Reconstructing Multivariate Trigonometric Polynomials from Samples Along Rank-1 Lattices

Lutz Kämmerer

Abstract The approximation of problems in d spatial dimensions by trigonometric polynomials supported on known more or less sparse frequency index sets $I \subset \mathbb{Z}^d$ is an important task with a variety of applications. The use of rank-1 lattices as spatial discretizations offers a suitable possibility for sampling such sparse trigonometric polynomials. Given an arbitrary index set of frequencies, we construct rank-1 lattices that allow a stable and unique discrete Fourier transform. We use a component-by-component method in order to determine the generating vector and the lattice size.

1 Introduction

Given a spatial dimension $d \in \mathbb{N}$, we consider Fourier series of continuous functions $f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$ mapping the d-dimensional torus $[0,1)^d$ into the complex numbers $\mathbb{C}$, where $\hat{f}_{\mathbf{k}} \in \mathbb{C}$ are the Fourier coefficients. A sequence $\left(\hat{f}_{\mathbf{k}}\right)_{\mathbf{k}\in\mathbb{Z}^d}$ with a finite number of nonzero elements specifies a trigonometric polynomial. We call the index set of the nonzero elements the frequency index set of the corresponding trigonometric polynomial. For a fixed index set $I \subset \mathbb{Z}^d$ with a finite cardinality $|I|$, $\Pi_I = \mathrm{span}\{\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}} : \mathbf{k} \in I\}$ is called the space of trigonometric polynomials with frequencies supported on I.

Assuming the index set I is of finite cardinality and a suitable discretization in frequency domain for approximating functions, e.g. functions of specific smoothness, cf. [8, 5], we are interested in evaluating the corresponding trigonometric polynomials at sampling nodes and reconstructing the Fourier coefficients $\left(\hat{f}_{\mathbf{k}}\right)_{\mathbf{k}\in I}$ from sample values. Accordingly, we consider (sparse) multivariate trigonometric polynomials

Lutz Kämmerer
Technische Universität Chemnitz, Faculty of Mathematics, 09107 Chemnitz, Germany
e-mail: kaemmerer@mathematik.tu-chemnitz.de

$$f(\mathbf{x}) = \sum_{\mathbf{k}\in I} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

and assume the frequency index set I is given.

For different specific index sets I there has been done some related work using rank-1 lattices as spatial discretizations [7, 4]. A multivariate trigonometric polynomial evaluated at all nodes of a rank-1 lattice essentially simplifies to a one-dimensional fast Fourier transform (FFT) of the length of the cardinality of the rank-1 lattice, cf. [6]. Allowing for some oversampling one can find a rank-1 lattice, which even allows the reconstruction of the trigonometric polynomial from the samples at the rank-1 lattice nodes. A suitable strategy to search for such reconstructing rank-1 lattices can be adapted from numerical integration. In particular, a modification of the component-by-component constructions of lattice rules based on various weighted trigonometric degrees of exactness described in [3] allows one to find adequate rank-1 lattices in a relatively fast way. We already showed the existence and upper bounds on the cardinality of reconstructing rank-1 lattices for hyperbolic crosses as index sets , cf. [4].

In this paper we generalize these results considering arbitrary frequency index sets I and suggest some strategies for determining reconstructing rank-1 lattices even for frequency index sets containing gaps. To this end, we present corresponding component–by–component (CBC) algorithms, where the frequency index set I is the only input.

In Section 2, we introduce the necessary notation and specify the relation between exact integration of trigonometric polynomials and reconstruction of trigonometric polynomials using rank-1 lattices. Section 3 contains the main results, i.e., a component-by-component algorithm searching for reconstructing rank-1 lattices for given frequency index sets I and given rank-1 lattice sizes M. In detail, we determine conditions on M guaranteeing the existence of a reconstructing rank-1 lattice of size M for the frequency index set I. The proof of this existence result describes a component-by-component construction of a corresponding generating vector $\mathbf{z} \in \mathbb{N}^d$ of the rank-1 lattice, such that we obtain directly a component-by-component algorithm. In Section 4, we give some simple improvements of the component-by-component construction, such that the corresponding algorithms automatically determine suitable rank-1 lattice sizes. Accordingly, the only input is the frequency index set I here. Finally, we give some specific examples and compare the results of our different algorithms in Section 5.

2 Rank-1 Lattices

For given $M \in \mathbb{N}$ and $\mathbf{z} \in \mathbb{N}^d$ we define the *rank-1 lattice*

$$\Lambda(\mathbf{z}, M) := \{\mathbf{x}_j = \frac{j\mathbf{z}}{M} \bmod 1, j = 0, \ldots, M-1\}$$

as discretization in the spatial domain. Following [6], the evaluation of the trigonometric polynomial $f \in \Pi_I$ with frequencies supported on I simplifies to a one-dimensional discrete Fourier transform (DFT), i.e.,

$$f(\mathbf{x}_j) = \sum_{\mathbf{k}\in I} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} j \mathbf{k}\cdot\mathbf{z}} = \sum_{l=0}^{M-1} \left(\sum_{\mathbf{k}\cdot\mathbf{z}\equiv l \pmod M} \hat{f}_{\mathbf{k}} \right) \mathrm{e}^{2\pi \mathrm{i} \frac{jl}{M}}.$$

We evaluate f at all nodes $\mathbf{x}_j \in \Lambda(\mathbf{z},M),\ j = 0,\ldots,M-1$, by the precomputation of all $\hat{g}_l := \sum_{\mathbf{k}\cdot\mathbf{z}\equiv l \pmod M} \hat{f}_{\mathbf{k}}$ and a one-dimensional (inverse) FFT in $\mathscr{O}(M \log M + d|I|)$ floating point operations, cf. [2], where $|I|$ denotes the cardinality of the frequency index set I.

As the fast evaluation of trigonometric polynomials at all sampling nodes $\mathbf{x}_j$ of the rank-1 lattice $\Lambda(\mathbf{z},M)$ is guaranteed, we draw our attention to the reconstruction of a trigonometric polynomial f with frequencies supported on I using function values at the nodes $\mathbf{x}_j$ of a rank-1 lattice $\Lambda(\mathbf{z},M)$. We consider the corresponding Fourier matrix $\mathbf{A}$ and its adjoint $\mathbf{A}^*$,

$$\mathbf{A} := \left(\mathrm{e}^{2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{x}}\right)_{\mathbf{x}\in\Lambda(\mathbf{z},M),\,\mathbf{k}\in I} \in \mathbb{C}^{M\times|I|} \quad\text{and}\quad \mathbf{A}^* := \left(\mathrm{e}^{-2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{x}}\right)_{\mathbf{k}\in I,\,\mathbf{x}\in\Lambda(\mathbf{z},M)} \in \mathbb{C}^{|I|\times M},$$

in order to determine necessary and sufficient conditions on rank-1 lattices $\Lambda(\mathbf{z},M)$ allowing for a unique reconstruction of all Fourier coefficients of $f \in \Pi_I$. The reconstruction of the Fourier coefficients $\hat{\mathbf{f}} = (\hat{f}_{\mathbf{k}})_{\mathbf{k}\in I} \in \mathbb{C}^{|I|}$ from sampling values $\mathbf{f} = (f(\mathbf{x}))_{\mathbf{x}\in\Lambda(\mathbf{z},M)} \in \mathbb{C}^M$ can be realized by solving the normal equation $\mathbf{A}^*\mathbf{A}\hat{\mathbf{f}} = \mathbf{A}^*\mathbf{f}$, which is equivalent to solve the least squares problem

$$\text{find } \hat{\mathbf{f}} \in \mathbb{C}^{|I|} \text{ such that } \|\mathbf{A}\hat{\mathbf{f}} - \mathbf{f}\|_2 \to \min,$$

cf. [1]. Assuming $\mathbf{f} = (f(\mathbf{x}))_{\mathbf{x}\in\Lambda(\mathbf{z},M)}$ being a vector of sampling values of the trigonometric polynomial $f \in \Pi_I$, the vector $\mathbf{f}$ belongs to the range of $\mathbf{A}$ and we can find a possibly non-unique solution $\hat{\mathbf{f}}$ of $\mathbf{A}\hat{\mathbf{f}} = \mathbf{f}$. We compute a unique solution of the normal equation, iff the Fourier matrix $\mathbf{A}$ has full column rank.

Lemma 1. *Let $I \subset \mathbb{Z}^d$ of finite cardinality and $\Lambda(\mathbf{z},M)$ a rank-1 lattice be given. Then two distinct columns of the corresponding Fourier matrix $\mathbf{A}$ are orthogonal or equal, i.e., $(\mathbf{A}^*\mathbf{A})_{\mathbf{h},\mathbf{k}} \in \{0,M\}$ for $\mathbf{h},\mathbf{k} \in I$.*

Proof. The matrix $\mathbf{A}^*\mathbf{A}$ contains all scalar products of two columns of the Fourier matrix $\mathbf{A}$, i.e., $(\mathbf{A}^*\mathbf{A})_{\mathbf{h},\mathbf{k}}$ is the scalar product of column $\mathbf{k}$ with column $\mathbf{h}$ of the Fourier matrix $\mathbf{A}$. We obtain

$$(\mathbf{A}^*\mathbf{A})_{\mathbf{h},\mathbf{k}} = \sum_{j=0}^{M-1} \left(\mathrm{e}^{2\pi \mathrm{i}\frac{(\mathbf{k}-\mathbf{h})\cdot\mathbf{z}}{M}}\right)^j = \begin{cases} M, & \text{for } \mathbf{k}\cdot\mathbf{z} \equiv \mathbf{h}\cdot\mathbf{z} \pmod M, \\ \frac{\mathrm{e}^{2\pi \mathrm{i}(\mathbf{k}-\mathbf{h})\cdot\mathbf{z}}-1}{\mathrm{e}^{2\pi \mathrm{i}\frac{(\mathbf{k}-\mathbf{h})\cdot\mathbf{z}}{M}}-1} = 0, & \text{else.} \end{cases}$$

□

According to Lemma 1 the matrix $\mathbf{A}$ has full column rank, iff

$$\mathbf{k}\cdot\mathbf{z}\not\equiv\mathbf{h}\cdot\mathbf{z} \pmod M, \qquad \text{for all} \quad \mathbf{k}\neq\mathbf{h};\, \mathbf{k},\mathbf{h}\in I, \tag{1}$$

or, equivalent,

$$\mathbf{k}\cdot\mathbf{z}\not\equiv 0 \pmod M, \qquad \text{for all} \quad \mathbf{k}\in\mathscr{D}(I)\setminus\{\mathbf{0}\} \tag{2}$$

with $\mathscr{D}(I):=\{\mathbf{h}=\mathbf{l}_1-\mathbf{l}_2\colon \mathbf{l}_1,\mathbf{l}_2\in I\}$. We call the set $\mathscr{D}(I)$ *difference set* of the frequency index set I and a rank-1 lattice $\Lambda(\mathbf{z},M)$ ensuring (1) and (2) *reconstructing rank-1 lattice* for the index set I. In particular, condition (2) ensures the exact integration of all trigonometric polynomials $g\in\Pi_{\mathscr{D}(I)}$ applying the lattice rule given by $\Lambda(\mathbf{z},M)$, i.e., the identity $\int_{\mathbb{T}^d} g(\mathbf{x})\mathrm{d}\mathbf{x}=\frac{1}{M}\sum_{j=0}^{M-1} g(\mathbf{x}_j)$ holds for all $g\in\Pi_{\mathscr{D}(I)}$, cf. [9]. Certainly, $f\in\Pi_I$ and $\mathbf{k}\in I$ implies that $f\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\circ}\in\Pi_{\mathscr{D}(I)}$ and we obtain

$$\frac{1}{M}\sum_{j=0}^{M-1} f\left(\frac{j\mathbf{z}}{M} \bmod \mathbf{1}\right)\mathrm{e}^{-2\pi\mathrm{i}j\frac{\mathbf{k}\cdot\mathbf{z}}{M}}=\int_{\mathbb{T}^d} f(\mathbf{x})\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\mathrm{d}\mathbf{x}=:\hat{f}_{\mathbf{k}},$$

where the right equality is the usual definition of the Fourier coefficients.

Another fact, which comes out of Lemma 1, is that the matrix $\mathbf{A}$ fulfills $\mathbf{A}^*\mathbf{A}=M\mathbf{I}$ in the case of $\Lambda(\mathbf{z},M)$ being a reconstructing rank-1 lattice for I. The normalized normal equation simplifies to

$$\hat{\mathbf{f}}=\frac{1}{M}\mathbf{A}^*\mathbf{A}\hat{\mathbf{f}}=\frac{1}{M}\mathbf{A}^*\mathbf{f},$$

and in fact we reconstruct the Fourier coefficients of $f\in\Pi_I$ applying the lattice rule

$$\hat{f}_{\mathbf{k}}=\frac{1}{M}\sum_{j=0}^{M-1} f(\mathbf{x}_j)\mathrm{e}^{-2\pi\mathrm{i}\frac{j\mathbf{k}\cdot\mathbf{z}}{M}}=\frac{1}{M}\sum_{j=0}^{M-1} f(\mathbf{x}_j)\mathrm{e}^{-2\pi\mathrm{i}\frac{jl}{M}}$$

for all $\mathbf{k}\in I$ and $l=\mathbf{k}\cdot\mathbf{z} \bmod M$. In particular, one computes all Fourier coefficients using one one-dimensional FFT and the unique inverse mapping of $\mathbf{k}\mapsto\mathbf{k}\cdot\mathbf{z} \bmod M$. The corresponding complexity is given by $\mathscr{O}(M\log M+d|I|)$.

Up to now, we wrote about reconstructing rank-1 lattices without saying how to get them. In the following section, we prove existence results and give a first algorithm in order to determine reconstructing rank-1 lattices.

3 A CBC construction of reconstructing rank-1 lattices

A reconstructing rank-1 lattice for the frequency index set I is characterized by (1) and (2), respectively. Similar to the construction of rank-1 lattices for the exact integration of trigonometric polynomials of specific trigonometric degrees, see [3], we are interested in existence results and suitable construction algorithms for reconstructing rank-1 lattices. In order to prepare the main theorem of this paper, we define the projection of an index set $I \subset \mathbb{Z}^d$ on $\mathbb{Z}^s$, $d \geq s \in \mathbb{N}$,

$$I_s := \{(k_j)_{j=1}^s : \mathbf{k} = (k_j)_{j=1}^d \in I\}. \tag{3}$$

Furthermore, we call a frequency index set $I \subset \mathbb{Z}^d$ *symmetric to the origin* iff $I = \{-\mathbf{k} \colon \mathbf{k} \in I\}$, i.e., $\mathbf{h} \in I$ implies $-\mathbf{k} \in I$ for all $\mathbf{k} \in I$.

Theorem 1. *Let $s \in \mathbb{N}$, $d \geq s \geq 2$, $\tilde{I} \subset \mathbb{Z}^d$ be an arbitrary d-dimensional set of finite cardinality that is symmetric to the origin, and M be a prime number satisfying*

$$M \geq \frac{|\{\mathbf{k} \in \tilde{I}_s : \mathbf{k} = (\mathbf{h}, h_s), \mathbf{h} \in \tilde{I}_{s-1} \setminus \{\mathbf{0}\} \text{ and } h_s \in \mathbb{Z} \setminus \{0\}\}|}{2} + 2.$$

Additionally, we assume that each nonzero element of the set of the s-th component of $\tilde{I}_s$ and M are coprime, i.e., $M \nmid l$ for all $l \in \{h_s \in \mathbb{Z} \setminus \{0\} : \mathbf{k} = (\mathbf{h}, h_s) \in \tilde{I}_s, \mathbf{h} \in \tilde{I}_{s-1}\}$, and that there exists a generating vector $\mathbf{z}^* \in \mathbb{N}^{s-1}$ that guarantees

$$\mathbf{h} \cdot \mathbf{z}^* \not\equiv 0 \pmod{M} \quad \text{for all } \mathbf{h} \in \tilde{I}_{s-1} \setminus \{\mathbf{0}\}.$$

Then there exists at least one $z_s^* \in \{1, \ldots, M-1\}$ such that

$$(\mathbf{h}, h_s) \cdot (\mathbf{z}^*, z_s^*) \not\equiv 0 \pmod{M} \quad \text{for all } (\mathbf{h}, h_s) \in \tilde{I}_s \setminus \{\mathbf{0}\}.$$

Proof. We adapt the proof of [3, Theorem 1]. Let us assume that

$$\mathbf{h} \cdot \mathbf{z}^* \not\equiv 0 \pmod{M} \text{ for all } \mathbf{h} \in \tilde{I}_{s-1} \setminus \{\mathbf{0}\}.$$

Basically, we determine an upper bound of the number of elements $z_s \in \{1, \ldots, M-1\}$ with

$$(\mathbf{h}, h_s) \cdot (\mathbf{z}^*, z_s) \equiv 0 \pmod{M} \text{ for at least one } (\mathbf{h}, h_s) \in \tilde{I}_s \setminus \{0\}$$

or, equivalent,

$$\mathbf{h} \cdot \mathbf{z}^* \equiv -h_s z_s \pmod{M} \text{ for at least one } (\mathbf{h}, h_s) \in \tilde{I}_s \setminus \{0\}.$$

Similar to [3] we consider three cases:

- $h_s = 0$: With $(\mathbf{h}, h_s) \in \tilde{I}_s \setminus \{\mathbf{0}\}$ we have $\mathbf{0} \neq \mathbf{h} \in \tilde{I}_{s-1} \setminus \{\mathbf{0}\}$. Consequently, $\mathbf{h} \cdot \mathbf{z}^* \equiv -0 z_s \pmod{M}$ never holds because of $\mathbf{h} \cdot \mathbf{z}^* \not\equiv 0 \pmod{M}$ for all $\mathbf{h} \in \tilde{I}_{s-1} \setminus \{\mathbf{0}\}$.
- $\mathbf{h} = \mathbf{0}$: We consider $z_s \in \{1, \ldots, M-1\}$. We required M being prime, so z_s and M are coprime. Due to $(\mathbf{h}, h_s) \in \tilde{I} \setminus \{\mathbf{0}\}$, we obtain $h_s \neq 0$ and we assumed M and $h_s \neq 0$ are coprime. Consequently, we realize $z_s h_s \neq 0$ and $z_s h_s$ and M are relatively prime. So $\mathbf{0}\mathbf{z}^* \equiv -h_s z_s \pmod{M}$ never holds for $(\mathbf{0}, h_s) \in \tilde{I}_s \setminus \{\mathbf{0}\}$ and $z_s \in \{1, \ldots, M-1\}$.
- else: Since $0 \neq h_s$ and M are coprime and $\mathbf{h} \cdot \mathbf{z}^* \not\equiv 0 \pmod{M}$, there is at most one $z_s \in \{1, \ldots, M-1\}$ that fulfills $\mathbf{h} \cdot \mathbf{z}^* \equiv -h_s z_s \pmod{M}$. Due

to the symmetry of the considered index set $\{(\mathbf{h},h_s)\in\tilde{I}_s\setminus\{\mathbf{0}\}:\mathbf{h}\in\tilde{I}_{s-1}\setminus\{\mathbf{0}\}$ and $h_s\in\mathbb{Z}\setminus\{0\}\}$ we have to count at most one z_s for the two elements $(\mathbf{h},h_s)$ and $-(\mathbf{h},h_s)$.

Hence, we have at most

$$\frac{|\{(\mathbf{h},h_s)\in\tilde{I}_s\setminus\{\mathbf{0}\}:\mathbf{h}\in\tilde{I}_{s-1}\setminus\{\mathbf{0}\}\text{ and }h_s\in\mathbb{Z}\setminus\{0\}\}|}{2} \tag{4}$$

elements of $\{1,\ldots,M-1\}$ with

$$\mathbf{h}\cdot\mathbf{z}^*\equiv -h_sz_s \pmod M \text{ for at least one } (\mathbf{h},h_s)\in\tilde{I}_s\setminus\{\mathbf{0}\}.$$

If the candidate set $\{1,\ldots,M-1\}$ for z_s^* contains more elements than (4) we can determine at least one z_s^* with

$$\mathbf{h}\cdot\mathbf{z}^*\not\equiv -h_sz_s^* \pmod M \text{ for all } (\mathbf{h},h_s)\in\tilde{I}_s\setminus\{\mathbf{0}\}.$$

Consequently, the number of elements in $\{1,\ldots,M-1\}$ with

$$|\{1,\ldots,M-1\}|\geq\frac{|\{(\mathbf{h},h_s)\in\tilde{I}_s\setminus\{\mathbf{0}\}:\mathbf{h}\in\tilde{I}_{s-1}\setminus\{\mathbf{0}\}\text{ and }h_s\in\mathbb{Z}\setminus\{0\}\}|}{2}+1$$

and M is prime guarantees that there exists such a z_s^*. Since we assumed M being prime and

$$\begin{aligned}M&=|\{1,\ldots,M-1\}|+1\\&\geq\frac{|\{(\mathbf{h},h_s)\in\tilde{I}_s\setminus\{\mathbf{0}\}:\mathbf{h}\in\tilde{I}_{s-1}\setminus\{\mathbf{0}\}\text{ and }h_s\in\mathbb{Z}\setminus\{0\}\}|}{2}+2\end{aligned}$$

we can find at least one z_s by testing out all possible candidates $\{1,2,\ldots,M-1\}$. □

Theorem 1 outlines one step of a component-by-component construction of a rank-1 lattice, guaranteeing the exact integration of trigonometric polynomials with frequencies supported on index sets $\tilde{I}$ which are symmetric to the origin.

We obtain this symmetry of the difference sets $\mathscr{D}(I)_s$

$$\mathbf{h}\in\mathscr{D}(I)_s\Rightarrow\exists\mathbf{k}_1,\mathbf{k}_2\in I_s\colon\mathbf{h}=\mathbf{k}_1-\mathbf{k}_2\Rightarrow-\mathbf{h}=\mathbf{k}_2-\mathbf{k}_1\in\mathscr{D}(I)_s.$$

So, our strategy is to apply Theorem 1 to the difference set $\mathscr{D}(I)_s$ of the frequency index set I_s for all $2\leq s\leq d$. In order to use Theorem 1, we have to find sufficient conditions on rank-1 lattices of dimension $d=1$ guaranteeing that $hz_1\not\equiv 0 \pmod M$ for all $h\in\mathscr{D}(I)_1\setminus\{0\}$.

Lemma 2. *Let $I\subset\mathbb{Z}$ be a one-dimensional frequency index set of finite cardinality and M be a prime number satisfying $M\geq|I|$. Additionally, we assume M and h being coprime for all $h\in\mathscr{D}(I)\setminus\{0\}$. Then we can uniquely reconstruct the Fourier coefficients of all $f\in\Pi_I$ applying the one-dimensional lattice rule given by $\Lambda(1,M)$.*

Algorithm 1 Component-by-component lattice search

Input:	$M \in \mathbb{N}$ prime	cardinality of rank-1 lattice
	$I \subset \mathbb{Z}^d$	frequency index set

```
z = ∅
for s = 1, ..., d do
    form the set I_s as defined in (3)
    search for one z_s ∈ [1, M−1] ∩ ℤ with |{(z, z_s) · k mod M : k ∈ I_s}| = |I_s|
    z = (z, z_s)
end for
```

Output:	$\mathbf{z} \in \mathbb{Z}^d$	generating vector

Proof. Applying the lattice rule given by $\Lambda(1,M)$ to the integrands of the integrals computing the Fourier coefficient $\hat{f}_k$, $k \in I$, of $f \in \Pi_I$, we obtain

$$\frac{1}{M}\sum_{j=0}^{M-1} f\left(\frac{j}{M}\right)\mathrm{e}^{-2\pi\mathrm{i}\frac{kj}{M}} = \frac{1}{M}\sum_{j=0}^{M-1}\sum_{h\in I}\hat{f}_h \mathrm{e}^{2\pi\mathrm{i}\frac{hj}{M}}\mathrm{e}^{-2\pi\mathrm{i}\frac{kj}{M}}$$
$$= \frac{1}{M}\sum_{h\in I}\hat{f}_h\sum_{j=0}^{M-1}\mathrm{e}^{2\pi\mathrm{i}\frac{(h-k)j}{M}} = \hat{f}_k = \int_0^1 f(x)\mathrm{e}^{-2\pi\mathrm{i}kx}dx$$

due to $h-k \in \mathscr{D}(I)\setminus\{0\}$ and M are coprime. □

We summarize the results of Theorem 1 and Lemma 2 and figure out the following

Corollary 1. *Let $I \subset \mathbb{Z}^d$ be an arbitrary d-dimensional index set of finite cardinality and M be a prime number satisfying*

$$M \geq \max\left(|I_1|, \max_{s=2,\ldots,d}\frac{|\{\mathbf{k}\in\mathscr{D}(I)_s : \mathbf{k}=(\mathbf{h},h_s), \mathbf{h}\in\mathscr{D}(I)_{s-1}\setminus\{\mathbf{0}\} \text{ and } h_s\in\mathbb{Z}\setminus\{0\}\}|}{2}+2\right).$$

In addition we assume that $M \nmid l$ for all $l \in \{k = \mathbf{e}_s\cdot\mathbf{h} : \mathbf{h}\in\mathscr{D}(I), s=1,\ldots,d\}\setminus\{0\}$, where $\mathbf{e}_s \in \mathbb{N}^d$ is a d-dimensional unit vector with $e_{s,j} = \begin{cases}0, & \text{for } j\neq s\\ 1, & \text{for } j = s.\end{cases}$ Then there exists a rank-1 lattice of cardinality M that allows the reconstruction of all trigonometric polynomials with frequencies supported on I by sampling along the rank-1 lattice. Furthermore, once we determined a suitable M the proof of Theorem 1 verifies that we can find at least one appropriate generating vector component-by-component. Algorithm 1 indicates the corresponding strategy.

Once one has discovered a reconstructing rank-1 lattice $\Lambda(\mathbf{z},M)$ for the index set I, the condition

$$\mathbf{k}\cdot\mathbf{z} \neq \mathbf{h}\cdot\mathbf{z}, \qquad \text{for all} \quad \mathbf{k}\neq\mathbf{h};\ \mathbf{k},\mathbf{h}\in I,$$

holds and one can ask for $M' < M$ fulfilling

$$\mathbf{k}\cdot\mathbf{z} \not\equiv \mathbf{h}\cdot\mathbf{z} \pmod{M'}, \qquad \text{for all} \quad \mathbf{k}\neq\mathbf{h};\ \mathbf{k},\mathbf{h}\in I.$$

Algorithm 2 Lattice size decreasing

Input:	$I \subset \mathbb{Z}^d$	frequency index set
	$M_{\max} \in \mathbb{N}$	cardinality of rank-1 lattice
	$\mathbf{z} \in \mathbb{N}^d$	$\Lambda(\mathbf{z}, M_{\max})$ is reconstructing rank-1 lattice for I

```
for j = |I|, ..., M_max do
    if |{z·k mod (j) : k ∈ I}| = |I| then
        M_min = j
    end if
end for
```

Output:	$M_{\min}$	reduced lattice size

For a fixed frequency index set I and a fixed generating vector $\mathbf{z}$ we assume the rank-1 lattice $\Lambda(\mathbf{z}, M_{\max})$ being a reconstructing rank-1 lattice. Then, Algorithm 2 computes the smallest lattice size M' guaranteeing the reconstruction property of the rank-1 lattice $\Lambda(\mathbf{z}, M')$.

Finally, we give a simple upper bound on the cardinality of the difference set $\mathscr{D}(I)$ depending on the cardinality of I

$$|\mathscr{D}(I)| = |\{\mathbf{k} - \mathbf{h} \colon \mathbf{k}, \mathbf{h} \in I\}| = |\{\mathbf{k} - \mathbf{h} \colon \mathbf{k}, \mathbf{h} \in I, \mathbf{k} \neq \mathbf{h}\} \cup \{\mathbf{0}\}| \leq |I|(|I| - 1) + 1.$$

According to this and applying Bertrand's postulate, the prime number M from Corollary 1 is bounded from above by $|I|^2$, provided that $|I| \geq 4$.

4 Improvements

There are two serious problems concerning Corollary 1. In general, the computational costs of determining the cardinality of the difference sets $\mathscr{D}(I)_s$, $2 \leq s \leq d$, has a complexity of $\Omega(d|I|^2)$ and, maybe, the minimal M satisfying the assumptions of Corollary 1 is far away from a best possible reconstructing rank-1 lattice size. Accordingly, we are interested in somehow good estimations of the reconstructing rank-1 lattice size for the index set I.

In this section, we present another strategy to find reconstructing rank-1 lattices. We search for rank-1 lattices using a component-by-component construction determining the generating vectors $\mathbf{z} \in \mathbb{Z}^d$ *and* suitable rank-1 lattice sizes $M \in \mathbb{N}$.

Theorem 2. *Let $d \in \mathbb{N}$, $d \geq 2$, and $I \subset \mathbb{Z}^d$ of finite cardinality $|I| \geq 2$ be given. We assume that $\Lambda(\mathbf{z}, M)$ with $\mathbf{z} = (z_1, \ldots, z_{d-1})^\top$ is a reconstructing rank-1 lattice for the frequency index set $I_{d-1} := \{(h_s)_{s=1}^{d-1} : \mathbf{h} \in I\}$. Then the rank-1 lattice $\Lambda((z_1, \ldots, z_{d-1}, M)^\top, MS)$ with*

$$S := \min\{m \in \mathbb{N} : |\{h_d \bmod m : \mathbf{h} \in I\}| = |\{h_d : \mathbf{h} \in I\}|\}$$

is a reconstructing rank-1 lattice for I.

Proof. We assume the rank-1 lattice $\Lambda((z_1,\ldots,z_{d-1})^\top,M)$ is a reconstructing rank-1 lattice for I_{d-1} and $\Lambda((z_1,\ldots,z_{d-1},M)^\top,MS)$ is not a reconstructing rank-1 lattice for I, i.e., there exist at least two different elements $(\mathbf{h},h_d),(\mathbf{k},k_d)\in I$, $(\mathbf{h},h_d)\neq(\mathbf{k},k_d)$, such that

$$\mathbf{h}\cdot\mathbf{z}+h_dM\equiv\mathbf{k}\cdot\mathbf{z}+k_dM \pmod{MS}.$$

We distinguish three different possible cases of $(\mathbf{h},h_d),(\mathbf{k},k_d)\in I$, $(\mathbf{h},h_d)\neq(\mathbf{k},k_d)$:

- $\mathbf{h}=\mathbf{k}$ and $h_d\neq k_d$
  We consider the corresponding residue classes
  $$0\equiv\mathbf{k}\cdot\mathbf{z}+k_dM-\mathbf{h}\cdot\mathbf{z}-h_dM\equiv(k_d-h_d)M \pmod{MS}$$
  and obtain $S\mid(k_d-h_d)$, i.e., $k_d\equiv h_d \pmod{S}$. Thus, we determine the cardinality $|\{h_d \bmod S:\ \mathbf{h}\in I\}|<|\{h_d:\ \mathbf{h}\in I\}|$, which is in contradiction to the definition of S.
- $\mathbf{h}\neq\mathbf{k}$ and $h_d=k_d$
  Accordingly, we calculate
  $$0\equiv\mathbf{k}\cdot\mathbf{z}+k_dM-\mathbf{h}\cdot\mathbf{z}-h_dM\equiv(\mathbf{k}-\mathbf{h})\cdot\mathbf{z} \pmod{MS}$$
  and obtain $MS\mid(\mathbf{k}-\mathbf{h})\cdot\mathbf{z}$ and $M\mid(\mathbf{k}-\mathbf{h})\cdot\mathbf{z}$ as well. According to that, we obtain $\mathbf{h}\cdot\mathbf{z}\equiv\mathbf{k}\cdot\mathbf{z} \pmod{M}$, which is in contradiction to the assumption $\Lambda(\mathbf{z},M)$ is a reconstructing rank-1 lattice for I_{d-1}.
- $\mathbf{h}\neq\mathbf{k}$ and $h_d\neq k_d$
  Due to $\Lambda(\mathbf{z},M)$ is a reconstructing rank-1 lattice for I_{d-1} we have
  $$0\not\equiv\mathbf{k}\cdot\mathbf{z}-\mathbf{h}\cdot\mathbf{z} \pmod{M}.$$
  Thus, we can find uniquely specified $a_{\mathbf{k},\mathbf{h}}\in\mathbb{Z}$ and $b_{\mathbf{k},\mathbf{h}}\in\{1,\ldots,M-1\}$ such that $\mathbf{k}\cdot\mathbf{z}-\mathbf{h}\cdot\mathbf{z}=a_{\mathbf{k},\mathbf{h}}M+b_{\mathbf{k},\mathbf{h}}$. We calculate
  $$0\equiv\mathbf{k}\cdot\mathbf{z}+k_dM-\mathbf{h}\cdot\mathbf{z}-h_dM\equiv(a_{\mathbf{k},\mathbf{h}}+k_d-h_d)M+b_{\mathbf{k},\mathbf{h}} \pmod{MS}$$
  and obtain $MS\mid(a_{\mathbf{k},\mathbf{h}}+k_d-h_d)M+b_{\mathbf{k},\mathbf{h}}$. As a consequence, we deduce $M\mid b_{\mathbf{k},\mathbf{h}}$, which is in conflict with $b_{\mathbf{k},\mathbf{h}}\in\{1,\ldots,M-1\}$.

Extending the reconstructing rank-1 lattice $\Lambda(\mathbf{z},M)$ for I_{d-1} to $\Lambda((\mathbf{z},M),MS)$ with S as defined above, we actually get a reconstructing rank-1 lattice for the frequency index set $I\subset\mathbb{Z}^d$. □

In addition to the strategy provided by Theorem 2 and the corresponding Algorithm 3, we bring the following heuristic into play. We assume small components of the vector $\mathbf{z}$ being better than large ones. Therefore we tune Algorithm 3 and additionally search for the smallest possible component z_s fulfilling

$$|\{(\mathbf{z},z_s)\cdot\mathbf{h} \bmod SM_{s-1}:\ \mathbf{h}\in I_s\}|=|I_s|.$$

Algorithm 3 Component-by-component lattice search (unknown lattice size M)

Input: $I \subset \mathbb{Z}^d$ frequency index set

$M_1 = \min\{m \in \mathbb{N} : |\{k_1 \bmod m : \mathbf{k} \in I\}| = |\{k_1 : \mathbf{k} \in I\}|\}$
$z_1 = 1$
for $s = 2, \ldots, d$ **do**
  $S = \min\{m \in \mathbb{N} : |\{k_s \bmod m : \mathbf{k} \in I\}| = |\{k_s : \mathbf{k} \in I\}|\}$
  $\mathbf{z} = (\mathbf{z}, z_s)$
  $z_s = M_{s-1}$
  form the set I_s as defined in (3)
  search for $M_s = \min\{m \in \mathbb{N} : |\{\mathbf{z} \cdot \mathbf{k} \bmod m : \mathbf{k} \in I_s\}| = |I_s|\} \leq SM_{s-1}$ using Algorithm 2
end for
Output: $\mathbf{z} \in \mathbb{N}^d$ generating vector
$\mathbf{M} \in \mathbb{N}^d$ rank-1 lattice sizes for dimension $s = 1, \ldots, d$

Algorithm 4 Component-by-component lattice search (unknown lattice size M, improved)

Input: $I \subset \mathbb{Z}^d$ frequency index set

$M_1 = \min\{m \in \mathbb{N} : |\{k_1 \bmod m : \mathbf{k} \in I\}| = |\{k_1 : \mathbf{k} \in I\}|\}$
$z_1 = 1$
for $s = 2, \ldots, d$ **do**
  $S = \min\{m \in \mathbb{N} : |\{k_s \bmod m : \mathbf{k} \in I\}| = |\{k_s : \mathbf{k} \in I\}|\}$
  form the set I_s as defined in (3)
  search for the smallest $z_s \in [1, M_{s-1}] \cap \mathbb{Z}$ with $|\{(\mathbf{z}, z_s) \cdot \mathbf{k} \bmod SM_{s-1} : \mathbf{k} \in I_s\}| = |I_s|$
  $\mathbf{z} = (\mathbf{z}, z_s)$
  search for $M_s = \min\{m \in \mathbb{N} : |\{\mathbf{z} \cdot \mathbf{k} \bmod m : \mathbf{k} \in I_s\}| = |I_s|\}$ using Algorithm 2
end for
Output: $\mathbf{z} \in \mathbb{N}^d$ generating vector
$\mathbf{M} \in \mathbb{N}^d$ rank-1 lattice sizes for dimension $s = 1, \ldots, d$

Due to Theorem 2 the integer M_{s-1} is an upper bound for the minimal z_s we can find. Algorithm 4 indicates the described strategy in detail. Algorithms 3 and 4 provide deterministic strategies to find reconstructing rank-1 lattices for a given index set I. We would like to point out that in both algorithms the only input we need is the frequency index set I.

5 Numerical Examples

Our numerical examples treat frequency index sets of type

$$I_{p,N}^d := \left\{\mathbf{k} \in \mathbb{Z}^d : \|\mathbf{k}\|_p \leq N\right\},$$

where $\|\cdot\|_p$ is the usual p–(quasi–)norm

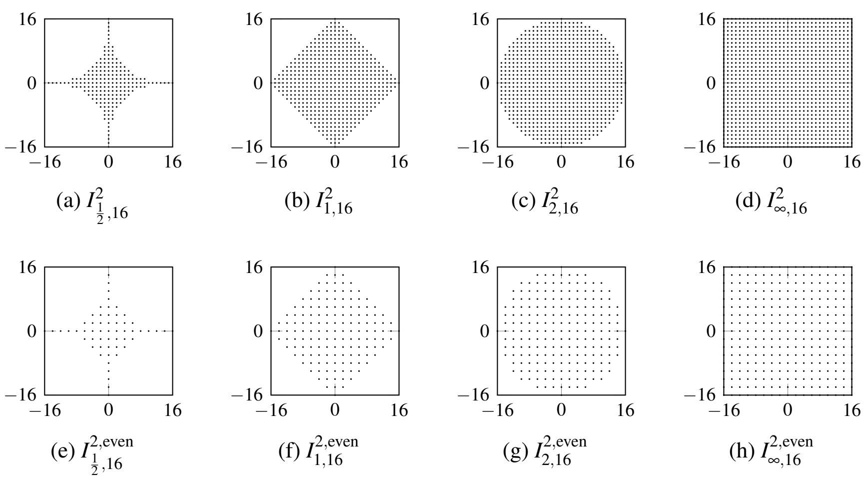
$$\|\mathbf{k}\|_p := \begin{cases} \left(\sum_{s=1}^{d} |k_s|^p\right)^{1/p} & \text{for } 0 < p < \infty \\ \max_{s=1,\ldots,d} |k_s| & \text{for } p = \infty. \end{cases}$$

In particular, trigonometric polynomials with frequencies supported on the index sets $I_{p,N}^d$ are useful in order to approximate functions of periodic Sobolev spaces $H^{\alpha,p}(\mathbb{T}^d)$ of isotropic smoothness

$$H^{\alpha,p}(\mathbb{T}^d) := \{f\colon \mathbb{T}^d \to \mathbb{C} | \sum_{\mathbf{k}\in\mathbb{Z}^d} \max(1, \|\mathbf{k}\|_p)^{\alpha} |\hat{f}_{\mathbf{k}}|^2\},$$

where $\alpha \in \mathbb{R}$ is the smoothness parameter. In [5], detailed estimates of the approximation error for $p = 1,2$ are given. Furthermore, tractability results are specified therein.

According to [5], our examples deal with $p = 1$, $p = 2$, and, in addition, $p = 1/2$, $p = \infty$, see Figures 1a – 1d for illustrations in dimension $d = 2$. We construct corresponding frequency index sets $I_{p,N}^d$ and apply Algorithms 1, 3, and 4 in order to determine reconstructing rank-1 lattices. We have to determine suitable rank-1 lattice sizes M for using Algorithm 1. For this, we compute the minimal prime number M_{Cor1} fulfilling Corollary 1. Since this computation is of high costs, we only apply Algorithm 1 to frequency index sets $I_{p,N}^d$ of cardinalities not larger than 20 000. We apply Algorithm 1 using the lattice size M_{Cor1} and the frequency index set $I_{p,N}^d$ as input. With the resulting generating vector, we apply Algorithm 2 in order to determine the reduced lattice size $M_{\mathrm{Alg1+Alg2}}$. Additionally, we use Algorithms 3 and 4 computing rank-1 lattices $\Lambda(\mathbf{z}_{\mathrm{Alg3}}, M_{\mathrm{Alg3}})$ and $\Lambda(\mathbf{z}_{\mathrm{Alg4}}, M_{\mathrm{Alg4}})$, respectively. For reasons of clarity, we present only the rank-1 lattice sizes M_{Cor1}, $M_{\mathrm{Alg1+Alg2}}$, M_{Alg3}, and M_{Alg4} but not the generating vectors $\mathbf{z} \in \mathbb{N}^d$ in our tables.



(a) $I_{\frac{1}{2},16}^2$ (b) $I_{1,16}^2$ (c) $I_{2,16}^2$ (d) $I_{\infty,16}^2$

(e) $I_{\frac{1}{2},16}^{2,\mathrm{even}}$ (f) $I_{1,16}^{2,\mathrm{even}}$ (g) $I_{2,16}^{2,\mathrm{even}}$ (h) $I_{\infty,16}^{2,\mathrm{even}}$

Fig. 1 two-dimensional frequency index sets $I_{p,16}^2$ and $I_{p,16}^{2,\mathrm{even}}$ for $p \in \{\frac{1}{2}, 1, 2, \infty\}$

First, we interpret the results of Table 1. In most cases, the theoretical result of Corollary 1 give a rank-1 lattice size M_{Cor1} which is much larger than the rank-1 lattice sizes found by applying the different strategies in practice. For $p=\infty$, all our algorithms determined a rank-1 lattice of best possible cardinalities, i.e., $|I_{\infty,N}^d| = M_{\mathrm{Alg1+Alg2}} = M_{\mathrm{Alg3}} = M_{\mathrm{Alg4}}$. The outputs M_{Alg3} of Algorithm 3 are larger than these of Algorithm 1 in tandem with Algorithm 2 and Algorithm 4, with a few exceptions. Considering the non-convex frequency index sets $I_{\frac{1}{2},N}^d$, Algorithm 3 brings substantially larger rank-1 lattice sizes M_{Alg3} than the two other approaches. Maybe, we observe the consequences of the missing flexibility in choosing the generating vector in Algorithm 3. Moreover, we observe the equality $M_{\mathrm{Alg1+Alg2}} = M_{\mathrm{Alg4}}$ in all our examples. We would like to point out that Algorithm 1 requires an input lattice size M, which we determined using Corollary 1. However, Algorithm 4 operates without this input.

Since our approach is applicable for frequency index sets with gaps, we also consider frequency index sets $I_{p,N}^{d,\mathrm{even}} := I_{p,N}^d \cap (2\mathbb{Z})^d$. These frequency index sets are suitable in order to approximate functions which are even in each coordinate, i.e., the Fourier coefficients $\hat{f}_{\mathbf{k}}$ are a priori zero for $\mathbf{k} \in \mathbb{Z}^d \setminus (2\mathbb{Z})^d$, cf. Figures 1e – 1h. Certainly, the gaps of the index sets $I_{p,N}^{d,\mathrm{even}}$ are homogeneously distributed. We stress the fact, that the theoretical results and the algorithms can also be applied to strongly inhomogeneous frequency index sets.

Analyzing the frequency index sets $I_{p,N}^{d,\mathrm{even}}$ in detail, we obtain

$$I_{p,N}^{d,\mathrm{even}} = \{2\mathbf{k} \colon \mathbf{k} \in I_{p,N/2}^d\}.$$

We assume $\Lambda(\mathbf{z},M)$ being a reconstructing rank-1 lattice for $I_{p,N/2}^d$. Accordingly, we know

$$\mathbf{k}_1 \cdot \mathbf{z} - \mathbf{k}_2 \cdot \mathbf{z} \not\equiv 0 \pmod{M}$$

for all $\mathbf{k}_1 \neq \mathbf{k}_2$, $\mathbf{k}_1, \mathbf{k}_2 \in I_{p,N/2}^d$. We determine $l_{\mathbf{k}_1,\mathbf{k}_2} \in \{1,\ldots,M-1\}$ and $t \in \mathbb{Z}$ such that

$$\mathbf{k}_1 \cdot \mathbf{z} - \mathbf{k}_2 \cdot \mathbf{z} = tM + l_{\mathbf{k}_1,\mathbf{k}_2}$$

and, furthermore,

$$2\mathbf{k}_1 \cdot \mathbf{z} - 2\mathbf{k}_2 \cdot \mathbf{z} = t2M + 2l_{\mathbf{k}_1,\mathbf{k}_2}.$$

This yields

$$2\mathbf{k}_1 \cdot \mathbf{z} - 2\mathbf{k}_2 \cdot \mathbf{z} \equiv 2l_{\mathbf{k}_1,\mathbf{k}_2} \pmod{M}, \tag{5}$$

where $2l_{\mathbf{k}_1,\mathbf{k}_2} \in \{2,4,\ldots,2M-2\}$. Assuming M being odd, we obtain $2l_{\mathbf{k}_1,\mathbf{k}_2} \not\equiv 0 \pmod M$ for all $\mathbf{k}_1 \neq \mathbf{k}_2$, $\mathbf{k}_1,\mathbf{k}_2 \in I_{p,N/2}^d$ and $\Lambda(\mathbf{z},M)$ is a reconstructing rank-1 lattice for $I_{p,N}^{d,\text{even}}$.

In Table 2 we present the reconstructing rank-1 lattice sizes we found for even frequency index sets. Comparing the two tables, we observe the same odd lattice sizes $M_{\text{Alg1+Alg2}}$ and M_{Alg4} for $I_{p,N/2}^d$ and $I_{p,N}^{d,\text{even}}$. In fact the corresponding generating vectors are also the same. In the case we found even reconstructing lattice sizes for $I_{p,N/2}^d$, we constructed some slightly larger reconstructing rank-1 lattice sizes for $I_{p,N}^{d,\text{even}}$. In these cases, we cannot use the found reconstructing rank-1 lattices for $I_{p,N/2}^d$ in order to reconstruct trigonometric polynomials with frequencies supported on $I_{p,N}^{d,\text{even}}$. The statement in (5) shows the reason for this observation. There exist at least one pair $\mathbf{k}_1,\mathbf{k}_2 \in I_{p,N}^d$, $\mathbf{k}_1 \neq \mathbf{k}_2$ with $\mathbf{k}_1\cdot\mathbf{z}-\mathbf{k}_2\cdot\mathbf{z} \equiv \frac{M}{2} \pmod M$. Consequently, doubling $\mathbf{k}_1$ and $\mathbf{k}_2$ leads to $2\mathbf{k}_1\cdot\mathbf{z}-2\mathbf{k}_2\cdot\mathbf{z} \equiv 0 \pmod M$ and, hence, $\Lambda(\mathbf{z},M)$ is not a reconstructing rank-1 lattice for $I_{p,N}^{d,\text{even}}$.

The fastest way for determining reconstructing rank-1 lattices is to apply Algorithm 1 with a small and suitable rank-1 lattice size M. As mentioned above, the biggest challenge is to determine this small and suitable rank-1 lattice size M. Consequently, estimating relatively small M using some a priori knowledge about the structure of the frequency index set I or some empirical knowledge, leads to the fastest way to reasonable reconstructing rank-1 lattices. We stress the fact, that this strategy fails if there exists no generating vector $\mathbf{z}$ which can be found using Algorithm 1.

All presented deterministic approaches use Algorithm 2. The computational complexity of Algorithm 2 is bounded by $\mathscr{O}((M_{\max}-|I|)|I|)$. However, some heuristic strategies can decrease the number of loop passes. The disadvantage of this strategy is that one does not find $M_{\min}$ but, maybe, an M with $M_{\min} \leq M \ll M_{\max}$. We do not prefer only one of the presented algorithms because the computational complexity mainly depends on the structure of the specific frequency index set and the specific algorithm which is used.

6 Summary

Based on Theorem 1, we determined a lattice size M_{Cor1} guaranteeing the existence of a reconstructing rank-1 lattice for a given arbitrary frequency index set I in Corollary 1. In order to proof this result, we used a component–by–component argument, which leads directly to the component–by–component algorithm given by Algorithm 1, that computes a generating vector $\mathbf{z}$ such that $\Lambda(\mathbf{z},M)$ is a reconstructing rank-1 lattice for the frequency index set I. Due to difficulties in determining M_{Cor1}, we developed some other strategies in order to compute reconstructing rank-1 lattices. The corresponding Algorithms 3 and 4 are also component–by–component algorithms. These algorithms compute complete reconstructing rank-1 lattices, i.e.,

p	N	d	$\lvert I_{p,N}^d \rvert$	M_{Cor1}	$M_{\text{Alg1+Alg2}}$	M_{Alg3}	M_{Alg4}
$\frac{1}{2}$	8	10	1 241	51 679	5 895	16 747	5 895
$\frac{1}{2}$	8	20	4 881	469 841	36 927	172 642	36 927
$\frac{1}{2}$	8	30	10 921	1 654 397	128 370	804 523	128 370
$\frac{1}{2}$	16	5	2 561	122 509	16 680	23 873	16 680
$\frac{1}{2}$	16	10	21 921	-	-	910 271	403 799
$\frac{1}{2}$	16	15	83 081	-	-	9 492 633	3 495 885
$\frac{1}{2}$	32	3	3 529	51 169	17 280	15 529	17 280
$\frac{1}{2}$	32	6	63 577	-	-	1 932 277	1 431 875
$\frac{1}{2}$	64	3	24 993	-	-	113 870	99 758
1	2	10	221	1 361	369	399	369
1	2	20	841	10 723	1 935	2 641	1 935
1	2	30	1 861	36 083	5 664	8 213	5 664
1	4	5	681	4 721	1 175	1 225	1 175
1	4	10	8 361	329 027	36 315	41 649	36 315
1	4	15	39 041	-	-	400 143	340 247
1	8	3	833	2 729	1 113	1 169	1 113
1	8	6	40 081	-	-	126 863	126 738
1	16	3	6 017	21 839	8 497	8 737	8 497
2	2	5	221	1 373	356	353	356
2	2	10	4 541	203 873	21 684	20 013	21 684
2	2	15	25 961	3 865079	259 517	280 795	259 571
2	2	20	87 481	-	-	1 634 299	1 481164
2	4	3	257	809	346	377	346
2	4	6	23 793	496 789	69 065	72 776	69 065
2	8	3	2 109	7 639	2 893	3 050	2 893
2	16	3	17 077	65 309	23 210	23 889	23 210
∞	1	3	27	53	27	27	27
∞	1	6	729	6 257	729	729	729
∞	1	9	19 683	781 271	19 683	19 683	19 683
∞	2	3	125	331	125	125	125
∞	2	6	15 625	236 207	15 625	15 625	15 625

Table 1 cardinalities of reconstructing rank-1 lattices of index sets $I_{p,N}^d$ found by applying Corollary 1, Algorithm 1 and 2, Algorithm 3, and Algorithm 4

generating vectors $\mathbf{z} \in \mathbb{N}^d$ and lattice sizes $M \in \mathbb{N}$, for a given frequency index set I. All the mentioned approaches are applicable for arbitrary frequency index sets of finite cardinality.

References

1. Björck, Å.: Numerical Methods for Least Squares Problems. SIAM, Philadelphia, PA, USA (1996)

p	N	d	$\lvert I_{p,N}^{d,\text{even}}\rvert$	M_{Cor1}	$M_{\text{Alg1+Alg2}}$	M_{Alg3}	M_{Alg4}
$\frac{1}{2}$	16	10	1 241	51 679	5 895	15 345	5 895
$\frac{1}{2}$	16	20	4 881	469 841	36 927	176 225	36 927
$\frac{1}{2}$	16	30	10 921	1 654 397	129 013	763 351	129 013
$\frac{1}{2}$	32	5	2 561	122 509	17 825	23 873	17 825
$\frac{1}{2}$	32	10	21 921	-	-	992 097	403 799
$\frac{1}{2}$	32	15	83 081	-	-	8 848 095	3 495 885
$\frac{1}{2}$	64	3	3 529	51 169	17 689	15 529	17 689
$\frac{1}{2}$	64	6	63 577	-	-	1 932 277	1 431 875
$\frac{1}{2}$	128	3	24 993	-	-	119 159	105 621
1	4	10	221	1 361	369	399	369
1	4	20	841	10 723	1 935	2 641	1 935
1	4	30	1 861	36 083	5 711	8 213	5 711
1	8	5	681	4 721	1 175	1 225	1 175
1	8	10	8 361	329 027	36 315	41 649	36 315
1	8	15	39 041	-	-	400 143	340 247
1	16	3	833	2 729	1 113	1 169	1 113
1	16	6	40 081	-	-	126 863	126 875
1	32	3	6 017	21 839	8 497	8 737	8 497
2	4	5	221	1 373	361	353	361
2	4	10	4541	203 873	22 525	20 013	22 525
2	4	15	25 961	-	-	280 795	259 571
2	4	20	87 481	-	-	1 634 299	1 497 403
2	8	3	257	809	347	13 309	347
2	8	6	23 793	-	-	72 777	69 065
2	16	3	2 109	7 639	2 893	3 063	2 893
2	32	3	17 077	65 309	23 243	23 915	23 243
∞	2	3	27	53	27	27	27
∞	2	6	729	6 257	729	729	729
∞	2	9	19 683	781 271	19 683	19 683	19 683
∞	4	3	125	331	125	125	125
∞	4	6	15 625	236 207	15 625	15 625	15 625

Table 2 cardinalities of reconstructing rank-1 lattices of index sets $I_{p,N}^{d,\text{even}}$ found by applying Corollary 1, Algorithm 1 and 2, Algorithm 3, and Algorithm 4

2. Cooley, J.W., Tukey, J.W.: An algorithm for machine calculation of complex Fourier series. Math. Comput. **19**, 297 – 301 (1965)
3. Cools, R., Kuo, F.Y., Nuyens, D.: Constructing lattice rules based on weighted degree of exactness and worst case error. Computing **87**, 63 – 89 (2010)
4. Kämmerer, L.: Reconstructing hyperbolic cross trigonometric polynomials by sampling along rank-1 lattices. SIAM J. Numer. Anal. (2013). URL http://www.tu-chemnitz.de/∼lkae/paper/Kae2012.pdf. In print
5. Kühn, T., Sickel, W., Ullrich, T.: Approximation numbers of Sobolev embeddings - sharp constants and tractability. J. Complexity (2013). In print
6. Li, D., Hickernell, F.J.: Trigonometric spectral collocation methods on lattices. In: Recent Advances in Scientific Computing and Partial Differential Equations, AMS Series in Contemporary Mathematics, vol. 330, pp. 121 – 132 (2003)

7. Munthe-Kaas, H., Sørevik, T.: Multidimensional pseudo-spectral methods on lattice grids. Appl. Numer. Math. **62**, 155 – 165 (2012). DOI 10.1016/j.apnum.2011.11.002. URL http://www.sciencedirect.com/science/article/pii/S0168927411002042
8. Sickel, W., Ullrich, T.: The Smolyak algorithm, sampling on sparse grids and function spaces of dominating mixed smoothness. East J. Approx. **13**, 387 – 425 (2007)
9. Sloan, I.H., Joe, S.: Lattice methods for multiple integration. Oxford Science Publications. The Clarendon Press Oxford University Press, New York (1994)