^t M_r} = \left( \mathbf{K}^{t,t}, \dots, \mathbf{K}^{t,l-1} \right) \begin{pmatrix} \boldsymbol{\lambda}_t \\ \vdots \\ \boldsymbol{\lambda}_{l-1} \end{pmatrix}, \quad t = 1, \dots, l-1$$

and the result  $\boldsymbol{\lambda}_t = \mathbf{0} \in \mathbb{C}^{|I_t|}$ ,  $t = 1, \dots, l-1$ , in (4.4), which implies  $\mathbf{a}_{\mathbf{k}} = \mathbf{0}$  and is in contradiction to  $\|\mathbf{a}_{\mathbf{k}}\|_1 = 1 - (l-1) + \sum_{r=1}^{l-1} M_r > 0$ .

Consequently, we apply Lemma 4.3 on the matrix  $\tilde{\mathbf{A}}_l$  and observe the full column rank of  $\tilde{\mathbf{A}}_l$ , in particular for  $l = s$ .  $\blacksquare$

Algorithm 5 determines a reconstructing sampling scheme for all multivariate trigonometric polynomials with frequencies supported on the index set  $I$ . Moreover, the idea behind Algorithm 5 is the stepwise reconstruction of trigonometric polynomials as mentioned above. Accordingly, we indicate the corresponding reconstruction strategy in Algorithm 6, where we use Algorithms 1 and 2 in order to compute the required single lattice based discrete Fourier transforms. As a consequence, we achieve a computational complexity of this algorithm which is bounded by  $C(\sum_{l=1}^s M_l \log M_l + s(d + \log |I|)|I|)$ , where the term  $C$  does not depend on the multiple rank-1 lattice  $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$ , the frequency index set  $I$ , or the spatial dimension  $d$ .

---

**Algorithm 6** Direct reconstruction of trigonometric polynomials  $p \in \Pi_I$  from samples along reconstructing multiple rank-1 lattices that are determined by Algorithm 5

---

Input: $M_1, \dots, M_s \in \mathbb{N}$ $\mathbf{z}_1, \dots, \mathbf{z}_s \in \mathbb{Z}^d$ $I \subset \mathbb{Z}^d$ $\mathbf{p} = \begin{pmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_s \end{pmatrix}$	lattice sizes of rank-1 lattices $\Lambda(\mathbf{z}_l, M_l)$ generating vectors of $\Lambda(\mathbf{z}_l, M_l)$ frequency index set sampling values of $p \in \Pi_I$ , $\mathbf{p}_l = \left( p\left(\frac{j}{M_l} \mathbf{z}_l\right) \right)_{j=1-\delta_{1,l}, \dots, M_l}$
1: $I_R = \{\}$ 2: $\hat{\mathbf{p}} = (\mathbf{0})_{\mathbf{k} \in I}$ 3: <b>for</b> $l = 1, \dots, s$ <b>do</b> 4: $I_l = \{\mathbf{k} \in I \setminus I_R : \mathbf{k} \cdot \mathbf{z}_l \not\equiv \mathbf{h} \cdot \mathbf{z}_l \pmod{M_l} \text{ for all } \mathbf{h} \in I \setminus (I_R \cup \{\mathbf{k}\})\}$ 5: $\mathbf{g}_l = (\mathbf{p}[1], \mathbf{p}[3-l + \sum_{r=1}^{l-1} M_r], \dots, \mathbf{p}[1-l + \sum_{r=1}^l M_r])^\top - \text{LFFFT}(M_l, \mathbf{z}_l, I_R, (\hat{\mathbf{p}}_{\mathbf{k}})_{\mathbf{k} \in I_R})$ 6: $(\hat{\mathbf{p}}_{\mathbf{k}})_{\mathbf{k} \in I_l} = \text{aLFFFT}(M_l, \mathbf{z}_l, I_l, \mathbf{g}_l)$ 7: $I_R = I_R \cup I_l$ 8: <b>end for</b>	
Output: $\hat{\mathbf{p}} = (\mathbf{A}^* \mathbf{A})^{-1} \mathbf{A}^* \mathbf{p}$	Fourier coefficients of $p \in \Pi_I$

---

Complexity:  $\mathcal{O}(\sum_{l=1}^s M_l \log M_l + s(d + \log |I|)|I|)$

---

## 5 Numerics

We use Algorithm 5 with oversampling parameter  $\sigma = 1$  and we choose  $n = 10d$  in order to determine the multiple rank-1 lattices that are used in our numerical examples. We computed the numerical tests on a compute server that has four Intel Xeon CPU E5-4640 2.40GHz 8-core, 512 GB RAM, running Matlab-R2015a on openSUSE Linux 13.1. In order to capture the computational times we restrict the number of used CPU to one using the parameter `-singleCompThread` for Matlab. The sparse grid Fourier transforms (HCFFT) are computed using the HCFFT from the NHCFFT toolbox, cf. [6].

### 5.1 Axis Crosses

We consider so-called axis crosses of a specific width  $N = 2^n$ ,  $n \in \mathbb{N}_0$ , defined by

$$I = I_{\text{ac}, N}^d := \{\mathbf{k} \in \mathbb{Z}^d : \|\mathbf{k}\|_\infty = \|\mathbf{k}\|_1 \leq 2^n\}$$

as frequency index sets. One can use sampling values along single rank-1 lattices  $\Lambda(\mathbf{z}, M)$  for the reconstruction of multivariate trigonometric polynomials supported on axis crosses, cf. [13, Ex. 3.27]. The main disadvantage of this approach is the necessary oversampling, i.e., one needs at least  $|\Lambda(\mathbf{z}, M)| \geq (N+1)^2$  sampling values in order to ensure a unique reconstruction of trigonometric polynomials  $p \in \Pi_{I_{\text{ac}, N}^d}$ . We compare this lower bound on the number of sampling values to the number of frequency indices within  $I_{\text{ac}, N}^d$ . This yields oversampling factors of

$$\frac{|\Lambda(\mathbf{z}, M)|}{|I_{\text{ac}, N}^d|} \geq \frac{(N+1)^2}{2dN+1} \geq \frac{N}{2d}$$

and thus the oversampling increases linearly in  $N$ .

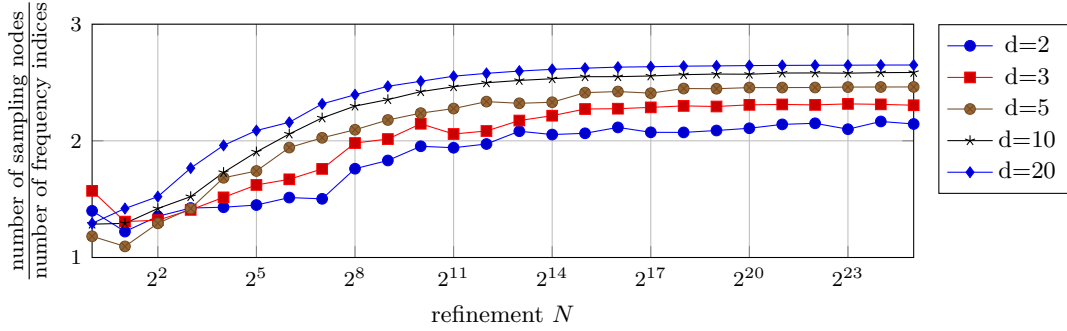


Figure 5.1: Oversampling factors of reconstructing multiple rank-1 lattices for axis crosses  $I_{ac,N}^d$ .

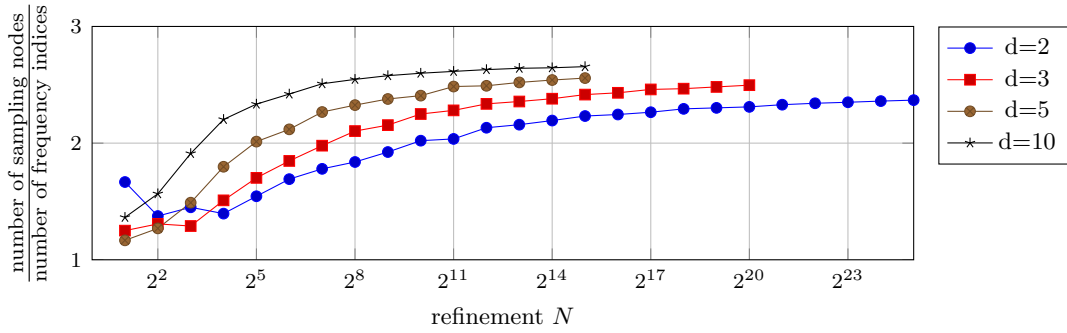


Figure 5.2: Oversampling factors of reconstructing multiple rank-1 lattices for dyadic hyperbolic crosses  $I_{dhc,N}^d$ .

On the other hand, we want to use multiple rank-1 lattices as spatial discretizations. Our numerical tests, illustrated in Figure 5.1, allows for the conjecture that the oversampling factors stagnate for growing width  $N$  of the considered axis crosses of fixed dimension  $d$ . Moreover, we observe small oversampling factors which are always below three. For comparison, sampling along a reconstructing single rank-1 lattice necessarily implies oversampling factors greater than 400 for  $N \geq 2^{14}$  and all dimensions  $2 \leq d \leq 20$ .

## 5.2 Dyadic Hyperbolic Crosses

In this section, we consider so-called dyadic hyperbolic crosses  $I = I_{dhc,N}^d$ , cf. (4.1), where the number  $N = 2^n$ ,  $n \in \mathbb{N}_0$ , is a power of two.

We applied Algorithm 5 to dyadic hyperbolic crosses of different refinements and dimensions in order to determine reconstructing multiple rank-1 lattices. The resulting oversampling factors slightly grows with respect to the refinement and dimension, cf. Figure 5.2. We observe that the oversampling factors seem to stagnate. More precisely, Euler's number  $e$  bounds them from above in all our numerical tests.

In the following, we compare three different sampling methods in order to reconstruct multivariate trigonometric polynomials with frequencies supported on dyadic hyperbolic cross index sets:



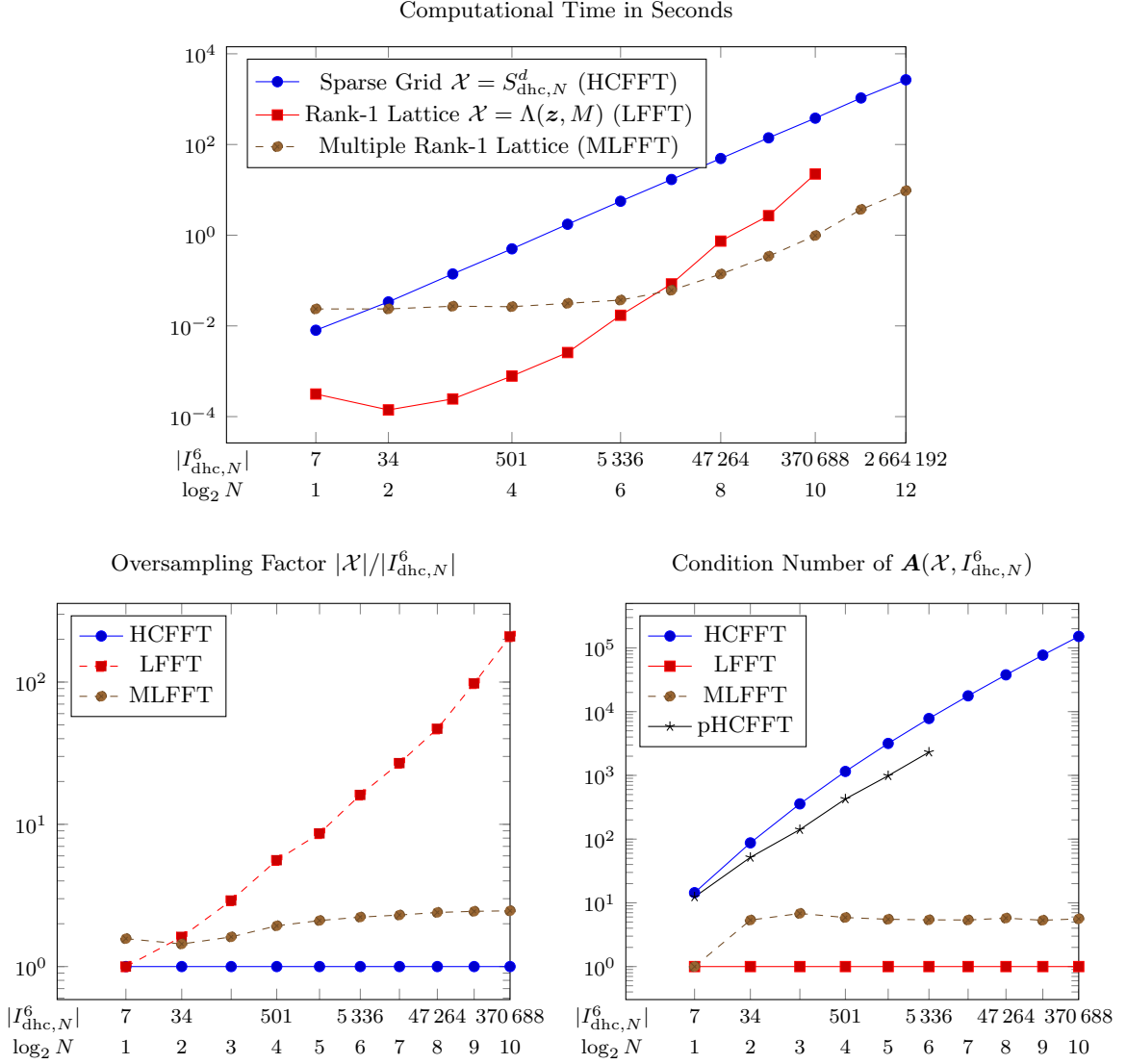


Figure 5.3: Six-dimensional hyperbolic cross fast Fourier transforms for comparison.

- Sampling along sparse grids (HCFFT, cf. [10]),
- sampling along reconstructing single rank-1 lattices (LFFT, cf. [12]),
- sampling along multiple rank-1 lattices found by Algorithm 5 (MLFFT).

Our numerical tests illustrate the characteristics of the discrete Fourier transforms (condition number of  $\mathbf{A}(\mathcal{X}, I)$ , spent oversampling  $|\mathcal{X}|/|I|$ ), and the computational times of the corresponding fast algorithms.

### Fixed Dimension $d = 6$

Since we are interested in numerical tests that visualize in some sense the asymptotic behavior of the three different sampling methods, we fix the moderate dimension  $d = 6$ . Thus, we can

compute condition numbers even for moderate refinements  $N = 2^n$  up to  $N = 1024$ , cf. Figure 5.3. We observe, that the discrete Fourier transform based on multiple rank-1 lattice sampling requires only low oversampling factors and that the corresponding Fourier matrices are well-conditioned. Moreover, the computational complexity of the fast Algorithm of the Fourier transform (MLFFT) is illustrated and at least almost as good as the computational complexity of the HCFFT with respect to  $N$ . We emphasize, that the computational complexity is described by the slopes of the given plots that show the computational times, independent from constant factors caused by hardware or software specifications. Accordingly, the MLFFT seems to avoid both the disadvantage of the HCFFT (growing condition numbers, cf. [14]) and the disadvantage of the LFFT (growing oversampling factors and the associated fast growing computational times, cf. [15]).

In addition, we tested the approach considered in Section 4.1 (pHCFFT), where we construct a sparse grid like sampling scheme consisting of boxes with edge lengths that are prime numbers. However, we plotted only the condition numbers of the Fourier matrices  $\mathbf{A}(S_{\text{pdhc},N}^6, I_{\text{dhc},N}^6)$ , cf. Figure 5.3. These condition numbers behaves similar to the condition numbers belonging to the HCFFT, and thus are the problematic properties of the pHCFFT. Moreover, the oversampling factors  $S_{\text{pdhc},N}^6/I_{\text{dhc},N}^6$ , that are not plotted, are relatively large up to 175.

#### Fixed Refinement $N = 2^2, \dots, 2^5$

In [15, Fig. 4.2] the authors compared the computational times of sampling along sparse grids to sampling along reconstructing single rank-1 lattices. The lesson of this figure is clear: Unique sampling along reconstructing single rank-1 lattices may be not optimal due to the fact that the number of necessarily used sampling values, cf. [12], may be not optimal, in general.

Similar to this, we illustrate the computational times of different fast algorithms in Figure 5.4. More precisely, we mapped the computational times of the

- hyperbolic cross fast Fourier transform (HCFFT), i.e., the fast algorithm for the evaluation of hyperbolic cross trigonometric polynomials at all nodes of a sparse grid and the fast algorithm for the reconstruction of hyperbolic cross trigonometric polynomials from the sampling values at sparse grid nodes,
- multiple lattice fast Fourier transform (MLFFT) applied to hyperbolic cross trigonometric polynomials, i.e., the fast algorithm for the evaluation of multivariate trigonometric polynomials at all nodes of a reconstructing multiple rank-1 lattice and different fast algorithms for the reconstruction of multivariate trigonometric polynomials from the sampling values at a reconstructing multiple rank-1 lattice.

In addition to the direct reconstruction, cf. Algorithm 6, we applied a CG method with starting vector  $\hat{\mathbf{p}} = \mathbf{0}$  and a CG method with starting vector  $\hat{\mathbf{p}}$  that is the result of the direct reconstruction. We are interested in the application of such a CG method since the direct reconstruction suffers from growing relative errors

$$\text{err}_2 := \frac{\|\tilde{\hat{\mathbf{p}}} - \hat{\mathbf{p}}_0\|_2}{\|\hat{\mathbf{p}}_0\|_2}, \quad (5.1)$$

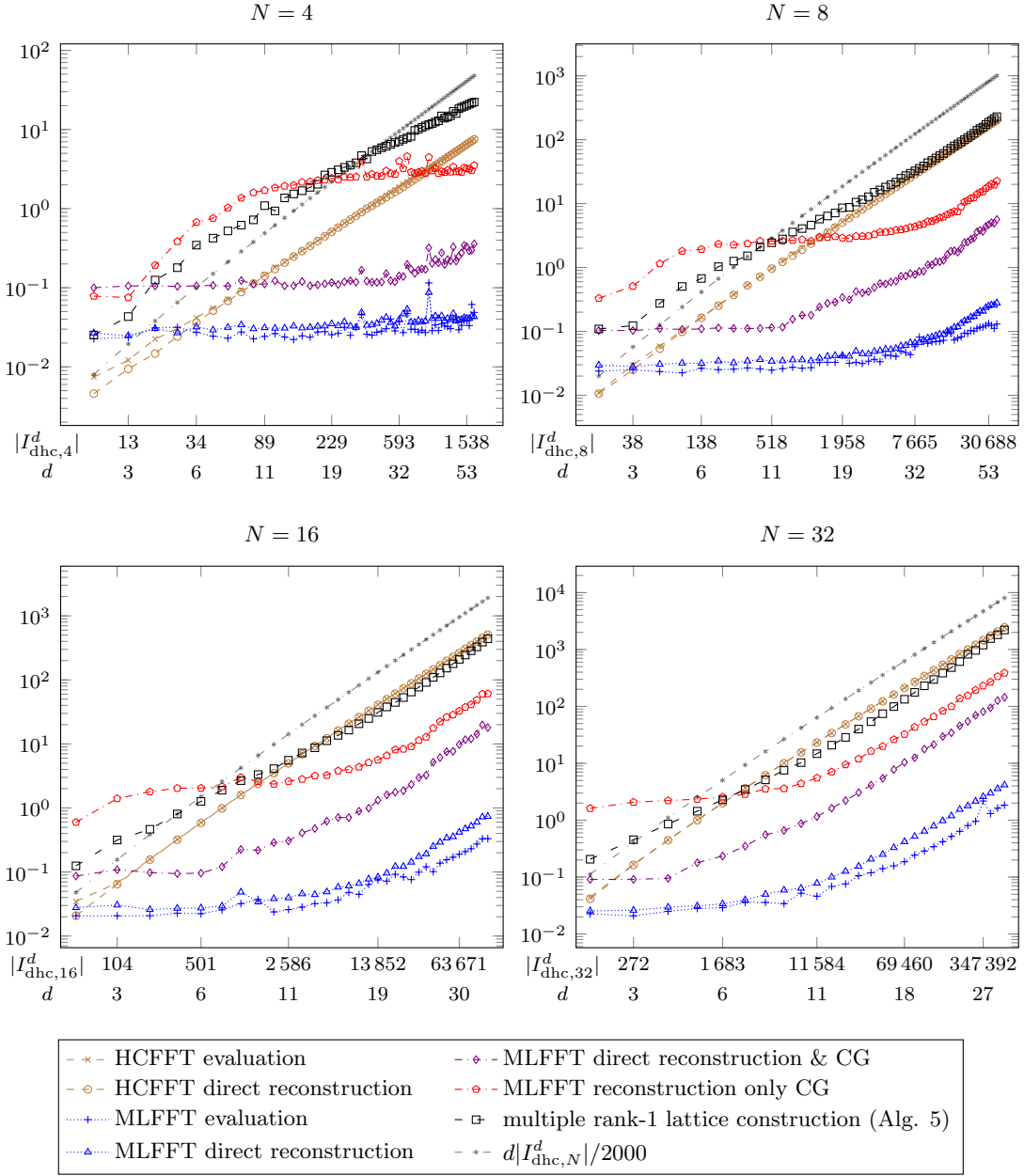


Figure 5.4: Computational times in seconds of the fast algorithms computing the dyadic hyperbolic cross discrete Fourier transforms with respect to the problem size  $|I_{dhc,N}^d|$ .

where  $\tilde{\mathbf{p}}$  is the result of the reconstruction from sampling values of the hyperbolic cross trigonometric polynomial with frequencies  $\hat{\mathbf{p}}_0$ , cf. Figure 5.5. This figure also contains the relative errors of the fast reconstruction method from sampling values at sparse grids (HCFFT). We point out that this fast algorithm also suffers from growing relative errors. We did not apply a conjugate gradient method on the HCFFT since we expect huge computational costs due to the expected number of iterations of the CG method which is indicated by the growing condition numbers of the corresponding Fourier matrices.

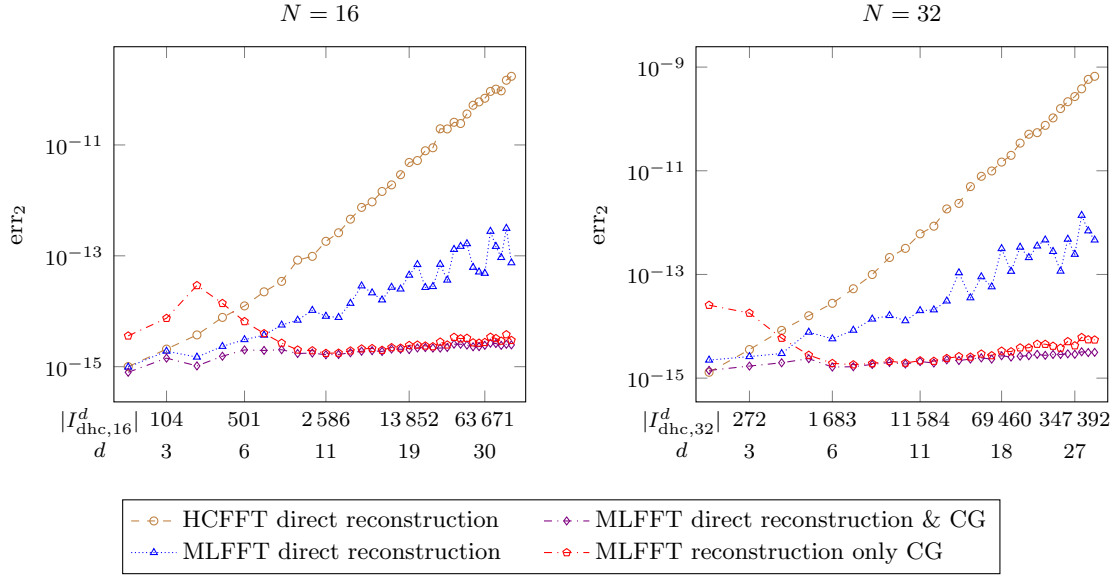


Figure 5.5: Relative errors, cf. (5.1), of the fast algorithms computing the dyadic hyperbolic cross discrete Fourier transforms with respect to the problem size  $|I_{\text{dhc},N}^d|$ .

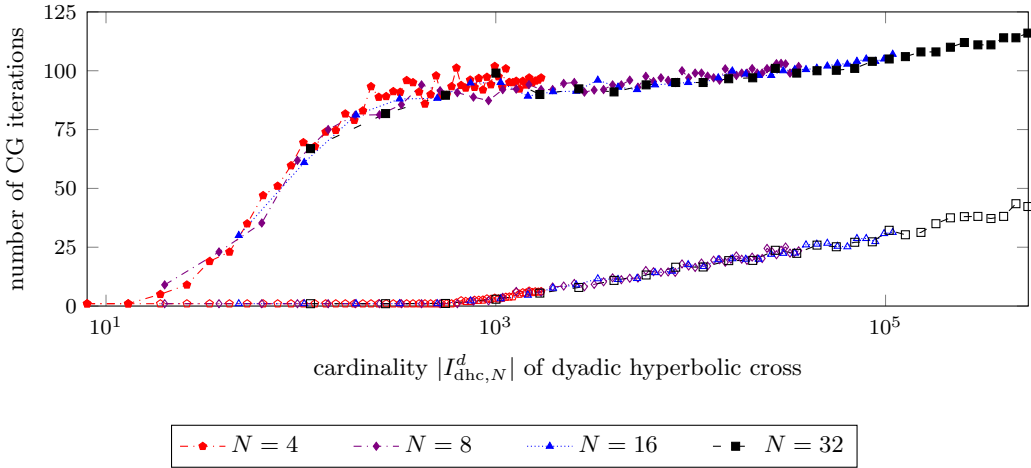


Figure 5.6: Number of iteration steps of the applied CG method against the cardinality of the hyperbolic cross. Each data point is the average of ten tests that differ of at most three (filled markers) and nine (unfilled markers). Filled markers: CG with starting vector zero, unfilled markers: CG with starting vector from Algorithm 6.

The runtimes of the new fast algorithms (MLFFT) behave similar to the runtimes of the HCFFT with respect to the dimension  $d$ , cf. Figure 5.4. Even the CG methods have a similar runtime behavior, which is caused by the extremely slow-growing number of iterations of the CG method, cf. Figure 5.6. Furthermore, Figure 5.6 indicates that the number of iterations slightly depends on the size of the matrices  $\mathbf{A}(\Lambda(z_1, M_1, \dots, z_s, M_s), I_{\text{dhc},N}^d)$  but not on the dimension  $d$  or the refinement parameter  $N$ , which is in accordance with the observation of stagnating condition numbers of the matrices  $\mathbf{A}(\Lambda(z_1, M_1, \dots, z_s, M_s), I_{\text{dhc},N}^d)$ , cf. Figure 5.7.

Condition Number of  $\mathbf{A}(\mathcal{X}, I_{\text{dhc},32}^d)$

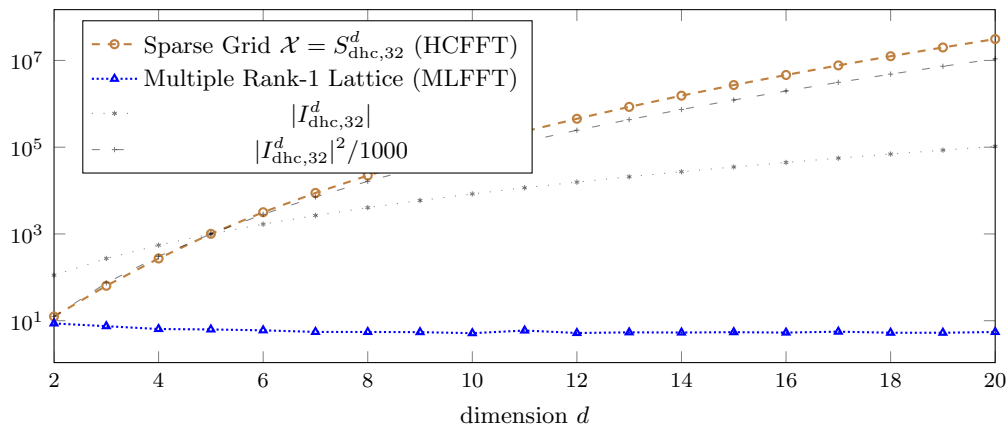


Figure 5.7: Condition numbers of the Fourier matrices using corresponding sparse grids and reconstructing multiple rank-1 lattices as sampling schemes.

### 5.3 Random frequency index sets

The complexity of the new fast Fourier transform algorithms, cf. Algorithms 3, 4, and 6, mainly depends on the number  $s$  of rank-1 lattices that are joined in order to build the multiple rank-1 lattice  $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$ . We demonstrate the behavior of  $s$  for ten-dimensional randomly chosen frequency index sets. Each component of the indices  $\mathbf{k} \in I \subset \mathbb{Z}^{10}$  are rounded values that are chosen from a normal distribution with mean zero and variance 10 000.

We considered frequency index sets  $I$  of different cardinalities which are powers of two, i.e.  $|I| \in \{2, 4, 8, \dots, 2^{20}\}$ . We produced 1 000 different frequency index sets for each of the cardinalities and constructed one reconstructing multiple rank-1 lattice for each of the frequency index sets. Figure 5.8 plots the cardinality of the frequency index set  $I$  against the maximal number  $s$ , the minimal number  $s$ , and the average of the numbers  $s$  that occurred in our numerical tests. We observe that the number  $s$  behave logarithmically with respect to the cardinality of  $I$ . We would like to point out that we observed a similar behavior in all numerical tests that we treated before.

Furthermore, the oversampling factors  $\sum_{r=1}^s M_r/|I|$  are less than Euler's number  $e$ . In addition, we computed the condition numbers of the Fourier matrices  $\mathbf{A}(\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s), I)$  for the cases  $|I| \leq 2^{12}$  and obtained low condition numbers less than 16. The average of the condition numbers were less than seven in each of the cases  $|I| = 2^n$ ,  $n = 1, \dots, 12$ .

### 5.4 Approximation experiment

The main focus of the paper is the reconstruction of trigonometric polynomials. Nevertheless, we would like to demonstrate the approximation properties of the new sampling method based on an example. We choose the scaled periodized (tensor product) kink function

$$g_d: \mathbb{T}^d \rightarrow \mathbb{R}, \quad g_d(\mathbf{x}) = \prod_{s=1}^d \left( \frac{5^{3/4} 15}{4\sqrt{3}} \max \left\{ \frac{1}{5} - \left( x_s - \frac{1}{2} \right)^2, 0 \right\} \right), \quad (5.2)$$

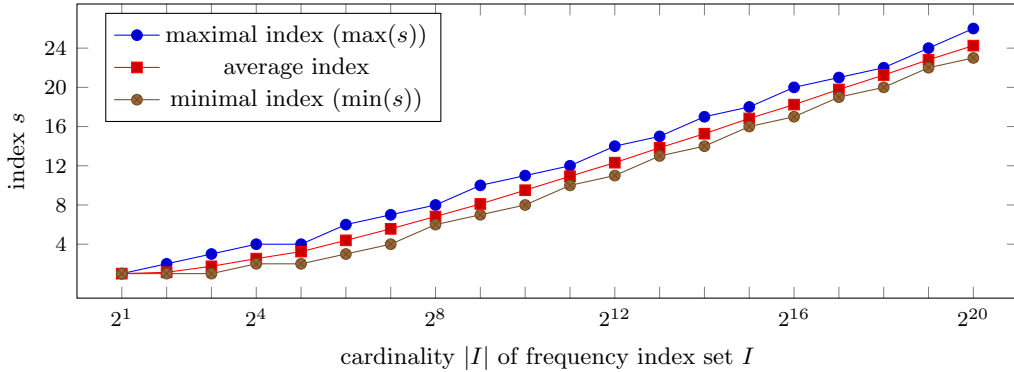


Figure 5.8: Indices  $s$  of reconstructing multiple rank-1 lattices for randomly chosen ten-dimensional frequency index sets  $I$ .

that is already treated in numerical tests for the single rank-1 lattice sampling methods in [4, Sec. 8] and the integration approach in [11, Sec. 6.2]. However, the function has norm one within  $L_2(\mathbb{T}^d)$  and belongs to the function space

$$\mathcal{H}_{\text{mix}}^\alpha(\mathbb{T}^d) := \left\{ f \in L_2(\mathbb{T}^d) : \|f\|_{\mathcal{H}_{\text{mix}}^\alpha(\mathbb{T}^d)}^2 := \sum_{\mathbf{k} \in \mathbb{Z}^d} |\hat{f}_{\mathbf{k}}|^2 \prod_{s=1}^d (1 + |k_s|^2)^\alpha < \infty \right\},$$

for each  $\alpha < 3/2$ . We consider the sampling operator  $\mathfrak{S}_I^\mathcal{X} : L_2(\mathbb{T}^d) \rightarrow \Pi_I$  that is given by

$$\mathfrak{S}_I^\mathcal{X} f := \min_{t \in \Pi_I} \|(f(\mathbf{x}) - t(\mathbf{x}))_{\mathbf{x} \in \mathcal{X}}\|_{\ell_2(|\mathcal{X}|)}, \quad f \in L_2(\mathbb{T}^d). \quad (5.3)$$

The solution of (5.3) is computed using the normal equation

$$\mathbf{A}^*(\mathcal{X}, I) \mathbf{A}(\mathcal{X}, I) \hat{\mathbf{t}} = \mathbf{A}^*(\mathcal{X}, I) \mathbf{g}, \quad \hat{\mathbf{t}} = (\hat{t}_{\mathbf{k}})_{\mathbf{k} \in I}, \quad \mathbf{g} = (g(\mathbf{x}))_{\mathbf{x} \in \mathcal{X}}$$

or, in the case of a full column rank matrix  $\mathbf{A}(\mathcal{X}, I)$ ,

$$\hat{\mathbf{t}} = (\mathbf{A}^*(\mathcal{X}, I) \mathbf{A}(\mathcal{X}, I))^{-1} \mathbf{A}^*(\mathcal{X}, I) \mathbf{g},$$

equivalently.

Since the function  $g_d \in \mathcal{H}_{\text{mix}}^{3/2-\varepsilon}(\mathbb{T}^d)$ ,  $\varepsilon > 0$  arbitrarily small, is of dominating mixed smoothness, hyperbolic cross trigonometric polynomials offer suitable approximations of the functions  $g_d$ . For that reason, we choose dyadic hyperbolic crosses  $I_{\text{dhc}, N}^d$  of different refinements  $N = 2^n$ ,  $n \in \mathbb{N}_0$ , as frequency index sets, search for reconstructing multiple rank-1 lattices for  $I_{\text{dhc}, N}^d$ , and compute both, the approximations  $\mathfrak{S}_{I_{\text{dhc}, N}^d}^{S_{\text{dhc}, N}^d}$  based on sampling values given at sparse grids  $S_{\text{dhc}, N}^d$  and the approximations  $\mathfrak{S}_{I_{\text{dhc}, N}^d}^{\Lambda(z_1, M_1, \dots, z_s, M_s)}$  based on sampling values given at the nodes along reconstructing multiple rank-1 lattices. We compare the results in Figure 5.9.

To this end, we fixed the dimensions  $d = 3, 6, 10$  and the parameters  $n$  up to 19 and plotted the  $L_2(\mathbb{T}^d)$  errors. On the one hand, the usage of multiple rank-1 lattices has a favorable pre-asymptotic behavior in our experiment. This effect is caused by the structure of the test

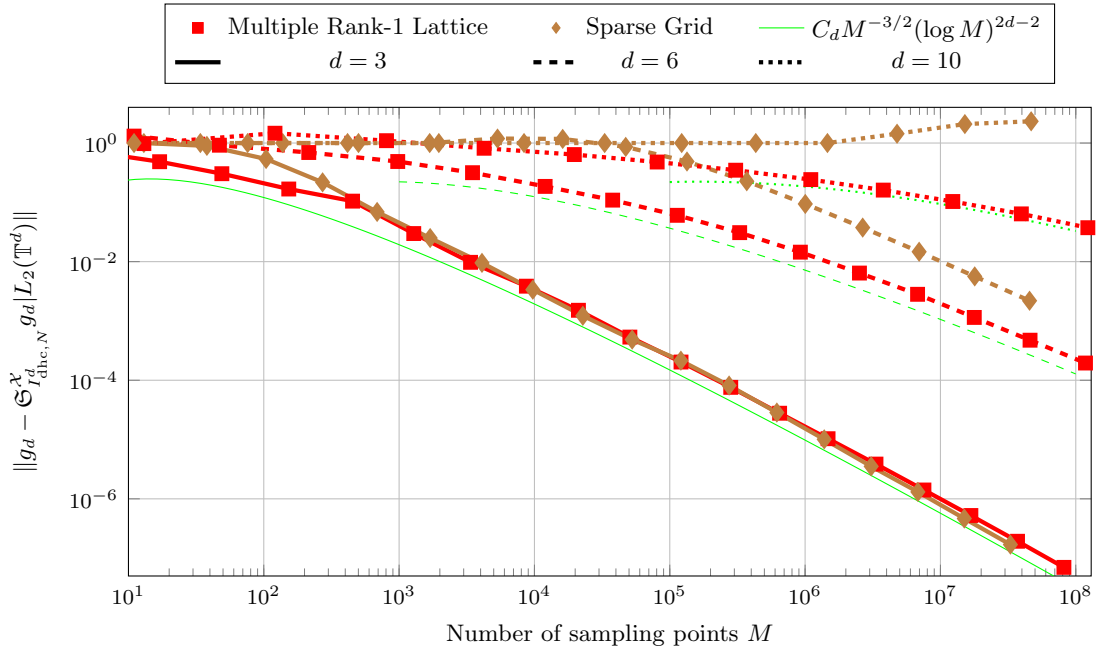


Figure 5.9:  $L_2(\mathbb{T}^d)$  sampling error and number of sampling points for the approximation of the kink function  $g_d$  from (5.2).

function, since it is zero at the boundary of  $\mathbb{T}^d$  and the sparse grid nodes are just on the boundary as long as the refinement parameter  $n$  is not at least as big as the dimension  $d$ . On the other hand, the multiple rank-1 lattice approach seems to be competitive even for higher refinements. The gray lines illustrate the best known upper bounds for the approximation error of the sparse grid sampling method up to constants that depend only on the dimension  $d$  and the smoothness parameter  $3/2$ , cf. [3, Thm. 6.10]. We stress the fact, that the approximation theory is primarily interested in the slopes of the plots in Figure 5.9. In our example, we notice that the slopes of the error curves of the sampling method along multiple rank-1 lattices are almost identical to those of the best known upper error bounds for sparse grids—at least for dimension  $d = 3$ .

At this point, we refer to the numerical tests in [4] and, in particular, for comparison to Fig. 8.1b and 8.2b therein. These plots show the sampling error of  $\mathfrak{S}_{I_{dhc,N}^d}^{\mathcal{X}} g_3$  and  $\mathfrak{S}_{I_{dhc,N}^d}^{\mathcal{X}} g_6$ , where  $\mathcal{X}$  is a reconstructing single rank-1 lattice for  $I_{dhc,N}^d$ . In accordance with the theoretical results of [4], the single rank-1 lattice approach is not able to compete with sparse grid methods in terms of the asymptotics of the approximation errors.

## 6 Pending issues

We presented a non-deterministic construction of sampling schemes for trigonometric polynomials in Algorithm 5. Various numerical tests promise excellent characteristics of this approach such as

- small oversampling factors  $|\mathcal{X}|/|I|$ ,

- fast evaluating as well as reconstructing algorithms,
- stability of the Fourier matrices,
- suitable approximation properties for functions of dominating mixed smoothness.

All those issues could not be clarified within this paper. Nevertheless, we would like to discuss known concepts that may lead to meaningful results in future research.

In order to achieve a reconstructing multiple rank-1 lattice for the frequency index set  $I$ , we need at most  $s \leq |I|$  rank-1 lattices, since each used rank-1 lattice uniquely reconstructs at least one frequency index of the current subset of  $I$  in Algorithm 5. The lattice size of each used rank-1 lattice is bounded by  $M_{\text{ub}} < 2 \max \left\{ \frac{|I|^2}{2} + 2, \max \{2\|\mathbf{k}\|_{\infty} : \mathbf{k} \in I\} \right\}$  due to Bertrand’s postulate. Assuming  $\max \{2\|\mathbf{k}\|_{\infty} : \mathbf{k} \in I\} \leq |I|^2/2 + 2$ , we determine a maximal size of the multiple rank-1 lattice of approximately  $|I|^3$ . This estimate is indeed unsatisfactory. A highly promising ansatz for lower, meaningful estimates of the number of needed sampling nodes may be adapted from recently published results dealing with straight-line programs, see e.g. [1]. These results are also necessary in order to clarify the complexity of a unique reconstruction using Algorithm 6.

Moreover, Algorithm 5 ensures that a matrix  $\mathbf{A}(\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s), I)$  consists of rows of the matrix  $\mathbf{A}(\Lambda(\mathbf{z}, M), I)$ , where  $\mathbf{z}$  and  $M$  are specified in Corollary 2.3 and each row of  $\mathbf{A}(\Lambda(\mathbf{z}, M), I)$  occurs at most once in  $\mathbf{A}(\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s), I)$ . The matrix  $\mathbf{A}(\Lambda(\mathbf{z}, M), I)$  has orthogonal columns, and thus we can understand the sampling along the multiple rank-1 lattice  $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$  as structured subsampling of the single rank-1 lattice  $\Lambda(\mathbf{z}, M)$ . Anyway, this point of view coincides with the considerations in [17, Sec. 2] called “random Fourier measurements” with the difference that multiple rank-1 lattices does not allow for a randomly chosen subset of the rows of  $\mathbf{A}(\Lambda(\mathbf{z}, M), I)$ . An appropriate adaption of the results in [17] may allow for a suitable estimate of the condition number of the Fourier matrix  $\mathbf{A}(\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s), I)$  applying its restricted isometry property, cf. [17, Def. 2.1].

The fourth item above addresses the quality of the sampling operator in specific approximation situations. In this context, the approximation of functions of dominating mixed smoothness is of main interest. Each sampling operator acting on all sampling values along single rank-1 lattices can not yield optimal results, cf. [4]. However, V. N. Temlyakov [28] shows the boundedness of the corresponding sampling operators already in 1986. Related results may be also possible for multiple rank-1 lattices. The estimates of the singular values—at least the smallest one—of the matrices  $\mathbf{A}(\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s), I)$  is one essential ingredient in order to proof a similar statement. Certainly, this estimate is closely related to the third item above that calls for usable upper bounds on the condition numbers of the matrices  $\mathbf{A}(\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s), I)$ .

## 7 Conclusion

The paper presents a new sampling method that allows for the unique sampling of sparse multivariate trigonometric polynomials. The main idea is to sample along more than one lattice grid similar to the sparse grids. In contrast to sparse grids we restrict the used lattices to rank-1 lattices. The constructive idea that determines reconstructing multiple rank-1 lattices, cf. Algorithm 5, allows for another point of view. The resulting multiple rank-1 lattices  $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$  are subsampling schemes of huge single reconstructing rank-1



lattices  $\Lambda(\mathbf{z}, M)$  for the frequency index set  $I$ , where  $\mathbf{z}$  and  $M$  are specified in Corollary 2.2. The Fourier matrix  $\mathbf{A}(\Lambda(\mathbf{z}, M), I)$  that belongs to this huge rank-1 lattice  $\Lambda(\mathbf{z}, M)$  has orthogonal columns. All numerical tests indicate that the reconstructing multiple rank-1 lattice  $\Lambda(\mathbf{z}_1, M_1, \dots, \mathbf{z}_s, M_s)$  is a stable subsampling scheme of the corresponding reconstructing rank-1 lattice  $\Lambda(\mathbf{z}, M)$ .

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