TECHNISCHE UNIVERSITÄT CHEMNITZ

Pia Weiß and Fritz Helmedag

From Bilateral Barter to Money Exchange:
Nash's Bargaining Problem Reconsidered

WWDP 87/2007

ISSN 1618-1352

TECHNISCHE UNIVERSITÄT
CHEMNITZ

FAKULTÄT FÜR WIRTSCHAFTSWISSENSCHAFTEN

Impressum:

Herausgeber:
Der Dekan der
Fakultät für Wirtschaftswissenschaften
an der Technischen Universität Chemnitz

Sitz:
Thüringer Weg 7
09126 Chemnitz
Postanschrift:
09107 Chemnitz
Telefon: (0371) 531-34208
Telefax: (0371) 531-34209
E-Mail: dekanat@wirtschaft.tu-chemnitz.de

Internet:
http://www.tu-chemnitz.de/wirtschaft/

ISSN 1618-1352 (Print)
ISSN 1618-1460 (Internet)

From Bilateral Barter to Money Exchange: Nash's Bargaining Problem Reconsidered

Pia Weiss and Fritz Helmedag*

Abstract

In this paper, the example that John Nash presented to explicate his famous solution to the negotiation problem is scrutinised. In his illustration of barter, an apparently unmotivated delivery of a good that the giver appreciates more than the receiver is necessary to realise the largest product of utility gains. Nevertheless, the maximal welfare attainable is not reached. Only the introduction of money and the formation of „fair" prices ensures that the solution to the bargaining problem yields the maximum of the utility increments' sum as well as its product and the equal distribution of the trades' advantages among the participants. Accordingly, even in a two-person economy, money is not neutral by any means.

JEL classification: C78, E40

Keywords: Bargaining, barter, Nash, money

---

* Address: Chemnitz University of Technology, Department of Economics, D-09107 Chemnitz, Germany, Tel.: +49 371 531-34180, Fax: +49 371 531-384180, Email: pia.weiss@wirtschaft.tu-chemnitz.de

1. An example fallen into oblivion

In 1950, John Nash suggested an approach to determine the outcome of bargaining situations which is acknowledged as the central contribution to cooperative game theory. The Nash solution is an agreement that maximises the product of both partners' advantages. Its attractive properties explain its popularity in many fields of economics: „The Nash solution is the only bargaining solution that is independent of utility origins and units, Paretian, symmetric, and independent of irrelevant alternatives" (Mas-Colell, Whinston, Green 1995, 843). According to Ariel Rubinstein the contribution epitomises pure economic theory:

> „This is my ideal paper in almost all respects but above all, it is just … beautiful. Every sentence is measured and appropriate. The construction of the model is so logical. The result is surprising. There are plenty of leftover issues" (Rubinstein 1995, 13).

This appraisal is especially remarkable as Nash augmented his theory with a numerical example covering more than a quarter of the paper – an unusual practice according to today's standards.[1] However, Nash's comments on the underlying mechanisms of the model economy are quite brief. Perhaps, this is the reason why the swap described by Nash's example does virtually play no role in the literature. This is unfortunate since it is excellently suited to investigate trade which can become quite complex in a seemingly simple situation. In addition, Nash mentioned the phenomenon that a transaction medium – money – generates additional gains as compared to pure barter. It seems worthwhile to examine this proposition. Here, it is demonstrated that the existence of a general exchange medium renders welfare improvements even in bilateral trade which do normally not arise. In this role, money achieves more than merely lowering transaction costs (cf. Helmedag 1999).

The following Table (Nash 1950, 161) depicts Bill's and Jack's valuations for the different goods measured in arbitrarily chosen utility units („utils"). The objects that are exchanged according to Nash, are reproduced below the Table.

---

[1] The statement refers to the reprint in Kuhn, H. W. and Nasar, S. (Eds), 2002, 37-46. In the original, the part in question appears to be longer because a figure appertaining to the preceding section is included.

Table 1: Nash's Bargaining Example

Bill's goods	Utility to Bill	Utility to Jack
book (B1)	2	4
whip (B2)	2	2
ball (B3)	2	1
bat (B4)	2	2
box (B5)	4	1
Jack's goods		
pen (J1)	10	1
toy (J2)	4	1
knife (J3)	6	2
hat (J4)	2	2

Bill gives Jack: book, whip, ball, bat

Jack gives Bill: pen, toy, knife

Nash's example indeed deserves scrutiny: Bill gives Jack two goods, the whip and the bat which are worth two utils to *both* persons. In addition, Bill parts with the ball contributing twice as much utils to his welfare than to Jack's well-being! At first glance, this seems mysterious.

Due to the voluntary nature of economically motivated barter, participants always benefit from such interactions: an object is acquired by the person who values it most as long as he or she is able to afford it. As a consequence, exchange acts raise the society's welfare (cf. Helmedag 1994, 51). The Pareto optimum is reached when all voluntary transactions have been carried out, i.e. nobody can be better off without making somebody worse off. The deal described by Nash seems to contradict this property: Although Bill and Jack receive the same utility from the whip and the bat, Bill gives away both. Moreover, Bill even relinquishes the ball to Jack although the ball is dearer to him than to Jack.

2. The whole exceeds the sum of its parts

Deals that seemingly contradict individual rationality result from the axiomatic approach of Nash's bargaining theory. In a world without money, a solution to the barter problem is derived which appears to be inconsistent at first. Table 2 illustrates the complex situation. It enumerates the potential transactions satisfying the (weak) Pareto criterion so that at least one participant benefits from exchange.

Table 2: Outcomes of Barter

No.	Bill's goods	Jack's goods	Bill's trade balance	Δu_B	Δu_J	Ω
1	J1, B3	B1, B2, B4, B5, J2, J3, J4	B1 (−2), B2 (−2), B4 (−2), B5 (−4), J1 (+10)	0	8	0
2	J1, J2, B5	B1, B2, B3, B4, J3, J4	B1 (−2), B2 (−2), B3 (−2), B4 (−2), J1 (+10), J2 (+4)	6	7	42
3	J1, J2, B3, B5	B1, B2, B4, J3, J4	B1 (−2), B2 (−2), B4 (−2), J1 (+10), J2 (+4)	8	6	48
4	J1, J2, J3, B5	B1, B2, B3, B4, J4	B1 (−2), B2 (−2), B3 (−2), B4 (−2), J1 (+10), J2 (+4), J3 (+6)	12	5	60
5	J1, J2, J3, B3, B5	B1, B2, B4, J4	B1 (−2), B2 (−2), B4 (−2), J1 (+10), J2 (+4), J3 (+6)	14	4	56
6	J1, J2, J3, B2, B5	B1, B3, B4, J4	B1 (−2), B3 (−2), B4 (−2), J1 (+10), J2 (+4), J3 (+6)	14	3	42
7	J1, J2, J3, B2, B3, B5	B1, B4, J4	B1 (−2), B4 (−2), J1 (+10), J2 (+4), J3 (+6)	16	2	32
8	J1, J2, J3, B2, B4, B5	B1, B3, J4	B1 (−2), B3 (−2), J1 (+10), J2 (+4), J3 (+6)	16	1	16
9	J1, J2, J3, B2, B3, B4, B5	B1, J4	B1 (−2), J1 (+10), J2 (+4), J3 (+6)	18	0	0

B1: book, B2: whip, B3: ball, B4: bat, B5: box

J1: pen, J2: toy, J3: knife, J4: hat

The second and third column list Bill's and Jack's possessions *after* the swap has taken place. The transactions from Bill's point of view are presented in the fourth column, where the welfare gains and losses are quoted in parentheses. Jack's utility changes would have to enter his trade balance (not displayed) with the opposite signs. The fifth and sixth column contain information on the participants' welfare improvement over the initial level, which amount to six and twelve utils for Bill and Jack according to Table 1. Transaction no. 1 results in a one-sided advantage for Jack who receives eight units; the barter described by no. 9 increases only Bill's utility by 18 utils. The Nash product (Ω) given in the last column is the product of the values from the two preceding columns. It vanishes for the situations described by the first and the last row, whereas it is strictly positive for all other cases.

The framed row marks the Nash solution since it is the one with the highest product of $\Omega_R = 60$. Deal no. 5 only differs from Nash's result in that the ball (B3) remains in Bill's possession who values it higher. In this situation, his utility increment exceeds the one of Nash's solution by two units while Jack's utility gain decreases only by one unit. Yet, the product of the two values is lower: $12 \cdot 5 = 60 > 14 \cdot 4 = 56$. The transfer of the ball becomes reasonable in this context. The exchange of the whip and the bat having the same utility for Bill and Jack can be explained by the same argument. Comparing this situation with transaction no. 9 reveals that the Nash product is non-negative because a swap of goods takes place which is counter-intuitive at first sight.

Another property of the suggested transactions may cause uneasiness. Apparently, barter potentially generates a maximum welfare gain of 18 utils. This sum is created by allocation no. 5 where the ball remains Bill's chattel. In the Nash solution, a utility increase of only 17 units is reached. Here, the question arises whether situation no. 5 can result as an actual business process. In general, there are two possibilities to transfer the goods: either whole bundles change the owner or a sequence of negotiations over a pair of objects takes place. In what follows, we consider step-by-step transactions where each deal entails an augmentation of utility.

3. Rational Bargaining

In a successive trading procedure, objects will only be exchanged if doing so benefits both individuals. Table 3 is based on the information of Table 1 and shows the possible constellations.

The header lists Bill's utility derived from his goods; the last column specifies the utils which he attributes to Jack's endowment itemised in the first column. By analogy, the last row displays Jack's valuation of his opponent's belongings. The respective transactions and the resulting utility alterations are displayed in each cell.

Consider for example the exchange of the book against the pen. Bill's utility rise of 8 utils is presented in the upper right corner of the corresponding box. It results as the difference between the 10 utils won by receiving the pen and the 2 utils lost by giving the book to his partner. In the same way, the lower left corner of the cell informs about Jack's utility increase: the book adds another 4 utils to his welfare and he sacrifices one util by handing the pen over to Bill. The difference amounts to 3 utility units. The sum of the the respective utility advantages is in the center of the cell.

Table 3: Gains and Losses of Barter

		Utility to Bill					
		Book 2	Whip 2	Ball 2	Bat 2	Box 4	
	Pen 1	8 11 3	8 9 1	8 8 0	8 9 1	6 6 0	10
	Toy 1	2 5 3	2 3 1	2 2 0	2 3 1	0 0 0	4
Utility to Jack	Knife 2	4 6 2	4 4 0	4 3 – 1	4 4 0	2 1 – 1	6
	Hat 2	0 2 2	0 0 0	0 – 1 – 1	0 0 0	– 2 – 3 – 1	2
		4	2	1	2	1	

A barter is regarded to be (strictly) rational if it benefits *every* participant so that positive numbers have to appear for such transactions in both corners. Focusing on such processes, it becomes apparent that Jack will never receive the box since none of the appropriate values in the penultimate column is positive. The same is true for the hat: no matter what Bill offers for it, he cannot profit from getting the hat because none of the upper right corners of the appropriate row exhibit a positive entry.

Clearly, only one element is eligible in every column and row because an object can be traded only once. This is the reason why the barter 'book against knife' is surely carried out; it represents the only transaction which leaves Jack better off after parting with his cutting device. Every other alternative diminishes his utility level. In addition, Jack offers the pen and the toy for the whip and the bat. The cells' shading indicates that the business sequence is indeterminate. However, the total effect remains unaffected. In the end, Bill receives the pen, the toy and the knife and hands out the book, the whip and the bat to the person opposite. Hence, transaction no. 5 proves to be the individually rational barter. Accordingly, Nash's solution cannot be replicated by a successive negotiation process.[2] The previous results are verified in the diagram below.

4. Gains graphically

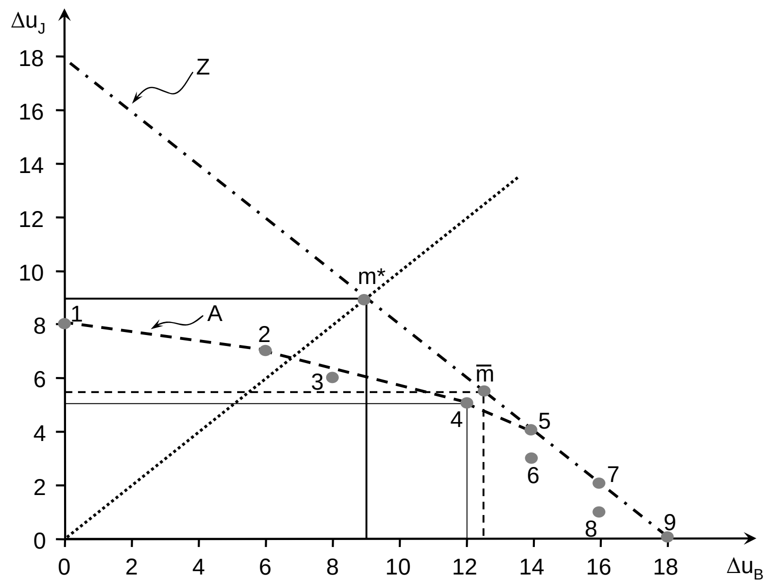
Figure 1 displays all conceivable, Pareto efficient barter solutions in the present economy. The axes measure the partners' utility increases. The nine numbers correspond to the respective events of Table 2. The graph *A* connecting the allocations to the farthest north-east shows that some of them dominate others. The „utility possibility frontier“ *Z* represents the linear combinations of Jack's and Bill's maximum rise in welfare of 18 utils.

Although case no. 4 maximises the Nash product under the prevailing circumstances, it fails to create the maximal total utility gain since it lies below the „efficiency curve“ *Z*. In contrast, transactions 5, 7 and 9 are located on *Z* but the associated Nash products are smaller. The reason for the disparities lies in the transfer of the ball which is less valued by the addressee than by the sender. In fact, Bill keeps this good in all deals situated on the *Z*-line. As it has been demonstrated above, only swap no. 5 is compatible with a sequential rational barter.

---

[2] The Nash solution can only be replicated by a sequence of deals if the pen is given in exchange of either the ball and the bat or the whip and the ball. The fact that the Nash solution being the result of a swap in *bundles* can be reproduced by a sequence of interactions is a property of the specific example and cannot be generalised.

Figure 1: The Utility Space



A general exchange medium offers new transaction possibilities. If a certain object becomes generally accepted as payment by the seller it is no longer necessary to dispense with something that is estimated higher by the owner than the business partner. Instead, a certain amount of money represents the reward. Consistently, the utility dimension is supposed to be denominated in monetary units: „In many cases the money equivalent of a good will serve as a satisfactory approximate utility function“ (Nash 1950, 161-2).

An inaccuracy appears in the corresponding Figure 3 in Nash's article. There, the optimal barter allocation seems to be located on the efficiency line. The subsequent analysis focuses on the symmetric distribution of the welfare gain among the participants by means of money: „Hence the solution has each bargainer getting the same money profit“ (Nash 1950, 162).

Yet, two problems have to be tackled prior. Firstly, among all the allocations placed on the „utility possibility frontier“, the feasible ones have to be determined which are supported by a real barter outcome. Moreover, it has to be examined whether and how the improvement can be evenly split so that the final distribution envisaged by Nash is achieved at all.

5. Money enters the stage

If trade no. 4 resulting from real exchange is regarded to be the starting point of further negotiations in which money is the accepted transaction medium, it forms the fall-back position. Hence, Bill and Jack refuse all proposals leaving them with less than 12 and 5 utils respectively derived from the referential Nash solution. Starting from no. 4, the two parties wish to realise the maximum welfare increase of 18 units.

Let x denote Bill's share. Then, the Nash product for further negotiation reads: $\Omega_M = (x-12)(18-x-5) = -x^2 + 25x - 156$. Differentiating this term with respect to x and setting the derivative equal to zero yields the allocation $\overline{m}$ in Figure 1. This agreement bestows an additional 0.5 money unit on both partners. The associated Nash product is $\Omega_M = 5.5 \cdot 12.5 = 68.75$. How can this transaction be realised in detail?

The book constitutes Bill's only object that is valued higher by Jack. On the other hand, Jack's pen, toy and knife are dearer to his opponent. Provided that money exists, only these goods change the owner. The new acquisitions increase Bill's utility by 20 utils whereas he dispenses with two units by giving away the book; Bill's welfare improvement from the real exchange amounts to 18 units. Certainly, Bill has to transfer Jack 5.5 money units for the monograph as allocation $\overline{m}$ shows. Receiving the book just compensates Jack for parting with the other objects. His net welfare increases by the 5.5 money units which Bill is ready to add to the book.[3] Both Bill and Jack benefit from this transaction compared to the Nash solution achieved without money.

However, this applies not necessarily to the final result Nash had in mind. The symmetry assumption requires to distribute the maximal welfare gain of 18 units evenly so that each person involved acquires nine units. Consequently, Bill would have to give back nine money units of his 18 units derived from the barter to Jack. Then, situation m^* in Figure 1 is reached. The corresponding allocation yields the maximal Nash product of $\Omega^* = 9 \cdot 9 = 81 > 68{,}75 = \Omega_M > 60 = \Omega_R$. With allocation m^* located on the 45°-line, the fall-back options for both participants are postulated to be the origin, i.e. the real Nash solution no. 4 does not form the reference point of a further negotiation. This raises the question why Bill should agree to such a business. Obviously, this bargain would leave him worse off as compared to a world without money. Therefore, Bill might not want to accept a general exchange medium.

---

[3] In case Bill's money resources are insufficient to balance the difference, the whip and the bat can be used as a substitute for remuneration as in barter. The minimum amount of cash required is one and a half money units in this situation.

However, there is one possibility to establish the Nash solution. The goods would have to be sold piece by piece using money for the transactions. In the first step, Bill e.g. offers a payment for the pen. According to Nash's „splitting the surplus"-rule, the price equals half of the buyer's and seller's surplus realised by the deal. Thus, Bill spends $p_{pen} = \frac{1}{2}(10+1) = 5.5$ for the pen. By analogy, Jack receives $p_{toy} = 2.5$ and $p_{knife} = 4$ for the other two goods and, therefore, 12 money units in total. In a corresponding sale, Bill obtains $p_{book} = 3$ for the book. After all transactions have been carried out, the net welfare advantage amounts to nine units for both persons. Ultimately, Nash's result arises; yet subject to the condition that the exchange of whole good bundles does not take place. Instead, „fair" prices guarantee that the maximal Nash product is put into practice.

At any rate, a generally accepted exchange medium provides a new dimension to conduct the seemingly simple transactions even in a two-person world. Apparently, money performs far more than just lubricating barter.

References

Helmedag, F. (1994) Warenproduktion mittels Arbeit, Zur Rehabilitation des Wertgesetzes. 2nd edition, Marburg: Metropolis.

Helmedag, F. (1999) 'Geld: Einführung und Überblick', Enzyklopädisches Lexikon des Geld-, Bank- und Börsenwesens, vol. 1, Frankfurt a.M.: Knapp, 739-45.

Kuhn, H. W. and Nasar, S. (Eds) (2002) The Essential John Nash. Princeton University Press.

Mas-Colell, A., Whinston, M. D. and Green, J. R. (1995) Microeconomic Theory. Oxford University Press.

Nash, J. (1950) 'The Bargaining Problem', Econometrica 18, 155-62. Reprinted in Nash, J. (1996), Essays on Game Theory. Edward Elgar, 1-8 and in Kuhn, H. W. and Nasar, S. (Eds) (2002) The Essential John Nash. Princeton University Press, 37-46.

Rubinstein, A. (1995) 'John Nash: The Master of Economic Modeling', Scandinavian Journal of Economics 97, 9-13.