Barter, efficiency, and money prices: dissecting Nash's bargaining example

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Abstract: John Nash's own illustration of his famous bargaining solution has fallen into oblivion. There, a good is traded that the giver appreciates more than the taker. Although this transaction contributes to the largest (weighted) product of utility gains, their sum falls below the attainable maximum which indicates efficiency. In addition, it is shown that with a medium of exchange and 'fair' prices both criteria can be met. The participants then enjoy the same benefits from exchange. Accordingly, even with only two persons, money can improve their welfare. The insights presented in this paper deserve to find their way into classrooms.

Keywords: bargaining; barter; Nash; money.

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Biographical notes: Fritz Helmedag studied Economics in Tübingen and Stuttgart-Hohenheim from 1974 until 1980 when he received his diploma. Subsequently, he worked at the RWTH Aachen University as a Scientific Assistant, earning his doctorate in 1986. The habilitation took place in 1991. The third edition of his postdoctoral thesis entitled 'Warenproduktion mittels Arbeit' ('Production of commodities by means of labour') was published in 2018. After an Assistant Professorship with a focus on Public Finance, he moved to Chemnitz University of Technology where he was appointed as a Full Professor in 1993. He holds a Chair in Economics, preferring the research fields of competition, employment and history of economic thought. His list of publications comprises more than 100 entries (mostly in German).

1 A forgotten case in point

In 1950, John Nash characterised an optimal result of the negotiations between two parties. His concept is acknowledged as a central contribution to cooperative game theory and praised as "the cornerstone of modern bargaining theory" [Rubinstein, (1995), p.9].[1] Nash proposed an agreement that maximises the mathematical product of both participants' advantages. The attractive properties of the approach explain its popularity in economics: "the Nash solution is the only bargaining solution that is independent of utility origins and units, Paretian, symmetric, and independent of irrelevant alternatives" [Mas-Colell et al., (1995), p.843].[2]

In contrast to the usual practice, Nash augmented his theory with a numerical example covering more than a quarter of the article.[3] But at first sight, the elucidation raises more questions than it answers. Perhaps, this is why Nash's illustration of the bargaining problem is virtually absent from the literature.[4]

This is most unfortunate, because the exemplification is well suited to investigate the explicit execution of trade, which can become quite complex even in a seemingly simple world where only two agents are involved. Yet, Nash remains silent about the behaviour of the negotiators and the detailed haggling process leading to the advocated solution. Moreover, Nash's suggested outcome proves to be sub-optimal. In fact, a different post-barter-allocation is efficient, but the welfare advantages are rather asymmetrically distributed, i.e., the transaction can be deemed unfair.

However, Nash hinted how things may change for the better. He mentioned that a transaction medium – money – generates additional gains vis-a-vis pure barter. Thus, from an educational point of view, Nash's numerical example serves instructively to demonstrate the power of money by opening up additional possibilities not only to increase welfare but also to attain fair results.[5]

Table 1 [Nash, (1950), p.161] depicts Bill's and Jack's valuations of different goods, expressed in utility units ('utils'). These magnitudes, however, must be dimensional homogeneous because Nash employs the product as well as the sum of the protagonists' utility (ibid, p.159). The initial position is 12 utils for Bill (2 (book) + 2 (whip) + 2 (ball) + 2 (bat) + 4 (box)) and 6 utils for Jack (1 (pen) + 1 (toy) + 2 (knife) + 2 (hat)).

The exchanged objects according to Nash are reported in Table 1. The proposed trade increases Bill's satisfaction (Δu_{BN}) by 12 utils (10 (pen) + 4 (toy) + 6 (knife) – 2 (book) – 2 (whip) – 2 (ball) – 2 (bat)) and adds to Jack's welfare (Δu_{JN}) 5 utils (4 (book) + 2 (whip) + 1 (ball) + 2 (bat) – 1 (pen) – 1 (toy) – 2 (knife)). For equally skilled bargainers [Nash, (1950), p.159], the optimal Nash-product amounts to $\Omega_N = \sqrt{\Delta u_{BN}}\sqrt{\Delta u_{JN}} = \sqrt{12 \cdot 5} \approx 7.746$.

Table 1 Nash's bargaining example

Bill's goods	*Utility to Bill*	*Utility to Jack*
Book	2	4
Whip	2	2
Ball	2	1
Bat	2	2
Box	4	1
Jack's goods		
Pen	10	1
Toy	4	1
Knife	6	2
Hat	2	2

Notes: *Bill gives Jack:* book, whip, ball and bat.
Jack gives Bill: pen, toy and knife.

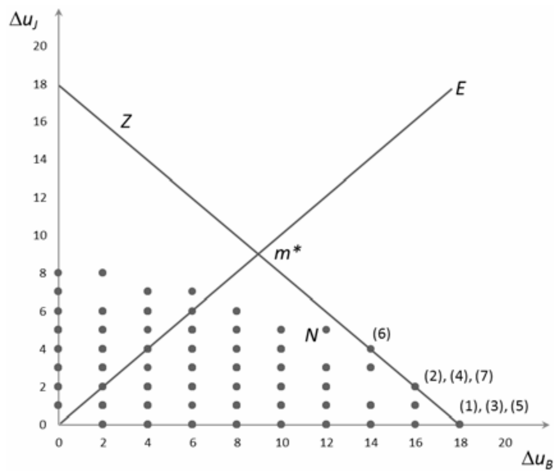
Finally, Nash argues that utility often can be quantified since in many cases "… the money equivalent of a good will serve as a satisfactory approximate utility function" (ibid, pp.161–162). Then utility corresponds to willingness to pay: "By the money equivalent it is meant the amount of money which is just as desirable as the good to the individual with whom we are concerned" (ibid, p.162). Now, due to the cardinal measurement of utility by a standard of value, its interpersonal transferability is provided [see Binmore, (1992), pp.175–176].

Against this backdrop, the example deserves scrutiny: Bill hands over two goods to Jack, the whip and the bat which are worth two utils (henceforth always expressed in money units) to both persons. In addition, Bill relinquishes the ball contributing twice as much to his welfare than to Jack's well-being! This seems mysterious. Due to the non-compulsory nature of economically motivated barter, participants always benefit from such interactions: an item is acquired by the person who appreciates it most as long as he or she is able to afford it. As a consequence, exchange increases society's welfare. The Pareto optimum is reached when all voluntary transactions have been carried out, i.e., nobody's welfare can be enhanced without making someone worse off. In Nash's solution, Jack becomes the owner of the ball, even though he ascribes less utility to the object than Bill. Obviously, such an outcome should stimulate further investigation.

2 Barter in action

Consider a person with n goods. Then $\sum_{k=1}^{n}\binom{n}{k}=\sum_{k=0}^{n}\binom{n}{k}-\binom{n}{0}=2^n-1$ possibilities exist to part with one or more of the items. In Nash's example, Bill has $2^5 - 1 = 31$ alternatives to pay in kind; Jack faces $2^4 - 1 = 15$ different options. Thus, $31 \cdot 15 = 465$ deals are conceivable. But some of them do not satisfy the (weak) Pareto criterion for at least one participant benefiting from an exchange. Figure 1 exhibits the (non-negative) net gains Δu_B and Δu_J of the 284 swaps which raise total utility in the given two-person economy. Clearly, some points represent more than one trade. N denotes the Nash-solution which is located below the 'efficiency-line' (Z) connecting exchanges generating the highest sum of net gains ($\prod$). The maximum amounts to 18 utils and is achieved by seven different trades.

Figure 1 Potential trades



The numbers in brackets refer to Table 2 where the superior bargains are described in detail. Compared to the initial situation, for both traders the goods given and the final situation are reported. The last two columns report the Nash-product ($\Omega = \sqrt{\Delta u_B}\sqrt{\Delta u_J}$) and efficiency ($\prod = \Delta u_B + \Delta u_J$).

The allocations below generate the maximum welfare gain of 18 utils. In contrast, the Nash proposal proves to be Pareto-inferior since it is associated with a utility increase of only $\prod_N = \Delta u_{BN} + \Delta u_{JN} = 12 + 5 = 17$ units. However, Bill and Jack should not accept this inferior solution when "… each individual wishes to maximise the utility to himself of the ultimate bargain" (ibid, p.159). Thus, the question arises whether and how situations leading to efficiency can emerge from actual business negotiations.

Table 2 Efficient barter outcomes

No.	*Bill gives*	*Jack gives*	*Bill's goods after trade*	*Jack's goods after trade*	*Ω*	*Π*
1	Book	Pen, toy, knife	Whip, ball, bat, box, pen, toy, knife	Book, hat	0	18
			$u_B = 30$ $\Delta u_B = 18$	$u_J = 6$ $\Delta u_J = 0$		
2	Book, whip	Pen, toy, knife	Ball, bat, box, pen, toy, knife	Book, whip, hat	$\sqrt{32} \approx 5.657$	18
			$u_B = 28$ $\Delta u_B = 16$	$u_J = 8$ $\Delta u_J = 2$		
3	Book, whip	Pen, toy, knife, hat	Ball, bat, box, pen, toy, knife, hat	Book, whip	0	18
			$u_B = 30$ $\Delta u_B = 18$	$u_J = 6$ $\Delta u_J = 0$		
4	Book, bat	Pen, toy, knife	Whip, ball, box, pen, toy, knife	Book, bat, hat	$\sqrt{32} \approx 5.657$	18
			$u_B = 28$ $\Delta u_B = 16$	$u_J = 8$ $\Delta u_J = 2$		
5	Book, bat	Pen, toy, knife, hat	Whip, ball, box, pen, toy, knife, hat	Book, bat	0	18
			$u_B = 30$ $\Delta u_B = 18$	$u_J = 6$ $\Delta u_J = 0$		
6	Book, whip, bat	Pen, toy, knife	Ball, box, pen, toy, knife	Book, whip, bat, hat	$\sqrt{56} \approx 7.483$	18
			$u_B = 26$ $\Delta u_B = 14$	$u_J = 10$ $\Delta u_J = 4$		
7	Book, whip, bat	Pen, toy, knife, hat	Ball, box, pen, toy, knife, hat	Book, whip, bat	$\sqrt{32} \approx 5.657$	18
			$u_B = 28$ $\Delta u_B = 16$	$u_J = 8$ $\Delta u_J = 2$		

Generally, there are two possibilities to transfer the goods: either whole bundles change owners, or a sequence of bargaining over several objects occurs. In what follows, we consider step-by-step transactions where each deal entails an augmentation of utility. In such successive barter activity, goods will only be exchanged if both individuals benefit from trade. Figure 2 is based on the information from Table 1 indicating possible constellations.

The header lists Bill's utility derived from his goods; the last column specifies the utils which he attributes to Jack's possessions. By analogy, the last row displays Jack's valuation of his opponent's belongings. The respective transactions and the ensuing utility alterations are displayed in each cell.

Figure 2 Welfare changes of successive barter (see online version for colours)

		Utility to Bill					
		BOOK 2	WHIP 2	BALL 2	BAT 2	BOX 4	
Utility to Jack	PEN 1	8 11 3	8 9 1	8 8 0	8 9 1	6 6 0	10
	TOY 1	2 5 3	2 3 1	2 2 0	2 3 1	0 0 0	4
	KNIFE 2	4 6 2	4 4 0	4 3 − 1	4 4 0	2 1 − 1	6
	HAT 2	0 2 2	0 0 0	0 − 1 − 1	0 0 0	− 2 − 3 − 1	2
		4	2	1	2	1	

Barter is (strictly) rational if it benefits every participant so that positive numbers have to occur for such transactions in both corners. In Nash's solution as depicted in Table 1 this is true for the swaps of Jack's pen and toy against Bill's book and whip. Yet, once these changes in ownership were carried out, the above-mentioned problem arises: in order to compensate Jack for parting with the knife, Nash has to resort to the assertion that Bill passes on the bat and the ball. However, if trade is done step-by-step, Jack will neither receive the box nor the ball. Figure 2 shows that for these goods none of the lower left corner values is positive. Accordingly, Nash's solution, where Bill transfers the ball to Jack, cannot be replicated by a successive negotiation process. Likewise, the hat remains in Jack's chattel: no matter what Bill offers, he cannot profit from getting it because none of the upper right corners of the appropriate row exhibit a positive entry. Thus, deals nos. 3, 5 and 7 have to be excluded.

Clearly, just one element is eligible in every column and row because an object can be traded only once. This explains why the exchange 'book against knife' seems most likely since it represents the unique transaction which leaves Jack better off. Every other alternative diminishes his utility level. Furthermore, Jack offers the pen and the toy (thereby losing 2 utils) for the whip and the bat (thus gaining 4 utils) in order to increase his welfare. The cells' shading indicates that this business sequence is indeterminate. However, the total effect remains unaffected. In the end, Bill receives the pen, the toy and the knife; and hands over the book, the whip and the bat to Jack. Hence, transaction no. 6 is the sole individually rational barter.

3 With money to fair trading

On condition that the persons involved also have some amount of a universal exchange medium in their pockets, new transaction possibilities arise.[6] Whenever a certain object has become generally accepted as payment by the seller it is no longer necessary to dispense with something that is estimated higher by the owner than the business partner. Instead, a certain amount of money represents the reward. Furthermore, Nash envisaged a symmetric distribution of the advantages from trade, i.e., the participants are "… getting the same money profit" (ibid, p.162). Therefore, we must examine whether the intended outcome can be established.

Now the initial endowments are regarded to form the basis of negotiations in which goods or money are accepted as remuneration. Efficiency demands that in the end the maximum welfare increase of 18 utils is materialised. It has been shown that transaction no. 6 complies with rational barter without money. But the welfare improvement is asymmetrically split: Bill receives 14 utils whilst Jack must be content with 4 utils. According to the uniformity assumption, the total gain should be shared evenly. These allocations are placed in Figure 1 on the bisecting 'equal-utility-line' (*E*) emanating from the origin. Obviously, the intersection (*m**) with the efficiency-line (*Z*) provides an optimum. How can this solution be attained?

The book represents Bill's single object that is valued higher by Jack. On the other hand, Jack's pen, toy and knife are dearer to Bill. Provided that money exists, only these goods change hands. The new acquisitions increase Bill's utility by 20 utils whereas he dispenses with 2 units by giving away the book; thus his welfare improvement from the real exchange amounts to 18 units. But this allocation does not constitute a 'fair' deal between identically skilled bargainers with the same negotiating power. Equity requires distributing the maximal welfare gain of 18 units into halves so that each person acquires 9 units. Consequently, Bill would have to give back 9 units of his 18 units derived from the barter to Jack. Then, situation *m** in Figure 1 is reached. The corresponding allocation yields the maximal Nash product of $\Omega^* = \sqrt{9 \cdot 9} = 9 > \sqrt{60} = \Omega_N \approx 7.746$.

In practice, a general medium of exchange opens up the way to the superior outcome. The goods would have to be sold piece by piece using, e.g., dollars ($) for the transactions. For instance, Bill offers a payment for the pen. According to Nash's 'splitting the surplus'-rule, the 'fair' price equals half of the buyer's and seller's surplus realised by the deal. Thus, Bill spends $p_{pen} = \frac{1}{2}(10+1) = 5.5$ ($) for the pen. By analogy, Jack receives $p_{toy} = 2.5$ ($) and $p_{knife} = 4$ ($) for the other two goods. Yet, in order to foot the bill of 12 ($), Bill needs no more than 9 money units since he obtains $p_{book} = 3$ ($) for the book. Table 3 summarises the final result.

Table 3 Trade balances

Bill	*Jack*
+10 (*pen*)	+5.5 (p_{pen})
+4 (*toy*)	+2.5 (p_{toy})
+6 (*knife*)	+4 (p_{knife})
–2 (*book*)	–1 (*pen*)
–5.5 (p_{pen})	–1 (*toy*)
–2.5 (p_{toy})	–2 (*knife*)
–4 (p_{knife})	–3 (p_{book})
+3 (p_{book})	+4 (*book*)
= 9	= 9

Alternatively, the necessary quantity of money for compensation purposes can be reduced if Bill pays in kind with the whip and/or the bat. Since both traders ascribe the same

2 utils to these goods, they are a partial substitute for money. Depending on whether Bill hands over the whip or the bat, bargain no. 2 or no. 4 ensues. Then 7 money units instead of 9 suffice. In case both goods are used for remuneration, deal no. 6 emerges. Now just 5 money units are enough for business. Anyway, both agents enjoy a net welfare advantage of 9 utils whenever fair prices prevail.

4 Conclusions

According to the preceding analysis, Nash's numerical example, if appropriately interpreted, suits well to illustrate core processes of a market economy on an elementary level. Even in a two-person world with only nine goods, numerous barter possibilities arise. It has been shown that an efficient allocation different from Nash's solution exists. Furthermore, the course of step-by-step barter leading to the maximal total welfare improvement was expounded. However, the result distributes the emerging advantage from trade asymmetrically. Finally, a generally accepted exchange medium provides a new dimension to conduct transactions efficiently. To boot, if both trading partners negotiate on an equal footing, the maximal attainable gain expressed as a sum is shared evenly. Under these circumstances, 'perfect' competition ensures fairness. Thus, money as a social tool accomplishes far more than merely lubricating barter. This insight should find its way into classes, even when mainstream textbooks are used.

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Notes

1 Basic concepts of game theory are clearly explained by Rasmusen (2007).

2 For a criticism of Nash's analysis see Luce and Raiffa (1957, pp.128–134); Kalai and Smorodinsky (1975) suggested another outcome.

3 The statement refers to the reprint in Kuhn and Nasar (2002, pp.37–46). In the original journal publication, the appropriate part appears to be longer because a figure appertaining to the preceding section is included.

4 A rare exception (without addressing efficiency) is Kennedy (1998, pp.10–15).

5 It goes without saying that the adumbrated positive potentialities of a general medium of exchange do not shield finance-dominated capitalism from qualified criticism.

6 Ehnts and Helmedag (2018, pp.157–160) provide a skeptical assessment regarding the treatment of money in established economics.