Mathematical Dictionary

Preliminary Version
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A

Abel's Inequality: ► [10, Prop. 2.47]: If $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences such that for some $m \in \mathbb{N}$, and $\alpha \in \mathbb{R}_+$ $b1 \geq b2 \geq \ldots \geq b_{m-1} \geq b_m \geq 0$ and $|s_n| \equiv |\sum_{i=1}^{n} a_i| \leq \alpha$ for $n = 1, 2, \ldots, m$, then $|\sum_{n=1}^{m} a_n b_n| \leq \alpha b_1$. ► [10, Cor. 2.48] If $\langle b_n \rangle$ is a sequence such that for some $m \in \mathbb{N}$ $b1 \geq b2 \geq \ldots \geq b_m \geq 0$, then $\sum_{n=1}^{m} (-1)^n b_n \leq b_1$.
absolute value: ► [10, p. 11] if $x \in \mathbb{R}$, $|x| = \{x \text{ if } x \geq 0 \text{ and } -x \text{ if } x < 0\} = \max\{x, -x\}$. Then for $a, b \in \mathbb{R}$: (1) $|a| \leq b \Leftrightarrow -b \leq a \leq b$, (2) $|a| = |-a|$, $|a|^2 = a^2$ and $\sqrt{a^2} = |a|$, (3) $|ab| = |a||b|$ and $a^2 \leq b^2 \Leftrightarrow |a| \leq |b|$, (4) $|a+b| \leq |a|+|b|$, [10, Prop. 1.8] (5) $||x|-|y|| \leq |x-y|$.
accumulation point: *see limit point*

B

ball, ϵ–: ► [2, p. 4] For $\epsilon > 0$ the ϵ–ball around $x \in X$ is the set $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$, with $d(x, y)$ being a metric.
bargaining game: ► *see game, bargaining*
bimonotone hulls: ► *see hulls, bimonotone*
bimonotone sets: ► *see set, bimonotone*
binary relation: ► [16, p. 11] A binary relation G on a set X specifies for all $x, y \in X$ either that xGy is true or that xGy is false. ► *Properties*: [10, Def. 1.19] The binary relation G on a set X is (a) antisymmetric iff: $(\forall x, y \in X) : [xGy \wedge yGx] \Rightarrow x = y$, (b) asymmetric iff: $(\forall x, y \in X) : xGy \Rightarrow \neg yGx$, (c) irreflexive iff $(\forall x \in X) : \neg xGx$, (d) reflexive iff $(\forall x \in X) : xGx$, (e) symmetric iff: $(\forall x, y \in X) : xGy \Rightarrow yGx$, (f) transitive iff: $(\forall x, y, z \in X) : [xGy \wedge yGz] \Rightarrow xGz$, (g) total iff: $(\forall x, y \in X) : xGy$ or yGx or $x = y$. ► [10, Def. 1.78] If P is a binary relation on a non–empty subset of $\mathbb{R}^n$, X, P is *upper semi–continuous* (on X) iff $(\forall \mathbf{x} \in X) : \mathbf{x}P := \{\mathbf{y} \in X | \mathbf{x}P\mathbf{y}\}$ is open relative to X, *lower semi–continuous* (on X) iff $(\forall \mathbf{x} \in X) : P\mathbf{x} := \{\mathbf{y} \in X | \mathbf{y}P\mathbf{x}\}$, *continuous* (on X) iff P is both, upper and lower semi–continuous on X.
bivariate subset: ► *see subset, bivariate*
Blackwell's contraction mapping: ► *see contraction mapping*.
Bolzano–Weierstrass Theorem ► [10, Theorem 2.27] If $\langle x_n \rangle$ is a bounded sequence, then there exists a subsequence $\langle x_{n_i} \rangle$ of $\langle x_n \rangle$, and a number $z \in \mathbb{R}$ such that $\lim_{i \to \infty} x_{n_i} = z$.
bound, greatest lower: *see infimum of A*
bound, least upper: *see supremum of A*
bound, lower: ► [10, Def. 1.9] If $A \subseteq \mathbb{R}$ $\alpha \in \mathbb{R}$ is an lower bound for A iff $(\forall x \in A) : x > \alpha$. ► [16, p. 12] Let $(X, \preceq)$ be a partially ordered set and $X' \subseteq X$. If $x' \in X$ and $x' \preceq x$ for each $x \in X'$, then x' is a lower bound for X'.
bound, upper: ► [10, Def. 1.9] If $A \subseteq \mathbb{R}$ $\alpha \in \mathbb{R}$ is an upper bound for A iff $(\forall x \in A) : x < \alpha$. ► [16, p. 12] Let $(X, \preceq)$ be a partially ordered set and $X' \subseteq X$. If $x' \in X$ and $x \preceq x'$ for each $x \in X'$, then x' is an upper bound for X'.[1]
: $(B(A))$ ► [10, Def. 1.67] The set of all boundary points of A is the boundary of A. ► [10, Prop. 1.68] *Properties*: If A is a subset of $\mathbb{R}^n$, then (1) $\text{int}A \cap B(A) = \emptyset$ and (2) $\bar{A} = \text{int}A \cup B(A)$. ► [12, ch. 2, p. 3] Let (X, d) be a metric space and $S \subseteq X$, let $\text{int}(S)$ be the interior and $\bar{S}$ the closure of S, then the boundary of S is defined as $B(S) := \bar{S} \setminus \text{int}(S)$.
boundary point: ► [10, Def. 1.67] Let A be a subset of $\mathbb{R}^n$. A point $\mathbf{x}^* \in \mathbb{R}^n$ is said to be a boundary point of A iff for each $\epsilon \in \mathbb{R}_{++}$ $N(\mathbf{x}^*, \epsilon) \cap A \neq \emptyset$ and $N(\mathbf{x}^*, \epsilon) \cap A^C \neq \emptyset$.

C

Carathéodory Extension Theorem: ► [2, p. 9] states that if P is a countably additive set function (probability measure) on a field $\mathfrak{G}$ with $P\Omega = 1$ then there is a unique probability measure $\bar{P}$ on the smallest σ–field $\sigma(\mathfrak{G})$ such that $\bar{P}F = PF$ for all $F \in \mathfrak{G}$.
Cauchy–Schwarz Inequality: ► For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $|\mathbf{x} \cdot \mathbf{y}| \leq ||\mathbf{x}|| \cdot ||\mathbf{y}||$.
Cauchy sequence: ► [10, Def. 2.28] A sequence $\langle x_n \rangle$ is called a Cauchy sequence iff, for every $\epsilon > 0$, there exists a $p \in \mathbb{N}$ such that for all $m, n \geq p$, $|x_n - x_m| < \epsilon$. ► [10, Prop 2.29] If $\langle x_n \rangle$ is a Cauchy sequence, then $\langle x_n \rangle$ is bounded. ► [10, Theorem 2.30] A sequence $\langle x_n \rangle$ is convergent iff it is a Cauchy sequence.
chain ► [15] A partially ordered set is a chain, if it does not contain an unordered pair of elements. ► [16, p. 20] If X_1, X_2 are chains, then any bimonotone subset of $X_1 \times X_2$ and hence any bimonotone hull of any subset of $X_1 \times X_2$ is a sublattice of $X_1 \times x_2$.
: $(\bar{A})$ ► [10, Def. 1.64] Let $A \subset \mathbb{R}^n$. A point $\mathbf{x} \in \mathbb{R}^n$ is a point of closure of A iff for each positive real number ϵ, $N(\mathbf{x}, \epsilon) \cap A \neq \emptyset$. ► [10, Prop. 1.65] *Properties*: If A is a subset of $\mathbb{R}^n$ then (1) $A \subseteq \bar{A}$, (2) $\overline{(\bar{A})} = \bar{A}$ (3) $\bar{A}$ is closed. ► [12, ch. 2, p. 6] Let (X, d) be a metric space and $S \subseteq X$. The smallest closed set containing S

[1] Note that the upper bound x' need not to be an element of X'.

is called closure of S. ► [16, p. 30] For a set $X \in \mathbb{R}^n$ the closure of X is the intersection of all closed sets containing X.

cluster point: *see also limit point* ► [10, Def. 2.24] $x^* \in \mathbb{R}$ is a cluster point of the sequence $\langle x_n \rangle$ iff, given any $\epsilon > 0$, and any positive integer m, there exists $n > m$ such that $|x_n - x^*| < \epsilon$. ► [10, Prop. 2.25] The real number x^* is a cluster point of the sequence $\langle x_n \rangle$ iff there exists a subsequence of $\langle x_n \rangle$, $\langle x_{n_i} \rangle$, such that $x_{n_i} \to x^*$. ► [10, Cor. 2.26] If the sequence $\langle x_n \rangle$ converges to $x^* \in \mathbb{R}$, then x^* is a cluster point; in fact the only cluster point of $\langle x_n \rangle$. ► *Bolzano–Weierstrass Theorem* [10, Theorem 2.27] If $\langle x_n \rangle$ is a bounded sequence, then there exists a subsequence $\langle x_{n_i} \rangle$ of $\langle x_n \rangle$, and a number $z \in \mathbb{R}$ such that $\lim_{i\to\infty} x_{n_i} = z$.

collection of non–empty sublattices $\mathcal{L}(X)$: *see sublattice, collection of non–empty*.

comparative statics: ► [16, p. 7] Comparative static is concerned with the dependencies of optimal solutions on the parameter.

comparative statics, monotone: ► [16, p. 7] Monotone comparative static is concerned the optimal solutions varying monotonically with the parameter. ► [16, pp. 7] Monotone comparative statics considerers conditions under which the set of optimal solutions $\text{argmax}_{x\in S_t} f(x,t)$ is increasing in t and one can select an optimal solution x_t in $\text{argmax}_{x\in S_t} f(x,t)$ for each $t \in T$ such that if $t', t'' \in T$ and $t' \le t''$ it follows that $x_{t'} \le x_{t''}$.

comparison test: ► *see convergency tests*.

connected space: ► *see metric space, connected*

continuity: *Uniform continuity* ► [12, ch. 3, p. 6] A function $f : X \to Y$ is uniformly continuous if, for all $\epsilon > 0$, there exists a $\delta > 0$, such that $f(N(x,\delta)) \subseteq N(f(x),\epsilon)$ for all $x \in X$. *Hölder continuity* ► [12, ch. 2, p. 6] A function $f : (X,d) \to (Y,d_Y)$ is said to be Hölder continuous if there exists a $K > 0$ and a $\alpha > 0$ such that $d_Y(f(x),f(y)) \le Kd(x,y)^\alpha$ for all $x,y \in X$. *Lipschitz continuity* ► [12, ch. 3, p. 7] A function $f : (X,d) \to (Y,d_Y)$ is said to be Lipschitz continuous if there exists a $K > 0$ such that $d_Y(f(x),f(y)) \le Kd(x,y)$ for all $x,y \in X$. *Contraction* ► [12, ch. 3, p. 7]A function $f : (X,d) \to (Y,d_Y)$ is said to be a contraction if there exists a $K \in (0,1)$ such that $d_Y(f(x),f(y)) \le Kd(x,y)$. ► [12, ch. 3, p. 7] Any differentiable function is Lipschitz continuous if its derivative is bounded; it is non–expansive if $\sup\{|f'(f)| : t \in \mathbb{R}\} \le 1$; and it is a contraction if $\sup\{|f'(f)| : t \in \mathbb{R}\} \le K \le 1$ for some $K \in (0,1)$. ► [12, ch. 3, p. 8] If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with a constant $K > 0$, then the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) := f(x) + Kx$ is increasing. Thus, g, and hence f, are differentiable almost everywhere by Lebesgue's theorem. ► [12, ch. 3, p. 43] Let T be any non–empty set in a metric space X. Any Lipschitz continuous function $f \in \mathbb{R}^T$ can be extended to a Lipschitz continuous function $F \in \mathbb{R}^X$.

continuous: see also *function, continuous* and *binary relation, continuous*

contraction mapping: ► [9, Def. 7.63] If (M,d) is a metric space, a function $f : M \to M$ such that there exists a constant $k \in [0,1]$ satisfying $d(f(x),f(y)) \le kd(x,y)$ for all $x,y \in M$ is called a contraction mapping. ► [17, p. 47] *Sufficiency condition*: Consider a game $(A_i, \pi_i, i \in \mathbb{N})$, with A_i as a product of k_i compact intervals and π_i be a smooth function on A and strictly quasi–concave in a_i. When A_i is one–dimensional, $r(\cdot)$ is a contraction if $\partial^2\pi_i/\partial a_i^2 + \sum_{j\neq} |\partial^2\pi_i/\partial a_i \partial a_j| < 0$. Similar dominant diagonal conditions on the second derivatives of π_i yield the result for the multidimensional case. [12, ch. 2, p.34] Let T be a non–empty set and let S be any non–empty set $B(T)$ which is closed under addition by positive constant functions. Assume that $\Phi : S \to S$ has the following properties: (i) $f \le g$ implies $\Phi(f) \le \Phi(g)$ for all $f,g \in S$; (ii) there exists a $\delta \in (0,1)$ such that $\Phi(f+\alpha) \le \Phi(f) + \delta\alpha$ for all $f \in S$ and all $\alpha \ge 0$. Then Φ is a contraction.

contraction mapping theorem: ► [9, Theorem 7.64] If (M,d) is a complete metric space and $f : M \to M$ is a contraction mapping, then f has a unique fixed point, i.e. there exists a $x^* \in M$ satisfying $x^* = f(x^*)$ and if $\hat{x} \in M$ also satisfies $\hat{x} = f(\hat{x})$, then $x^* = \hat{x}$.

convergency tests: ► [10, Theorem 2.43] COMPARISON TEST: Suppose $\sum_{n=1}^{\infty} b_n$ is a non–negative series, and let $\sum_{n=1}^{\infty} a_n$ be a second series such that there exists a constant $\alpha > 0$, and an integer m satisfying $|a_n| \le \alpha b_n$ for $n = m+1, m+2, \ldots$. Then (1) if $\sum_{n=1}^{\infty} bn$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and (2) if $\sum_{n=1}^{\infty} |a_n|$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is also divergent. ► [10, Prop. 2.45] D'ALEMBERT'S RATIO TEST: Suppose $a_n > 0$ for $n = 1, 2, \ldots$, and that $\lim\sup(a_{n+1}/a_n) = r < 1$. Then $\sum_{n=1}^{\infty} a_n$ converges. If on the other hand, $\lim\inf(a_{n+1}/a_n) = s > 1$, then the series diverges. ► [10, Prop. 2.49] DIRICHLET'S TEST Suppose $\langle a_n \rangle$ is a sequence such that the corresponding sequence of partial sums $\langle s_n \rangle$ is bounded; and let $\langle b_n \rangle$ be a no–negative monotone decreasing sequence such that $b_n \to 0$. Then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

correspondence, compact–valued: ▶ [12, ch. 4, p. 4] A correspondence $\Gamma : X \to Y$ is compact–valued if the image of each point is compact in the codomain Y of the correspondence,

correspondence, continuous: ▶ [12, ch. 4, p. 9] Let X and Y be metric spaces. A correspondence $\Gamma : X \to Y$ is said to be continuous at x in X if it is upper and lower hemi–continuous at x.

correspondence, increasing (decreasing): ▶ [16, p. 33] A correspondence S_t is increasing (decreasing) in t on T if the domain T is a poset, the range $\{S_t : t \in T\} \in \mathcal{L}(X)$, where X is a lattice and $\mathcal{L}(X)$ is a poset with ordering relation $\sqsubseteq$, and S_t is an increasing (decreasing) from T into $\mathcal{L}(X)$, i.e. $t' \preceq t'' \in T$ implies $S_{t'} \sqsubseteq S_{t''}$ ($S_{t''} \sqsubseteq S_{t'}$). ▶ [16, p. 35, T. 2.4.2] If X is a lattice, T is a poset, the correspondence $S_{\alpha t}$ is an increasing function of t from T into $\mathcal{L}(X)$ for each $\alpha \in A$, and $\cap_{\alpha\in A} S_{\alpha t}$ is non–empty for each $t \in T$ then the correspondence $\cap_{\alpha\in A} S_{\alpha t}$ is an increasing function of t from T into $\mathcal{L}(X)$.

correspondence, increasing family of: ▶ [4] ($\phi_t : t \in T$): Let X be a complete lattice and T be a partially ordered set. An increasing family of correspondences is a correspondence $\phi : X \times T \to X$ such that (1) $(\forall t \in T) : x \mapsto \phi_t(x)$ is weakly increasing, upper hemi–continuous and subcomplete sublattice valued and (2) $(\forall x \in X) : t \mapsto \phi_t(x)$ is weakly increasing.

correspondence, lower hemi–continuous: ▶ [12, ch. 4, p. 8] Let X and Y be metric spaces. A correspondence $\Gamma : X \to Y$ is said to be lower hemi–continuous at $x \in X$, if, for any sequence $(x_m) \in X$ with $x_m \to x$ and $y \in \Gamma(x)$, there exists a sequence $(y_m) \in Y$ such that $y_m \in \Gamma(x_m)$ for each m, and $y_m \to y$. ▶ [12, ch. 4, p. 8] Let X and Y be metric spaces. The correspondence $\Gamma : X \to Y$ is lower hemi–continuous if, and only if, for every open set $O \in Y$ with $\Gamma(x) \cap 0 \neq \emptyset$, there exists a $\delta > 0$ such that $\Gamma(z) \cap O \neq \emptyset$ for all $z \in N(x, \delta)$.

correspondence, upper hemi–continuous: ▶ [12, ch. 4, p. 3] Let X and Y be metric spaces. A correspondence $\Gamma : X \to Y$ is said to be upper hemi–continuous at $x \in X$ if for every open set O in Y with $\Gamma(x) \subset O$, there exists a $\delta > 0$ such that $\Gamma(N(x, \delta)) \subseteq O$.[2] ▶ [12, ch. 4, p. 4] Let X and Y be metric spaces. If $\Gamma : X \to Y$ is a compact–valued and upper hemi–continuous correspondence, then $\Gamma(S)$ is compact in Y whenever S is compact in X. ▶ [12, ch. 4, p. 4] Let X and Y be metric spaces, and let $\Gamma : X \to Y$ be any correspondence. Γ is upper hemi–continuous at x if, for any sequence $(x_m) \in X^\infty$ and $(y_m) \in Y^\infty$ with $x_m \to x$ and $y_m \in Y$ for each m, there exists a subsequence y_m that converges to a point in $\Gamma(x)$. If Γ is compact–valued, then the converse is also true. ▶ [12, ch. 4, p. 5] If Γ_1 and Γ_2 are upper hemi–continuous then so is $\Phi := \Gamma_1 \cup \Gamma_2$. If, in addition, Γ_1 and Γ_2 are compact valued and $\Psi := \Gamma_1 \cap \Gamma_2 \neq \emptyset$ then Ψ is also upper hemi–continuous. ▶ [12, ch. 4, p. 5] Let Γ_1 and Γ_2 be any two correspondences that map a metric space X into $\mathbb{R}^n$. Define $\Phi = (\Gamma_1(x), \Gamma_2(x))$ and $\Psi := \{y_1 + y_2 : y_i \in \Gamma_i\}$, then $\Phi : X \to \mathbb{R}^{2n}$ and $\Psi : X \to \mathbb{R}^n$ are compact–valued and upper hemi–continuous. ▶ [12, ch. 4, p. 6] Let X and Y be metric spaces, and let $\Gamma : X \to Y$ be a correspondence. (a) If Γ has a closed graph, then it need not be upper hemi–continuous. But if Γ has a closed graph and Y is compact, then it is upper hemi–continuous. (b) If Γ is upper hemi–continuous, then it need not to have a closed graph. But if Γ is upper hemi–continuous and closed–valued (i.e. $\Gamma(x)$ is closed in Y for all $x \in X$), then it has a closed graph.

cover: ▶ [16, p. 11] Let $(X, \preceq)$ be a partially ordered set and $x', x'' \in X$. If $x' \prec x''$ and there does not exist a $z \in X$ with $x' \prec z \prec x''$ then x'' covers x' in X. ▶ *see also Set, partially ordered.*

cover, increasable: ▶ [16, p. 27] An increasable set of indices I is an increasable cover for $x' \in S$ if I is a cover for the empty set in the collection of all increasable sets of indices for x'; i.e. if I is a non–empty increasable set for x' and there does not exist an increasable set I' for x' with $\emptyset \subset I' \subset I$. ▶ [16, p. 27, T. 2.2.3.] Suppose $X_1, \dots, X_n$ are chains and S is a sublattice of $\times_{i=1}^n X_i$. (a) The distinct increasable covers for $x' \in S$ are disjoint. (b) If $x', x'' \in S$ correspond to distinct increasable covers for $x' \wedge x'' \in S$ then $S \cap (\times_{i=1}^n \{x'_i, x''_i\}) = \{x', x'', x' \wedge x'', x' \vee x''\}$. (c) If $x', x'' \in S$ are unordered and $S \cap (\times_{i=1}^n \{x'_i, x''_i\}) \neq \{x', x'', x' \wedge x'', x' \vee x''\}$, then there exist $z', z'' \in S \cap (\times_{i=1}^n \{x'_i, x''_i\})$ such that $S \cap (\times_{i=1}^n \{z'_i, z''_i\}) = \{z', z'', z' \wedge z'', z' \vee z''\}$, z', z'' correspond to distinct increasable covers for $x' \wedge x'' \in S \cap (\times_{i=1}^n \{x'_i, x''_i\})$, $z' \preceq x'$, $z'' \preceq x''$, $x' \prec x' \vee z''$, $x'' \prec x'' \vee z'$.

D

D'Alembert's ratio test: *see convergency tests*

DeMorgan's Laws: ▶ [10, Theorem 1.7] S, A non–

[2] A small perturbation of x does not cause the image set $\Gamma(x)$ to suddenly get large.

empty sets, $\mathcal{X} \equiv \{X_a | a \in A\}$ is a family of subsets of S, then (1) $(\cup_{a\in A} X_a)^C = \cap_{a\in A}(X_a)^C$ and (2) $(\cap_{a\in A} X_a)^C = \cup_{a\in A}(X_a)^C$.

differences, decreasing ▶ [17, p. 24] Let X be a lattice and T a poset. The function $g : X \times T \to \mathbb{R}$ has (strictly) decreasing differences in (x, t) if $g(x, t) - g(s, t')$ is strictly decreasing for all $t \geq t'$ $(t \geq t', t \neq t')$.

differences, increasing ▶ [17, p. 24] Let X be a lattice and T a poset. The function $g : X \times T \to \mathbb{R}$ has (strictly) increasing differences in (x, t) if $g(x, t) - g(s, t')$ is strictly increasing for all $t \geq t'$ $(t \geq t', t \neq t')$. ▶ [17, Remark 6] If g is (strictly) supermodular on $X \times T$, then it has (strictly) increasing differences on $X \times T$ and g is (strictly) supermodular on X for any $t \in T$.

Dirichlet's Test: *see convergency tests*.

distance function: *see metric*.

distribution, exponential: ▶ [2, p. 36] An $\mathbb{R}_+$–valued random variable T is exponentially distributed if its survivor function $F(t) = P[T > t]$ is given by $F(t) = e^{-\lambda t}$, $t \geq 0$ for some constant $\lambda = 0$. ▶ *Properties*: [2, p. 36] The mean and standard deviation is equal to $1/\lambda$. ▶ [2, p. 36] The conditional distribution of the remaining time given that $T > s$ is given by $P[T > t + s | T > s] = F(t + s)/F(s) = e^{-\lambda t}$.

dual of a partially ordered set: ▶ [16, p. 11] Let $(X, \preceq)$ be a partially ordered set. Then, the dual is the partially ordered set consisting of the same set X with the binary relation $\preceq'$, where $x' \preceq' x''$ for $x', x'' \in X$ if and only if $x'' \preceq x'$.

E

element, greatest: ▶ [16, p. 12] Let $(X, \preceq)$ be a partially ordered set and $X' \subseteq X$. If $x' \in X'$ is an upper bound for X', then x' is the greatest element of X'.[3]

element, least: ▶ [16, p. 12] Let $(X, \preceq)$ be a partially ordered set and $X' \subseteq X$. If $x' \in X'$ is an lower bound for X', then x' is the least element of X'.[4]

element,maximal: ▶ [16, p. 12] Let $(X, \preceq)$ be a partially ordered set and $X' \subseteq X$. If $x' \in X'$ and $(\nexists x'' \in X') : x' \prec x''$ then x' is a maximal element of X'.[5]

element, minimal: ▶ [16, p. 12] Let $(X, \preceq)$ be a partially ordered set and $X' \subseteq X$. If $x' \in X'$ and $(\nexists x'' \in X') : x'' \prec x'$ then x' is a minimal element of X'.[6]

elements, join ▶ [15] If two elements, x and y, on a partially ordered set have a least upper bound $(x \vee y)$ it is their join.[7]

elements, meet ▶ [15] If two elements, x and y, on a partially ordered set have a greatest lower bound $(x \wedge y)$ it is their meet.

elements, ordered pair of: ▶ [16, p. 11] Let $(X, \preceq)$ be a partially ordered set and $x', x'' \in X$. The elements x' and x'' are ordered if either $x' \preceq x''$ or $x'' \preceq x\prime$.

elements, unordered pair of ▶ [15] Two elements x and y of a partially ordered set are unordered if neither $x \leq y$ nor $y \leq x$.

embedding: ▶ [12, ch. 3, p. 11] If f is not necessarily surjective, but $f : X \to f(X)$ is a homeomorphism, then f is called an embedding.

envelope Theorem: ▶ [6] Let $K \subset X$ be non–empty and compact, suppose that for all t, $f(\cdot, t) : K \to \mathbb{R}$ is upper semi–continuous. Further assume that the partial derivative $f_t(x, t)$ exists and is a continuous function of (x, t). Define further $V(t) = \max_{x\in K} f(x, t)$ and $x^*(t) = \arg\max_{x\in K} f(x, t)$. Then, (a) V has bounded right-hand and left–hand derivatives on $[0, 1)$ and $(0, 1]$, respectively, and these are given by the formulas: $V'_+ = \max_{x\in x^*(t)} f_t(x, t)$ and $V'_- = \min_{x\in x^*(t)} f_t(x, t)$, (b) V is almost everywhere differential on $(0, 1)$ and whenever the derivative exists, $V'(t) = f_t(x(t), t)$ for any $x(t) \in x^*(t)$. (c) For every $t \in [0, 1]$ and any selection $x(t)$ from $x^*(t)$, $V(t) = V(0) + \int_0^t f_t(x(s), s)ds$.

equivalence Relation on X: ▶ [10, Def. 1.25] If X is a non–empty set and R is a binary relation on X, then R is an equivalence relation on X iff R is reflexive, symmetric, and transitive.

existence Theorem: ▶ [17, Theorem 2.5] TOPKIS In a supermodular game the equilibrium set E is non–empty and has a largest, $\bar{a} = \sup\{a \in A : \bar{\Psi}(a) \geq a\}$, and a smallest, $\underline{a} = \inf\{a \in A : \underline{\Psi} \leq a\}$, element. ▶ [17, Theorem 2.6] TARSKI Let $(S, \geq)$ be a completely and densely ordered lattice and $f : S \to S$ a quasi–increasing function. Then, the set of fixed points E is non–empty and $(E, \geq)$ is a completely ordered lattice. In particular, $\bar{x} = \sup\{x \in S : f(x) \geq x\}$ and $\underline{x} = \inf\{x \in S : f(x) \leq x\}$ are the largest and the smallest

[3] Note that there is at most one greatest element of X'.

[4] Note that there is at most one least element of X'.

[5] Unlike the greatest element, there may be several maximal elements of X'. The distinct maximal elements are unordered pairs of elements.

[6] Unlike the least element, there may be several minimal elements of X'. The distinct minimal elements are unordered pairs of elements.

[7] Note that this is usually determined by the *componentwise* comparison of x and y.

fixed points of f. ▶ [17, p. 40] AMIR Let S be a compact interval and $\psi : S \to S$ be a correspondence such that its slope is bounded below, i.e. if $x_1 \neq x_2$, $y_1 \in \psi(x_1)$, and $y_2 \in \psi(x_2)$, then $(y_1 - y_2)/(x_1 - x_2) \geq -k$, for some $k > 0$. Then ψ has a fixed point. ▶ [17, Theorem 2.7] Let A_i be a compact interval of the reals, and suppose that the best replies are upper hemi–continuous strongly decreasing correspondences of the type $\Psi_i(\sum_{j \neq i} a; j)$ for all i. A fixed point of the best reply map exists then. If in addition for all i Ψ_i has slopes strictly above -1, then the equilibrium is unique.

existence Theorem for mixed strategies: ▶ [17, Theorem 2.9] Consider a game $(A_i, \pi_i, i \in \mathbb{N})$. If the (pure) strategy sets are non–empty subsets of the Euclidean space and payoffs are continuous there is a mixed strategy NE.

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field: ▶ [2, p. 9] A class $\mathfrak{G}$ is a field if: (1) $G \in \mathfrak{G}$ implies $G^c := \Omega \setminus G \in \mathfrak{G}$, and (2) $G_1, G_2 \in \mathfrak{G}$ implies $G_1 \cup G_2 \in \mathfrak{G}$.

field, Borel σ**–**: ($\mathfrak{B}(X)$) ▶ [2, p. 8] If $(X, \mathcal{T})$ is a topological space then the Borel σ–field $\mathfrak{B}(X)$ in X is the σ–field generated by the open sets $\mathcal{T}$.

field, σ**–**: ($\mathfrak{F}$) ▶ [2, p. 8] A σ–field $\mathfrak{F}$ in Ω is a class of subsets such that: (1) $F \in \mathfrak{F}$ implies $F^c := \Omega \setminus F \in \mathfrak{F}$; and (2) if $F_i \in \mathfrak{F}$, $i = 1, 2, \ldots$ then $\cup_i F_i \in \mathfrak{F}$, i.e. $\mathfrak{F}$ is closed under countable unions. ▶ *Properties*: [2, p. 8] $\mathfrak{F}$ is also closed under countable intersections, i.e. if $(\mathfrak{F}_\alpha, \alpha \in J)$ is an indexed family of σ–fields with arbitrary index set J then $\mathfrak{G} = \cap_\alpha \mathfrak{F}_\alpha = \{F \in \mathfrak{F} : F \in \mathfrak{F}_\alpha \forall \alpha \in J\}$ is also a σ–field. ▶ [2, p. 8] For any collection $\mathfrak{C}$ of subsets of Ω there is a unique smallest σ–field containing $\mathfrak{C}$, namely the intersection of all σ–fields containing $\mathfrak{C}$.

filtration, natural: ▶ [2, p. 18] Let $\mathring{\mathfrak{F}}_t^X$ be the smallest σ–field in $\mathfrak{F}$ with respect to which all the random variables $\{X_t\}_{t \in [0,t]}$ are measurable. By definition $\mathring{\mathfrak{F}}_{t_1}^X \subset \mathring{\mathfrak{F}}_{t_2}^X$ if $t_1 \leq t_2$ is an increasing family of sub–σ–fields of $\mathfrak{F}$. It is called the natural filtration of X.

fixed point theorem, Brouwer's: ▶ [12, ch. 3, p. 14] Let S be a non–empty compact and convex set in $\mathbb{R}^n$. If $f : S \to S$ is continuous, then there exists an $x \in S$ such that $f(x) = x$.

fixed point theorem, Caristi's: ▶ [12, ch. 3, p. 23] Let Φ be a self–map on a complete metric space X. If $d(x, Phi(x)) \leq f(x) - f(\phi(x))$ for all $x \in X$ for some lower semi–continuous $f \in \mathbb{R}^X$ which is bounded from below then Φ has a fixed point in X.

fixed point theorem, Intermediate Value Theorem: ▶ [12, ch. 3, p. 14] Any continuous function $f : [a, b] \to [a, b]$ has a fixed point.

fixed point theorem, Kakutani–Fan–Glicksberg: ▶ [9, Cor. 12.24] Let K be a non–empty convex and compact subset of a locally convex Hausdorff space, and suppose that the correspondence $\psi : K \to K$ is closed, and is non–empty- and convex–valued. Then the set of fixed points ψ is compact and non–empty.

fixed point theorem, Tarski's: ▶ [8] If T is a complete lattice and $f : T \to T$ is a non–decreasing function, then f has a fixed point. Moreover, the set of fixed points of f has $\sup\{x \in T | f(x) \geq x\}$ as its largest element and $\inf\{x \in T | f(x) \leq x\}$ as its smallest element. ▶ [17, Remark 2] Tarski's theorem is not asserting that the set of fixed points E is a sublattice of T, i.e. if $x, y \in E$, $\sup_T\{x, y\}$ and $\inf_S\{x, y\}$ need not to be elements of E, but $\sup_E\{x, y\}$ and $\inf_E\{x, y\}$ belong to E. ▶ [17, Remark 3] The conclusion that the set of fixed points E is a complete lattice is stronger than the assertion that $\sup_S E$ and $\inf_S E$ belong to E, i.e. it may be true that $\sup_S E, \inf_S E \in E$ but E is not a complete lattice.

function, antitone: ▶ [15] A function f from a partially ordered set S to a partially ordered set T is antitone if $x \leq y$ in S implies $f(y) \leq f(x)$ in T.

function, bijective: ▶ [10] A function is bijective if it is both one–to–one and onto.

function, bimonotone: ▶ [16, p. 23] Let X_i be a poset, the range of $f(x)$ on $\times_{i=1}^n$ be a poset. If there are distinct i', i'' such that the function $f(x)$ does not depend on x_i, $i \neq i'$, $i \neq i''$, $f(x)$ is increasing in $x_{i'}$ and decreasing in $x_{i''}$, then $f(x)$ is bimonotone. ▶ [16, p. 23] A generalised indicator function of a bimonotone set is bimonotone. ▶ [16, p. 23] Each level set of a bimonotone function is bimonotone. ▶ [16, p. 25,L. 2.2.7.] A real–valued separable sublattice–generating function on the direct product of a finite collection of chains is either univariate or bimonotone.

function, bivariate: ▶ [16, p. 23] Let $X_1, \ldots, X_n$ be sets, $x = (x_1, \ldots, x_n) \in \times_{i=1}^n X_i$ with $x_i \in X_i$ and $f(x)$ a function on $\times_{i=1}^n$. If there are distinct i' and i'' such that $f(x)$ does not depend on x_i, $i \neq i'$, $i \neq i''$, then $f(x)$ is bivariate. ▶ [16, p. 23] A generalised indicator function of a bivariate set is bivariate.

function, bounded: ▶ [10, Def. 3.7] Let $f : X \to \mathbb{R}^m$, where $X \subset \mathbb{R}^n$ is non–empty and let $A \subset X$. f is bounded on A iff there exists $a \in \mathbb{R}_+$ such that $(\forall \mathbf{x} \in$

$A) : ||f(\mathbf{x})|| \leq a$. ▶ [10, Prop. 3.8] If $f : X \to \mathbb{R}^m$, where $X \subset \mathbb{R}^n$ is non–empty, $\mathbf{x}^*$ is a limit point of X, and $\mathbf{y} \in \mathbb{R}^m$ is such that $\lim_{\mathbf{x}\to\mathbf{x}^*} f(\mathbf{x}) = \mathbf{y}$, then there exists a positive real number δ such that f is bounded on $N(\mathbf{x}^*, \delta) \cap X$. ▶ [12, ch. 3, p. 15] A continuous real function defined on a compact metric space is bounded.

function, composite: ▶ [10, Lemma 3.23] Let $f : X \to Y$ and $g : Y \to Z$, where X, Y, and Z are non–empty sets, and define the composite function h on X by: $h(x) = g[f(x)]$ for $x \in X$. Then, for any $W \subset z$, $h^{-1}(W) = f^{-1}[g^{-1}(W)]$.

function, continuous: ▶ [10, Def. 3.1] Let $f : X \to \mathbb{R}^m$, where $X \subset \mathbb{R}^n$ is non–empty. f is continuous at $\mathbf{x}^* \in X$ iff for every $\epsilon \in \mathbb{R}_{++}$, there exists a $\delta \in \mathbb{R}_{++}$ such that $(\forall \mathbf{x} \in N(\mathbf{x}^*, \delta) \cap X) : ||f(\mathbf{x}) - f(\mathbf{x}^*)|| < \epsilon$.[8] ▶ [10, Theorem 3.2] If $f : X \to \mathbb{R}^m$, where $X \subset \mathbb{R}^n$ is non–empty, then f is continuous at a point $\mathbf{x}^* \in X$ iff each f_i is continuous at $\mathbf{x}^*$. ▶ [10, Prop. 3.6] Let $f : X \to \mathbb{R}^m$, where X is a non–empty subset of $\mathbb{R}^n$, and let $\mathbf{x}^* \in X$ be a limit point of X. Then, f is continuous at $\mathbf{x}^*$ iff $\lim_{\mathbf{x}\to\mathbf{x}^*} f(\mathbf{x}) = f(\mathbf{x}^*)$. ▶ [10, Lemma 3.19] Let $f : X \to \mathbb{R}^n$, where $X \subset \mathbb{R}^n$. Then f is continuous on X if, and only if, for every open set U in $\mathbb{R}^n$, $f^{-1}(U)$ is open relative to X. ▶ [10, Theorem 3.20] Let $f : X \to \mathbb{R}^n$, where $X \subset \mathbb{R}^m$, and suppose $f(X) \subset Y \subset \mathbb{R}^n$. Then f is continuous on X if, and only if, for each subset of Y, U, which is open relative to Y, we have that $f^{-1}(U)$ is open relative to X. ▶ [10, Theorem 3.22] Let $f : X \to \mathbb{R}^m$, where X is a subset of $\mathbb{R}^n$, and let Y be such that $f(X) \subset Y \subset \mathbb{R}^m$. Then f is continuous on X if, and only if, for each set $C \subset Y$ which is closed relative to Y, $f^{-1}(C)$ is closed relative to X. ▶ [10, Theorem 3.10] Suppose $f : X \to \mathbb{R}^m$ and $g : X \to \mathbb{R}^m$, where X is a non–empty subset of $\mathbb{R}^n$. If f and g are both continuous at $\mathbf{x}^* \in X$, then: (1) $f + g$ is continuous, (2) $(\forall \alpha \in \mathbb{R}) : \alpha f$ is continuous, (3) $f \cdot g$ is continuous at $\mathbf{x}^*$. ▶ [10, Theorem 3.11] Suppose $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$, where X is a non–empty subset of $\mathbb{R}^n$, and that both f and g are continuous at $\mathbf{x}^* \in X$. If $g(\mathbf{x}^*) \neq 0$, then the function $h(\mathbf{x}) = f(\mathbf{x})/g(\mathbf{x})$, is well–defined in a neighbourhood of $\mathbf{x}^*$, and is continuous at $\mathbf{x}^*$. ▶ [10, Theorem 3.12] Let $f : X \to \mathbb{R}^n$ and $g : Y \to \mathbb{R}^p$, where $X \subset \mathbb{R}^m$ and $f(X) \subset Y \subset \mathbb{R}^n$; and suppose f is continuous at $\mathbf{x}^* \in X$, and g is continuous at $\mathbf{y}^* \equiv f(\mathbf{x}^*) \in Y$. Then the composite function, $h = g \circ f$ is continuous at $\mathbf{x}^*$. ▶ [10, Theorem 3.24] Suppose $f : X \to \mathbb{R}^n$ and $g : Y \to \mathbb{R}^n$, where $X \subset \mathbb{R}^m$ and $f(X) \subset Y \subset \mathbb{R}^n$; and suppose that f is continuous on X, and that g is continuous on Y. Then the composite function $h = g \circ f$, defined by $h(x) = g[f(x)]$ for $x \in X$ is continuous on X. ▶ [10, Prop. 3.14] Suppose $f : X \to \mathbb{R}^m$, where X is a non–empty subset of $\mathbb{R}^n$ and let $\mathbf{x}^* \in X$. Then f is continuous at $\mathbf{x}^*$ if , and only if, for every sequence $\langle \mathbf{x}_q \rangle \subset X$, we have $\mathbf{x}_q \to \mathbf{x}^* \Rightarrow f(\mathbf{x}_q) \to f(\mathbf{x}^*)$. ▶ [12, ch. 3, Example 2] The composition of two continuous functions is continuous. ▶ [12, ch. 3, p. 8] If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with a constant $K > 0$, then the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) := f(x) + Kx$ is increasing. Thus, g, and hence f, are differentiable almost everywhere by Lebesgue's theorem.

function, generalised indicator: ▶ [16, p. 15] Let X be a set, $(Y, \preceq)$ be a poset, $f : X \to Y$. The function $f(x)$ is a generalised indicator function for the subset $X' \subset X$ if

$$f(x) = \begin{cases} y'' & \text{for } x \in X' \\ y' & \text{for } x \in X, x \notin X' \end{cases}$$

where $y' \preceq y''$; i.e. if the only level sets of $f(x)$ on X are X and X'. ▶ [16, p. 21] A generalised indicator function for a subset X' of a lattice X is a sublattice–generating function iff X' is a sublattice of X. ▶ *see also Set, level.* ▶ [16, p. 21] A generalised indicator function for a subset X' of a lattice X is a sublattice–generating function iff X' is a sublattice of X. ▶ [16, p. 23] A generalised indicator function of a bivariate set is bivariate. ▶ [16, p. 23] A generalised indicator function of a bimonotone set is bimonotone.

function, identity: is defined on an arbitrary set X by: $\mathbf{i}(x) = x$.

function, indicator: ▶ [16, p. 15] An indicator function is a generalised indicator function with $Y = \mathbb{R}$, $y' = 0$, and $y'' = 1$.

function, injective: *see function, one–to–one.*

function, isotone: ▶ [15] A function f from a partially ordered set S to a partially ordered set T is antitone if $x \leq y$ in S implies $f(x) \leq f(y)$ in T.

function, linear: ▶ [10, Def. 1.80] A function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is linear iff for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all $a \in \mathbb{R}$ (1) $\mathbf{f}(\mathbf{x} + \mathbf{y}) = \mathbf{f}(\mathbf{x}) + \mathbf{f}(\mathbf{y})$ and (2) $\mathbf{f}(a\mathbf{x}) = a\mathbf{f}(\mathbf{x})$. ▶ [10, Prop. 1.81] Suppose $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is linear, that $\mathbf{x}_1, \ldots, \mathbf{x}_q \in \mathbb{R}^n$ and that $a_1, \ldots, a_q$ are real numbers. Then $\mathbf{f}(\sum_{i=1}^q a_i x_i) = \sum_{i=1}^q a_i \mathbf{f}(x_i)$. ▶ [10, Prop. 1.82] A function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and

[8] Notice that if $f : X \to \mathbb{R}^m$ and $\mathbf{x}^* \in X$ is not a limit point of X, then f is necessarily continuous at $\mathbf{x}^*$, i.e. any such function f is continuous at $\mathbf{x}^*$. Hence, this is only a necessary, but no sufficient condition for f to be continuous at $\mathbf{x}^*$.

only if there exists a $m \times n$ matrix $\mathbf{A}$ such that for every $\mathbf{x} \in \mathbb{R}^n$. $\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x}$. ▶ [10, 1.83] Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear, and let $\mathbf{A}$ be the matrix of transformation. Then f is one–to–one if and only if $r(\mathbf{A}) = n$. ▶ [10, 1.83] Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear and let $\mathbf{A}$ be the matrix of transformation. Then f is onto $\mathbb{R}^m$ if and only if $r(\mathbf{A}) = m$. ▶ [10, 1.83] Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear and let $\mathbf{A}$ be the matrix of transformation. Then f is both one–to–one and onto $\mathbb{R}^m$ if and only if $m = n$ and $r(\mathbf{A}) = n$ and thus $\mathbf{A}$ is non–singular.

function, log–concave (convex): ▶ [17, p. 366, FN: 3] A function f is log–concave (convex) if $\log f$ is concave (convex).

function, lower semi–continuous: ▶ [14] A function is called lower semi–continuous (l.s.c.) if $-f$ is upper semi–continuous. ▶ [12, ch. 3 p. 13] Let X be a metric space. f is lower semi–continuous at $x \in X$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $d(x, y) > \delta$ implies that $f(y) \geq f(x) - \epsilon$.[9] ▶ [10, Def. 3.30] Let $f : X \to \mathbb{R}$, where X is a non–empty subset of $\mathbb{R}^n$. We shall say that f is lower–semi continuous at $x^* \in X$ iff the following holds: for each $\epsilon > 0$, there exists $\delta > 0$ such that: $(\forall x \in N(x^*, \delta) \cap X) : f(x) > f(x^*) - \epsilon$.

function, measurable: ▶ [2, p. 9] Let $(\Omega, \mathfrak{F})$ and $(Y, \mathfrak{Y})$ be measurable spaces and $f : \Omega \to Y$. f is a measurable function if $f^{-1}(Y') \in \mathfrak{F}$ for all $Y' \in \mathfrak{Y}$.

function, monotone: ▶ [16, p. 15] A function is monotone if it is either isotone (increasing) or antitone (decreasing).

function, onto: ▶ [10, Def. 1.3] A function is onto B iff for all $b \in B$ there exists a $a \in A$ such that $b = f(a)$. ▶ [10, 1.83] Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear and let $\mathbf{A}$ be the matrix of transformation. Then f is onto $\mathbb{R}^m$ if and only if $r(\mathbf{A}) = m$.

function, one–to–one: ▶ [10, Def. 1.3] A function is one–to–one iff for all $a, a^* \in A$ $f(a) = f(a^*) \Rightarrow a = a^*$. ▶ [10, 1.83] Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear, and let $\mathbf{A}$ be the matrix of transformation. Then f is one–to–one if and only if $r(\mathbf{A}) = n$.

function, order continuous: ▶ [8] Given a complete lattice S, a function $f : S \to \mathbb{R}$ is order continuous if it converges along every chain $C \subset S$ (in both the increasing and decreasing direction), that is, if $\lim_{x \in C, x \downarrow \sup C} f(x) = f(\inf C)$ and $\lim_{x \in C, x \uparrow \sup C} f(x) = f(\sup C)$.

function, order semi–continuous: ▶ [8] Given a complete lattice S, a function $f : S \to \mathbb{R}$ is order continuous if it converges along every chain $C \subset S$ (in both the increasing and decreasing direction), that is, if $\lim_{x \in C, x \downarrow \sup C} f(x) \leq f(\inf C)$ and $\lim_{x \in C, x \uparrow \sup C} f(x) \leq f(\sup C)$.

function, quasi–concave: ▶ [17, p. 149] For *Cournot* competition, the profit function is quasi–concave if $P_i(q)$ is log–concave (and downward sloping) in q_i and $\partial P_i / \partial q_i - C_i'' < 0$. Note: Log–concavity in q_i is equivalent to $p_i \partial^2 P_i / \partial q_i^2 - (\partial P_i / \partial q_i)^2$. ▶ [17, p. 149] With *Bertrand* competition the profit function is quasi–concave if (1) $1/D_i(p)$ is convex in p_i, (2) $D_i(p)$ is log–concave in p_i and the costs are convex or (3) if $\log D_i(p)$ is concave in $\log p_i$.

function, quasi–increasing: ▶ [17, p. 39] A function $f : S \to S$ is quasi–increasing if for every non–empty subset $X \subset S$, $f(\sup X) \geq \inf f(X)$ and $f(\inf X) \leq \sup f(X)$, where $f(X) = \{x \in S : y = f(x), x \in X\}$. ▶ Note: A quasi–increasing function can only have upwards jumps.

function, sample: ▶ [2, p. 17] Let $X(t, \omega)$ be a stochastic process. Then, the function $t \longrightarrow X(t, \omega)$ is called a sample function of the process.

function, separable: ▶ [15] A real–valued function f on $\times_{i=1}^n S_i$ is separable if $f(x) = \sum_{i=1}^n f_i(x_i)$ for all $x = (x_1, \ldots, x_n)$ with $x_i \in S_i$ for $i = 1, \ldots, n$. ▶ [15] If S_i is a chain for $i = 1, \ldots, n$ then f is separable on $\times_{i=1}^n$ iff f is a valuation on $\times_{i=1}^n$. ▶ [16, p. 25,L. 2.2.7.] A real–valued separable sublattice–generating function on the direct product of a finite collection of chains is either univariate or bimonotone.

function, sublattice–generating: ▶ [16, p. 21] Let X be a lattice and Z be a poset. If each level set of $f : X \to Z$ on X is a sublattice, then $f(x)$ is a sublattice–generating function. ▶ [16, p. 21] A generalised indicator function for a subset X' of a lattice X is a sublattice–generating function iff X' is a sublattice of X. ▶ [16, p. 22, L. 2.2.6.] Let X be a lattice, Y be a chain and $f : X \to Y$. (a) If $f(x)$ is the pointwise infimum of a collection of sublattice–generating functions, then $f(x)$ is a sublattice–generating function. (b) Suppose Y has at least two distinct elements. The function $f(x)$ is a sublattice–generating function and is bounded above iff it is the pointwise infimum of collection of sublattice–generating generalised indicator functions. ▶ [16, p. 23, T. 2.2.2] Let $n \geq 2$, $X_1, \ldots, X_n$ be lattices and $f(x)$ be a function from $\times_{i=1}^n$ into a chain in which each non–empty subset that is bounded below has a greatest lower bound, $x = (x_1, \ldots, x_n)$. (a) $f(x)$ is a sublattice–generating function and s bounded

[9] If f is lower semi–continuous, then the images of points nearby x do not fall below $f(x)$ "to much".

above iff it is the pointwise infimum of $n(n-1)/2$ bivariate sublattice–generating functions that are bounded above. (b) $f(x)$ is a sublattice–generating function and is bounded above iff it is the pointwise infimum of $n(n-1)$ bimonotone sublattice–generating functions that are bounded above and n univariate sublattice–generating functions that are bounded above. If X_i is a chain then (c) if $f(x)$ is univariate, then $f(x)$ is a sublattice–generating function. (d) If $f(x)$ is bimonotone, then $f(x)$ is a sublattice–generating function. (e) If $f(x)$ is the pointwise infimum of $n(n-1)$ bimonotone functions and n univariate functions, then $f(x)$ is a sublattice–generating function. (f) If $f(x)$ is a sublattice–generating function and is bounded above, then it is the pointwise infimum of $n(n-1)$ bimonotone functions that are bounded above and n univariate functions that are bounded above. ▶ [16, p. 25,L. 2.2.7.] A real–valued separable sublattice–generating function on the direct product of a finite collection of chains is either univariate or bimonotone.

function, surjective: *see function, onto*.

function, upper semi–continuous: ▶ [14] A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is upper semi–continuous (u.s.c) at $x \in \mathbb{R}_+$ if for any sequence x_k tending to x, $\limsup_{k\to\infty} f(x_k) \leq f(x)$. A function f is called u.s.c if it is u.s.c at every $x \in \mathbb{R}_+$. ▶ [12, ch. 3 p. 13] Let X be any metric space, and let $f \in \mathbb{R}^X$ be any function. f is upper semi–continuous at $x \in X$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $d(x, y)$ implies $f(y) \leq f(x) + \epsilon$.[10] ▶ [10, Def. 3.30] Let $f : X \to \mathbb{R}$, where X is a non–empty subset of $\mathbb{R}^n$. We shall say that f upper semi–continuous at $x^* \in X$ iff the following holds: for each $\epsilon > 0$ there exists $\delta > 0$ such that: $(\forall x \in N(x^*, \delta) \cap X) : f(x) < f(x^*) + \epsilon$.

function, uniformly continuous ▶ [10, Def. 3.16] Let $f : X \to \mathbb{R}^m$, where X is a non–empty subset of $\mathbb{R}^n$. f is uniformly continuous on X iff, given $\epsilon > 0$, there exists $\delta > 0$ such that for all $\mathbf{x}^*, \mathbf{x}' \in X$ it is true that $d(\mathbf{x}', \mathbf{x}^*) < \delta \Rightarrow ||f(\mathbf{x}') - f(\mathbf{x}^*)|| < \epsilon$.[11]

function, univariate: ▶ [16, p. 23] Let $X_1, \ldots, X_n$ be sets, $x = (x_1, \ldots, x_n) \in \times_{i=1}^n X_i$ with $x_i \in X_i$ and $f(x)$ a function on $\times_{i=1}^n$. If there is some i' such that $f(x)$ does not depend on x_i, $i \neq i'$, then $f(x)$ is univariate. ▶ [16, p. 25,L. 2.2.7.] A real–valued separable sublattice–generating function on the direct product of a finite collection of chains is either univariate or bimonotone.

function, vector–valued: ▶ [12, ch. 3,Example 2(iii)] Let $f_i : X \to \mathbb{R}$, $i = 1, \ldots, n$. A vector–valued function $f : X \to \mathbb{R}^n$ is defined by $f(x) := (f_1(x), \ldots, f_n(x))$.

G

game, bargaining: ▶ [13, p. 5] A bargaining game is described by a set $N = \{1, \ldots, n\}$ of players and a pair (S, d) where S is a compact, convex subset of $\mathbb{R}^n$ representing the feasible utility payoffs to the players, and d is an element of S corresponding to the disagreement outcome. For simplicity assume that there is a least point s in S such that $d < s$. Denote by B all such bargaining games.

Game, ordered normal form: ▶ [8] Let $N \neq \emptyset$ be the set of players. Each player n has a strategy set S_n with typical element x_n; the competitors' strategies are denoted by x_{-n} and a full strategy profile is denoted by $x = (x_n, x_{-n}) \in S$. Each strategy set S_n comes with a partial order $\geq_n$, and the strategy profiles are endowed with the product order, that is $X \geq x'$ means $x_n \geq_n x'_n$ for all $n \in N$. Player n's payoff function is $f_n(x_n, x_{-n})$. The object $\Gamma = \{N, (S_n, f_n, n \in N), \geq\}$ is a game in ordered normal form.

Game, smooth supermodular ▶ [17, p. 32] The game $(A_i, \pi_i, i \in \mathbb{N})$ is smooth supermodular, if each A_i is a compact cube in the Euclidean space, π_i is twice continuously differentiable and $\partial^2\pi_i/\partial a_{jh}\partial a_{ik} \geq 0$ for all $k \neq h$ and $\partial^2\pi_i/\partial a_{ih}\partial a_{jk} \geq 0$ for all $i \neq j$ and for all h, k.

Game, Solution to the bargaining game ▶ [13, p. 5] The solution to a bargaining problem is a function $f : B \to \mathbb{R}^n$ such that $f(S, d)$ is an element of S for any (S, d) in B.[12] ▶ [13, p. 6] *Nash's properties of a solution*: (1) Independence of Equivalent Utility Representation: For any bargaining game (S, d) and real numbers a_i and b_i for $i = 1, \ldots, n$ such that each $a_i > 0$, let the bargaining game (S', d') be defined by $S' = \{y \in \mathbb{R}^n : (\exists x \in S) : y_i = a_i x_i + b_i, i = 1, \ldots, n\}$ and $d'_i = a_i d_i + b_i$ for $i = 1, \ldots, n$. Then $f_i(S', d') = a_i f_i(S, d) + b_i$ for $i = 1, \ldots, n$.[13] (2) Symmetry: Suppose (S, d) is a symmetric bargaining

[10] If f is upper semi–continuous, the images nearby x do not exceed $f(x)$ "too much".

[11] Note that every uniformly continuous function is continuous and that the reverse is not in general true. Any constant function on $\mathbb{R}^n$ and mapping into $\mathbb{R}^m$ is uniformly continuous.

[12] That is, a solution is a rule which assigns to each bargaining game a feasible utility payoff of the game.

[13] As the utility function is unique only up to a positive monotone transformation, the solution should be the same if the utility function is transformed.

game, i.e. suppose that $d_1 = d_2 = \ldots = d_n$ and that if $x \in S$ then every permutation[14] of x is contained in S. Then $f_1(S,d) = f_2(S,d) = \ldots = f_n(S,d)$. (3) Independence of irrelevant alternatives: Suppose (S,d) and (T,d) are bargaining games such that $S \subset T$, and $f(T,d) \in S$. Then $f(S,d) = f(T,d)$. (4) Pareto Optimality: For any given game (S,d), if x and y are elements of S such that $y > x$ then $f(S,d) \neq x$. ▶ [13, p. 8] *Nash's Theorem*: There is a unique solution possessing Properties (1)–(4). It is the function $f = F$ defined by $F(S,d) = x$ such that $x \geq d$ and $\Pi_{i=1}^{n}(x_i - d_i) \geq \pi_{i=1}^{n}(y_i - d_i)$ for all $y \in S$ and $y \neq x$.[15]

Game, supermodular: ▶ [8] A game $\Gamma = \{N, (S_n, f_n, n \in N), \geq\}$ is supermodular if for each $n \in N$: (1) S_n is a complete lattice, (2) $f_n : S \to \mathbb{R} \cup \{-\infty\}$ is ordered upper semi–continuous in x_n (for fixed x_{-n}) and order continuous in x_{-n} (for fixed x_n) and has a finite upper bound, (3) f_n is supermodular in x_n (for fixed x_{-n}), (4) f_n has increasing differences in x_n and x_{-n}. ▶ [8, Theorem 4] Suppose that a typical strategy for player n is $(x_{nj}; j = 1, \ldots, k_n) \in \mathbb{R}^{k_n}$ and that $\geq$ is the usual componentwise ordering. Suppose there are finitely many players and the strategies and orders are described as in above. Then, Γ is supermodular if (1) S_n is in an interval in $\mathbb{R}^{k_i}$, that is $S_n = [\underline{y}_n, \bar{y}_n] = \{x | \underline{y}_n \leq x \leq \bar{y}_n\}$, (2) f_n is twice continuously differentiable on S_n. (3) $\partial^2 f_n / \partial x_{ni} \partial x_{nj} \geq 0$ for all n and all $1 \leq i \leq j \leq k_n$, (4) $\partial^2 f_n / \partial x_{ni} \partial x_{mj} \geq 0$ for all $n \neq m$, $1 \leq i \leq k_n$, and $1 \leq j \leq k_m$. ▶ [8, Theorem 5] Let Γ be a supermodular game. for each player n, there exist largest and smallest serially undominated strategies $\bar{x}_n$ and $\underline{x}_n$. Moreover, the strategy profiles $(\underline{x}_n; n \in N)$ and $(\bar{x}_n, n \in N)$ are pure Nash equilibrium profiles. *see also Existence Theorem, Topkis'* ▶ [8, Corollaries] Assume that the assumptions for a supermodular game are satisfied. Then (i) There exists a pure Nash equilibrium (NE). Moreover, there exists a largest and a smallest pure NE in the given order, (ii) if the game Γ has a unique pure NE, then Γ is dominance solvable, (iii) if in addition the game Γ is symmetric and Γ has a unique pure NE, then it is dominance solvable. ▶ [8, Theorem 6] Suppose that $\{N, (S_n, f_n(x_n, x_{-n}, \tau), n \in N), \geq\}$ is a family of supermodular games satisfying the condition that f_n has increasing differences in x_n, τ (for fixed x_{-n}) (note that this is equivalent to $\partial^2 f_n / \partial x_n \partial \tau \geq 0$ for smooth functions). Then, the smallest and largest serially undominated strategies $\underline{x}_n(\tau)$ and $\bar{x}_n(\tau)$ are nondecreasing in τ. ▶ [17, Remark 12] The set of NE of a supermodular game is a complete lattice.

Graph, closed: ▶ [12, ch. 4, p. 6] Let X and Y be metric spaces. A correspondence $\Gamma : X \to Y$ is said to be closed at $x \in X$, if, for any sequence $(x_m) \in X$ and $(y_m) \in Y$, $x_m \to x$, $y_m \in \Gamma(x_m)$ (for each m) and $y_m \to y$ imply $y \in \Gamma(x)$. Γ is said to have a closed graph if it is closed at every $x \in X$.[16] ▶ [12, ch. 4, p. 6] Let X and Y be metric spaces, and let $\Gamma : X \to Y$ be a correspondence. (a) If Γ has a closed graph, then it need not be upper hemi–continuous. But if Γ has a closed graph and Y is compact, then it is upper hemi–continuous. (b) If Γ is upper hemi–continuous, then it need not to have a closed graph. But if Γ is upper hemi–continuous and closed–valued (i.e. $\Gamma(x)$ is closed in Y for all $x \in X$), then it has a closed graph.

Graph of a function f: ▶ [10] $\mathcal{G}_f = \{(x,y) \in A \times B | y = f(x)\}$.

Graph of a correspondence Γ: ▶ [12, ch. 4, p. 5] The graph $\mathrm{Gr}(\Gamma) := \{(x,y) \in X \times Y : y \in \Gamma(x)\}$.

H

Homeomorphism: ▶ [12, ch. 3, p. 11] If $f : X \to Y$ is a bijection between two metric spaces such that both f and f^{-1} are continuous, then it is called homeomorphism between X and Y. ▶ *Properties* [12, ch. 3, p. 11] The set Y possesses any property that X possesses so long as this property is defined in terms of open sets.[17] ▶ [12, ch. 3, p. 12] Neither completeness nor boundedness are preserved by homeomorphism.

Homeomorphism Theorem: ▶ [12, ch. 3, p. 15] If X is a compact metric space and $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

Hulls, bimonotone: ▶ [16, p. 18] If X_1 and X_2 are posets and S is a subset of $X_1 \times X_2$, then the two bimonotone hulls generated by S on $X_1 \times X_2$ are $H_1(S) = \cup_{(x_1,x_2) \in S}([x_1, \infty) \times (-\infty, x_2])$ and $H_2(S) = \cup_{(x_1,x_2) \in S}((-\infty, x_1] \times [x_2, \infty))$. ▶ [16, p. 19, L. 2.2.5.] Suppose hat X_1, X_2 are lattices and S is a sublattice of $X_1 \times X_2$. (a) The bimonotone hulls

[14] Remember that x is a n–tupel. Then a permutation of x is when the elements change places.

[15] Just an odd formulation for the fact that the solution is function F which selects unique outcome that maximises the Nash product.

[16] Closedness at x simply says that if the images of points nearby x concentrate around a particular point in the codomain, this point must correspond to the image of x.

[17] If X is a connected space and Y is homeomorphic to X, then Y must be connected too.

of S are sublattices of $X_1 \times X_2$.[18] (b) If $x_2 \in X_2$ and $x'_1, x''_1 \in S_{x_2}$, then $[x'_1, x''_1] \cap \Pi_{X_1} S$ is contained in S_{x_2}. Hence, if $x_2 \in X_2$ and $\inf_{X_1} S_{x_2}, \sup_{X_1} S_{x_2} \in S_{x_2}$, then $S_{x_2} = [\inf_{X_1} S_{x_2}, \sup_{X_1} S_{x_2}] \cap \Pi_{X_1} S$. (c) The sublattice S is the intersection of its two bimonotone hulls and the direct product of its two projections. ► [16, p. 20] The bimonotone hulls are bimonotone sets. ► [16, p. 20] If X_1, x_2 are lattices and S is a sublattice of $X_1 \times X_2$, then the bimonotone hulls $H_1(S), H_2(S)$ are sublattices. ► [16, p. 20] Any bimonotone set containing S must also contain at least one of the two bimonotone hulls. ► [16, p. 20] If X_1, X_2 are chains, then any bimonotone subset of $X_1 \times X_2$ and hence any bimonotone hull of any subset of $X_1 \times X_2$ is a sublattice of $X_1 \times x_2$.

I

i′i′′–Generator: ► [16, p. 18] Suppose that $X_1, \ldots, X_n$ are sets and $x = (x_1, \ldots, x_n) \in \times_{i=1}^n X_i$ with $x_i \in X_i$. If L is a subset of $X_{i'} \times X_{i''}$ for $i' \neq i''$ and $S = \{x : x \in \times_{i=1}^n X_i, (x_{i'}, x_{i''}) \in L\}$. Then L is the $i'i''$–generator of S. ► *see also Subset, bivariate* ► [16, p. 20] If $X_1, \ldots X_n$ are posets, S is a bivariate subset of $\times_{i=1}^n X_i$, a subset L of $X_{i'} \times X_{i''}$ for $i' \neq i''$ is the $i'i''$–generator of S, and L is bimonotone, then S is bimonotone.

Image of A under f: ► [10, Def. 1.4] If $f : X \to Y$ and A and B are subsets of X and Y, the image of A under f denoted by $f(A)$ is $f(A) = \{y \in Y | (\exists x \in A) : y = f(x)\}$. ► [10, Def. 3.17] Let $f : X \to \mathbb{R}^m$, where $X \subset \mathbb{R}^n$. Then, given any subset, $A \subset \mathbb{R}^m$, the inverse image of A (under f) is defined by $f^{-1}(A)$ is defined by $f^{-1}(A) = \{x \in X : f(x) \in A\}$. ► [10, Prop. 3.18] Let $f : X \to Y$. Then, (1) for any $A \subset Y$; $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$, (2) for any family of subsets of Y, $\{B_a : a \in A\}$, $f^{-1}(\cup_{a \in A} B_a) = \cup_{a \in A} f^{-1}(A)$ and $f^{-1}(\cap_{a \in A} B_a) = \cap_{a \in A} f^{-1}(B_a)$. ► [10, Lemma 3.19] Let $f : X \to \mathbb{R}^n$, where $X \subset \mathbb{R}^n$. Then f is continuous on X if, and only if, for eery open set U in $\mathbb{R}^n$, $f^{-1}(U)$ is open relative to X.

Implicit Function Theorem: ► Suppose $h : \mathbb{R}^n \to \mathbb{R}^m$, where $n > m$, and h is in C^1. Further, suppose we can partition the variables, $x = (y, z)$, such that y is m–dimensional with $\text{grad}_y[h(x)]$ nonsingular at $x = (y^*, z^*)$. Then, there exists $\epsilon > 0$ for which there is an implicit function, f, on the neighbourhood, $N_\epsilon(z^*) = \{z : ||z - z^*|| < \epsilon\}$ such that $h(f(z), z) = 0$ for all z in $N_\epsilon(z^*)$. Further, f is in C^1 with $\text{grad}_f(z^*) = -\text{grad}_y[h(x)]^{-1}\text{grad}_z[h(z)]$. ► *From Harcourt Academic Press Dictionary of Science and Technology* A theorem that states the conditions under which an implicit equation $F(x_1, x_2, \ldots, x_n, y) = 0$ may be solved for y as an explicit function of $x_1, x_2, \ldots, x_n$. In particular, suppose (a) X, Y, and Z are Banach spaces (such as $\mathbb{R}^n$); (b) U is an open set of $X \times Y$; (c) $F : U \to Z$ is a continuously differentiable map such that $F(a, b) = 0$ for some point $(a, b) \in U$; and (d) the partial derivative $F''_y(a, b)$ is an isomorphism of Y onto Z. Then there exists an open set W in X with $a \in W$, an open set V contained in U with $(a, b) \in V$, and a continuously differentiable mapping $g : W \to Y$ such that, for all $(x, y) \in V$, $F(x, y) = 0$ if and only if $x \in W$ and $y = g(x)$. That is, if $x \in W$, the implicit equation $F(x, y) = 0$ has a continuously differentiable solution $y = g(x)$ with $(x, y) \in V$. This solution is unique on some open subset of W.

Inada conditions: ► [10, p. 98] For each t, f_t is twice differentiable at each $x > 0$ with $f'_t(x) > 0$ and $f''_t(x) < 0$, and $\lim_{x \to 0_+} f'_t > 1$, and $\lim_{x \to \infty} f'_t(x) < 1$.

Increasable set of indices: *see Set of indices, increasable*

Increasable cover: *see cover, increasable*

Increasing (decreasing) optimal selection: ► *see Selection, increasing (decreasing) optimal*.

Increasing (decreasing) optimal solution: ► *see Solution, increasing (decreasing) optimal*.

Increasing (decreasing) selection: ► *see Selection, increasing (decreasing)*.

Induced set ordering: *see Set ordering, induced*

Inequality, properties: ► [10, p. 11] $a, b, c, d \in \mathbb{R}$ then (1) if $a \leq b$ and $c \leq d$, then $a + c \leq b + d$, (2) if $c \geq 0$ then $a \leq b \Rightarrow ca \leq cb$, (3) if $c \leq 0$ then $a \leq b \Rightarrow cb \leq ca$, (4) if $a \geq b > 0$, then $0 < 1/a \leq 1/b$.

infimum of A: ► [10, Def. 1.11] $\alpha \in \mathbb{R}$ is a infimum of $A \subset \mathbb{R}$, $\inf A$, iff (1) α is an lower bound for A and (2) if β is an lower bound for A, then $\alpha \geq \beta$. ► [10, Theorem 1.12] Any non–empty set of real numbers that is bounded below has a infimum. ► [10, Prop. 1.15] Suppose $A, B \subset \mathbb{R}$ and non–empty satisfying $(\forall a \in A)(\forall b \in B) : a \geq b$ then $\in A$ and $\sup B$ both exist and $\inf A \geq \sup B$. ► [10, Prop. 1.17] Let $\lambda A = \{x \in \mathbb{R} | (\exists x' \in A) : x = \lambda x'\}$. If $A \subset \mathbb{R}$ and non–empty, which is bounded below, and $\lambda \in \mathbb{R}_+$, then $\inf(\lambda A)$

[18] Note that bimonotone hulls are not generally sublattices of $X_1 \times X_2$ if X_1, X_2 are lattices but not chains and S is not a sublattice of $X_1 \times X_2$.

exists and $\inf(\lambda A) = \lambda \inf A$. ▶ [10, Prop. 1.18] Let $A + B = \{x \in \mathbb{R} | (\exists a \in A, b \in B) : x = a + b\}$. If $A, B \subset \mathbb{R}$ and non–empty which are bounded from below, then $A+B$ is bounded below and $\inf(A+B) = \inf A + \sup B$. ▶ [15] If S is a complete lattice X, Y are in $P(S)$ and $X \leq^p Y$, then $\inf X \leq \inf Y$ and $\sup X \leq \sup Y$.

Inner Product: ▶ [16, p. 8] Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{t} = (t_1, t_2, \ldots, t_n)$, with $x_i, t_i \in \mathbb{R}$. Then, the inner product is given by $\mathbf{tx} = \sum_{i=1}^{n} t_i x_i$.

Intermediate Value Theorem: ▶ [12, ch. 3, p. 13] Let X be a connected metric space and let f be a continuous real function on X. If α belongs to the interval $(f(x), f(y))$ for some $x, y \in X$ then there exists a $z \in X$ such that $f(z) = \alpha$.

Interior: ▶ [12, ch. 2, p. 6] Let (X, d) be a metric space and $S \subseteq X$. The largest open set that is contained in S is called the interior of S and denoted by int(S).

Interior point: (int X) ▶ [10, Def. 1.55] Let $x \in \mathbb{R}^n$ and let X be a subset of $\mathbb{R}^n$. Then, x is an interior point of X iff there exists a positive real number ϵ such that $N(x, \epsilon) \subseteq X$. ▶ [10, Prop. 1.57] Let $x^* \in \mathbb{R}^n$, $\delta \in \mathbb{R}_{++}$. Then int$[N(x^*, \delta)] = N(x^*, \delta)$.

Intertemporal Efficiency: ▶ [10, Def. 2.52] If (x^*, y^*, c^*) is a feasible program for $(f, \bar{y})$, (x^*, y^*, c^*) is intertemporally efficient for $(f, \bar{y})$ iff there is no other feasible program for $(f, \bar{y})$ (x, y, c) such that $c > c^*$.

Interval Topology: ▶ [17, p. 30] Let $(S, \geq)$ be a lattice. Its interval Topology is defined by taking the sets of the type $\{z \in S : z \leq x\}$ and $\{z \in S : z \geq x\}$ to form a subbasis for closed sets. ▶ [16, p. 29] The interval topology on a poset $(X \preceq)$ is that topology for which each closed set is either X or the empty set or can be represented as the intersection of sets that are finite unions of closed intervals on X. ▶ [16, p. 29] A complete lattice is compact in its interval topology. ▶ [16, p. 29] A lattice that is compact in its interval topology is complete.

Inverse image of B under f: ▶ [10, Def. 1.4] If $f : X \to Y$ and A and B are subsets of X and Y, the inverse image of B under f denoted by $f^{-1}(B)$ is $f^{-1}(B) = \{x \in X | f(x) \in B\}$.

J

Join: ▶ *see Elements, join.*

K

Kakutani–Fan–Glicksberg Fixed Point Theorem ▶ *see Fixed Point Theorem*

Kernel, of the associated transformation: ▶ [10, 1.83] If $\mathbf{A}$ is an $m \times n$ matrix of rank r, then the kernel of the associated transformation, the set S is defined as $S = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{Ax} = 0\}$, and is a linear subspace of dimension $n - r$.

L

Lattice: ▶ [4] A partially ordered set X is a lattice if whenever $x, y \in X$, both $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$ exits in X. ▶ [16, p. 13] The direct product of lattices is a lattice. ▶ [15] For a lattice S, the relation $\leq^p$ is antisymmetric and transitive in $P(S)$. ▶ [15] If S is a lattice with relation $\leq$, then the set of non–empty sublattices $L(S)$ is a partially ordered set with the relation $\leq^p$. ▶ [16, p. 16, L. 2.2.1] If X' is a sublattice of a lattice X and X'' is a non–empty finite subset of X', then $\sup_X(X'')$ and $\inf_X(X'')$ exist and are contained in X'. Hence, if X is a non–empty finite lattice, then X has a greatest and a least element.

Lattice, compact: ▶ [17, p. 30] A lattice is compact in its interval topology iff it is complete.

Lattice, complete: ▶ [17] A lattice $(S, \geq)$ is complete if every non–empty subset of S has a supremum and infimum in S. ▶ [16, p. 29] By L. 2.2.1 any finite lattice is complete. ▶ [16, p. 29] A complete lattice is compact in its interval topology. ▶ [16, p. 29] A lattice that is compact in its interval topology is complete.

Lattice, completely and densely ordered: ▶ [17, p. 39] Let $(S, \geq)$ be a completely ordered lattice. Then it is densely ordered if for all $x, y \in S$ there is a $z \in S$ such that $x < z < y$.

Lebesgue's Theorem: ▶ [12, ch. 3, p.8] If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is differentiable (with a finite derivative) almost everywhere.[19]

Level set: ▶ *see Set, level*

limit: ▶ [10, p. 62] *of a sequence* Is the $x^* \in \mathbb{R}$ to which a sequence $\langle x_n \rangle$ converges. The following facts hold: (1) The limit of a sequence, if it exists, is unique. (2) The alteration of a finite number of terms of a sequence has no effect on convergence, divergence, or limit. (3) If all but a finite number of terms of a sequence are equal to some constant, then the sequence converges to that constant. ▶ [10, Def. 2.32] Let $\langle x_n \rangle$ be a bounded sequence and define $\langle y_m \rangle$ and

[19] Almost everywhere means that there may exist countable exceptions.

$\langle z_q \rangle$ by $y_m = \inf_{n\geq m} x_n$ and $z_q = \sup_{n\geq q} x_n$. Then $\liminf x_n = \lim_{m\to\infty} y_m$ and $\limsup x_n = \lim_{q\to\infty} z_q$. ▶ [10, Def. 3.3] Let $f : X \to \mathbb{R}^m$, where $X \subset \mathbb{R}^n$ is non–empty, and let $\mathbf{x}^* \in \mathbb{R}^n$ be a limit point of X. $\mathbf{y} \in \mathbb{R}^n$ is the limit of f as $\mathbf{x}$ approaches $\mathbf{x}^*$ iff for each $\epsilon > 0$ there exists a $\delta \in \mathbb{R}_{++}$ such that $(\forall \mathbf{x} \in N'(\mathbf{x}^*, \delta) \cap X) : ||f(\mathbf{x}) - \mathbf{y}|| < \epsilon$. ▶ [10, Def. 3.4] If $f : X \to \mathbb{R}^m$, where $X \subset \mathbb{R}^n$ is non–empty, and $\mathbf{x}^* \in \mathbb{R}^n$ is a limit point of X. The limit of f as $\mathbf{x}$ approaches $\mathbf{x}^*$ is $+\infty$ $(-\infty)$ iff for every $a \in \mathbb{R}_+$ there exists a $\delta \in \mathbb{R}_{++}$ such that $(\forall \mathbf{x} \in N'(\mathbf{x}^*, \delta) \cap X) : f(\mathbf{x}) \geq a$ $(f(\mathbf{x}) \leq -a)$. ▶ *Properties for vector–valued functions* [10, Theorem 3.9] Suppose $f : X \to \mathbb{R}^m$ and $g : X \to \mathbb{R}^m$, where $X \subset \mathbb{R}^n$, let $\mathbf{x}^*$ be a limit point of X and suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ are such that $\lim_{\mathbf{x}\to\mathbf{x}^*} f(\mathbf{x}) = \mathbf{y}$ and $\lim_{\mathbf{x}\to\mathbf{x}^*} g(\mathbf{x}) = \mathbf{z}$. Then (1) $\lim_{\mathbf{x}\to\mathbf{x}^*}[f(\mathbf{x}) + g(\mathbf{x})] = \mathbf{y} + \mathbf{z}$, (2) for any $\alpha \in \mathbb{R}$, $\lim_{\mathbf{x}\to\mathbf{x}^*} \alpha f(\mathbf{x}) = \alpha y$, (3) $\lim_{\mathbf{x}\to\mathbf{x}^*} f(\mathbf{x})g(\mathbf{x}) = \mathbf{yz}$.

limit point: ▶ [10, Def. 1.69] If A is a non–empty subset of $\mathbb{R}^n$, a point $\mathbf{x}^* \in \mathbb{R}^n$ will be said to be a limit point of A iff for each $\epsilon \in \mathbb{R}_{++}$ $N'(\mathbf{x}^*, \epsilon) \cap A \neq \emptyset$.

Log–supermodularity: ▶ [1, p. 3] A function f is log–supermodular if $\log f$ is supermodular. ▶ [1, p. 3] Sufficient conditions for the reaction correspondence of firm 1 to be globally non–increasing in p_2 is that either the profit function is log–supermodular or that the profit function satisfies the single–crossing property. ▶ [1, Lemma 2] Π^1 is log–supermodular in $(p_1, p_2) \in P^1 \times P^2$ if D^1 is log–supermodular, or equivalently if D^1 satisfies $\Delta_o := D^1 D^1_{p_1 p_2} - D^1_{p_1} D^1_{p_2} \geq 0$ for all $(p_1, p_2) \in P^1 \times P^2$.

Lower semi–continuous: see also *Function, lower semi–continuous* and *Binary relation, lower semi–continuous*

M

Matrix, non–singular: ▶ [10, 1.83] A square matrix of full rank is said to be non–singular. ▶ [10, 1.83] An $n \times n$ matrix is non–singular if and only if $|\mathbf{A}| \neq 0$.

Matrix, rank of: ▶ [10, 1.83] *row rank* The row rank of an $m \times n$ matrix $\mathbf{A}$ is the number of linearly independent row vectors of $\mathbf{A}$. *column rank* The column rank of an $m \times n$ matrix $\mathbf{A}$ is the number of linearly independent column vectors of $\mathbf{A}$. ▶ [10, 1.83] For any matrix $\mathbf{A}$ the row rank is necessarily equal to the column rank. The common rank is denoted by $r(\mathbf{A})$. ▶ [10, 1.83] If $\mathbf{A}$ is a $m \times n$ matrix, it is necessarily true that $r(\mathbf{A}) \leq \min\{m, n\}$. ▶ [10, 1.83] An $m \times n$ matrix $\mathbf{A}$ is said to have *full rank* iff $r(\mathbf{A}) = \min\{m, n\}$.

Matrix of the transformation: ▶ [10, 1.83] If $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear so that f can be written in the form $f * (\mathbf{x}) = \mathbf{Ax}$ for $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A}$ is called the matrix of the transformation f. ▶ [10, 1.83] Given two $m \times n$ matrices $\mathbf{A}, \mathbf{B}$ and the associated transformations $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^m$, the matrix $\mathbf{C} = \mathbf{A} + \mathbf{B}$ is the matrix of the transformation $f + g$. ▶ [10, 1.83] Given an $m \times n$ matrix $\mathbf{A}$ and a scalar $\alpha \in \mathbb{R}$ the matrix $\alpha\mathbf{A}$ is the matrix of the transformation αf, where f is defined by $f(\mathbf{x}) = \mathbf{Ax}$. ▶ [10, 1.83] Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ be defined by $f(\mathbf{x}) = \mathbf{Ax}$ for $\mathbf{x} \in \mathbb{R}^n$ and $g(\mathbf{y}) = \mathbf{By}$ for $\mathbf{y} \in \mathbb{R}^m$, then the matrix $\mathbf{C} = \mathbf{BA}$ is the matrix of the composite transformation $h = g \circ f$: $h(\mathbf{x}) = g[f(\mathbf{x})] = \mathbf{B}(\mathbf{Ax}) = (\mathbf{BA})\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$.

Matrix, sign of the elements: ▶ [17, p. 38] Consider $Dr_i = -D_{i1}^{-1} D_{i2}$. If the off–diagonal elements of D_{i1} are non–negative it follows that the all elements of $-D_{i1}^{-1}$ are non–negative and the diagonal elements are positive. A sufficient condition for the elements of Dr_i to be non–negative (non–positive) is that all the elements of D_{i2} are non–negative (non–positive).

Meet: ▶ *see Elements, meet.*

metric: ▶ [12, ch. 2, p.2] Let X be any non–empty set. A function $d : X \times X \to \mathbb{R}_+$ that satisfies the following properties is called a distance function or metric on X: for all $x, y, z \in X$ (1) $d(x, y) = 0$ if and only if $x = y$, (ii) (symmetry) $d(x, y) = d(y, x)$, (iii) (triangle inequality) $d(x, y) \leq d(x, y) + d(y, z)$.

Metric, discrete: ▶ [12, ch. 2, p. 2] The function $d : X \times X \to \mathbb{R}_+$ with $d := \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ is called the discrete metric on X.

Metric, Euclidean: ▶ [12, ch. 2, p. 2] $d(\cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is defined by $d(x, y) = [\sum_{i=1}^n (x_i - y_i)^2]^{1/2} = [(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})]^1/2 = ||x - y||$. ▶ [10, Def. 1.49] *Properties*: For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and a $\lambda \in \mathbb{R}$ $d(\cdot)$ satisfies (1) $d(x, y) \geq 0$ and $[d(x, y) = 0 \Rightarrow \mathbf{x} = \mathbf{y}]$, (2) (symmetry) $d(x, y) = d(y, x)$, (3) (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$, (4) (homogeneity) $d(\lambda x, \lambda y) = |\lambda| d(x, y)$, (5) (translation invariance) $d(x + z, y + z) = d(x, y)$. ▶ [10, Cor. 1.54] Denoting the Euclidean metric on $\mathbb{R}^n$ by $d_i(\cdot)$ for $i = 1, \ldots, n$ and for $\mathbb{R}^k$ with $k = \sum_{i=1}^n m_i$ by $d(\cdot)$ then, $\forall x, y \in \mathbb{R}^k$, $d_i(x_i, y_i) \leq d(x, y)$ and $d(x, y) \leq \sum_{i=1}^n d_i(x_i, y_i)$.

Metric, Hausdorff: ▶ [12, ch. 4, p. 10] Let Y be a metric space and let $c(Y)$ be the set of all non–empty compact sets in Y. For any two sets $A, B \in c(Y)$ let $\omega(A,B) := \max_{z\in A} d_Y(z,B)$, which is well defined. Define the function $d_H : c(Y)\times c(Y) \to \mathbb{R}_+$ by $d_H(A,B) := \max\{\omega(A,B), \omega(B,A)\}$, which is called the Hausdorff metric.[20]

Metric, p–m.: ▶ [12, ch. 2, p. 3] Let $x, y \in \mathbb{R}^n$, $p \geq 1$ $d_p(x,y) := \left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p}$.

Metric space: ▶ [10, Def. 1.48] If d is a distance function on X, (X,d) is called a metric space.

metric space, connected: ▶ [12, ch. 2, p. 11] A metric space (X,d) is said to be connected if there do not exist two nonempty and disjoint open sets A and B in X such that $A \cup B = X$. ▶ [12, ch. 2, p. 11] A metric space (X,d) is connected if, and only if, the only subsets of X that are both open and closed in X are $\emptyset$ and X. ▶ [12, ch. 3, p. 13] Let X and Y be metric spaces, and let $f : X \to Y$ be a continuous function. If X is connected, then $f(X)$ is a connected subset of Y.

Metric space, connected subset of X: ▶ [12, ch. 2, p. 11] A subset S of a metric space X is called connected in X if (S,d) is connected.

Metric, sup–m: ▶ [12, ch. 2, p. 4] Let $(x_m), (y_m) \in \mathbb{R}^\infty$ be sequences. Let l_∞ be the set of all sequences such that $\sup_{m\geq 1}|x_m| < \infty$. And let this set be endowed with the sup–metric defined by $d_\infty((x_m),(y_m)) := \sup_{m\geq 1}|x_m - y_m|$.

Minkowski's inequality: ▶ [12, ch. 2, p. 3] For any $a_i, b_i \in \mathbb{R}$, $i = 1,\dots,n$ and any $p \geq 1$, $\left(\sum_{i=1}^n |a_i + b_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^n |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^n |b_i|^p\right)^{1/p}$.

Mixed extension of the game: ▶ [17, p. 44] The mixed extension of a game $(A_i, \pi_i, i \in \mathbb{N})$ is defined by the strategy space $D(A_i)$, the set of all (Borel) probability measures on A_i and the payoff $V_i(\mu) = \int \pi_i(a)\ d\mu(a)$, where $d\mu(a) = (d\mu_1(a_1) \times \dots \times d\mu_n(a_n))$ and $\mu = (\mu_1, \dots, \mu_n)$, $\mu_i \in D(A_i)$ for all i. The mixed extension is denoted by $(D(A_i), V_i, i \in \mathbb{N})$.

Monotone comparative statics: ▶ *see Comparative statics, monotone*.

Monotonicity Theorem, Topkis's: ▶ [8] Let S_1 be a lattice and S_2 be a partially ordered set. Suppose $f(x,y) : S_1 \times S_2 \to \mathbb{R}$ is supermodular in x for given y and has increasing differences in x, y. Suppose that $y \geq y'$ and that $x \in M \equiv \arg\max f(x,y)$ and $x' \in M' \equiv \arg\max f(x,y')$. Then $x \wedge x' \in M'$ and $x \vee x' \in M$. In particular, (when $y = y'$), the set of maximisers of f is a sublattice.

[20] $d_Y(z,B)$ is defined by $d(x,S) := \inf\{d(z,x) : x \in B\}$.

N

Nash equilibrium in mixed strategies: ▶ [17, p. 44] A mixed strategy equilibrium of the game $(A_i, \pi_i, i \in \mathbb{N})$ is a NE of the mixed extension $(D(A_i), V_i, i \in \mathbb{N})$.

Nash equilibrium, of a game with strategic complements: ▶ [3] The set of Nash equilibria of a two player GSC, $\{S_1, S_2, u_1, u_2\}$ where S_1, S_2 are totally ordered, is a sublattice of $S_1 \times S_2$.

Neighbourhood, Euclidean: ▶ [10, Def. 1.51] For $\mathbf{x} \in \mathbb{R}^n$ and a $\delta \in \mathbb{R}_{++}$ the Euclidean neighbourhood of $\mathbf{x}$ with radius δ, $N(x,\delta)$ is defined by $N(x,\delta) = \{y \in \mathbb{R}^n | d(x,y) < \delta\}$. ▶ [10, Def. 1.51] *deleted Euclidean neighbourhood* with radius δ, $N'(x,\delta)$, by $N'(x,\delta) = \{y \in \mathbb{R}^n | 0 < d(x,y) < \delta\}$. ▶ [10, Lem. 1.56] If x^* is an element of $\mathbb{R}^n$, $\delta \in \mathbb{R}_{++}$ and $y \in N(x^*,\delta)$ and if $\epsilon := \delta - d(x^*,y)$, then $N(y,\epsilon) \subseteq N(x^*,\delta)$. ▶ [10, Prop. 1.57] Let $x^* \in \mathbb{R}^n$, $\delta \in \mathbb{R}_{++}$. Then int$[N(x^*,\delta)] = N(x^*,\delta)$.

Norm, Euclidean: ▶ [10, Def. 1.45] For $\mathbf{x} \in \mathbb{R}^n$, the Euclidean norm of x is $||x|| = \left(\sum_{i=1}^n x_i^2\right)^{1/2} = (\mathbf{x}\cdot\mathbf{x})^{1/2}$. ▶ [10, Theorem 1.46] *Cauchy–Schwarz Inequality*: For all $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$, $|\mathbf{x}\cdot\mathbf{y}| \leq ||\mathbf{x}|| \cdot ||\mathbf{y}||$. ▶ [10, Theorem 1.47] *Properties*: If $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^n$ and a $\lambda \in \mathbb{R}$ $||\cdot||$ satisfies (1) $||x|| \geq 0$ with $[||x|| = 0 \Rightarrow \mathbf{x} = 0]$, (2) (homogeneity) $||\lambda x|| = |\lambda|\ ||x||$, (3) (triangle inequality) $||x + y|| \leq ||x|| + ||y||$. ▶ [10, Prop. 1.53] Denoting the Euclidean norm on $\mathbb{R}^n$ by $||\cdot||_i$ for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ then, $||x_i||_i \leq ||x||\ \forall i = 1,\dots,n$, and $||x|| \leq \sum_{i=1}^n ||x_i||_i$.

O

Order, weak: ▶ [10, Def. 1.21] Let G be a binary relation on a non–empty set X. G is a weak order iff G is total, reflexive, and transitive. ▶ [10, Prop. 1.26] If G is a weak order on X, then (1) the asymmetric part of G, P, is irreflexive, asymmetric and transitive, and (2) the symmetric part of G, I is an equivalence relation.

P

Partially ordered set: ▶ *see Set, partially ordered* ▶ *see also Dual*

Poincaré–Hopf Index Theorem ▶ [17, p. 362, FN 32] Let $f : A \to \mathbb{R}^k$ be a smooth vector field on a compact cube $A \subset \mathbb{R}^k$. Suppose that f has only a finite number of zeros $x_1, \dots, x_n$ and that it points in on the boundary of A. Then $\sum_{i=1}^n I(x_i) = 1$, where $I(x_i)$ is the index of x_i. $I(x_i) = +1$ if $\det[-Df(x_i)] > 0$, $I(x_i) = -1$ if $\det[-Df(x_i)] < 0$ and $I(x_i)$ equals an integer depending on further topological conditions if $\det[-Df(x_i)] = 0$.

Power set: ► [16, p. 12] The power set $\mathcal{P}(X)$ of a set X is the set of all subsets of X.

Process, stochastic: ► [2, p. 16] A stochastic process is a family $X := \{X_t\}_{t\in J}$ of random variables with some index set $J \subset \mathbb{R}$. If $J = \mathbb{Z}_+$then X is a discrete–time process, while if $J = \mathbb{R}_+$ X is a continuous–time process. ► [2, p. 17] X is measurable if the function $(t,\omega) \longrightarrow X(t,\omega)$ is measurable on the product space $(\mathbb{R}\times\Omega, \mathfrak{B}\otimes\mathfrak{F})$, where $\mathfrak{B}$ is the Borel set.

Product, inner: ► *see Inner product.*

Production function, Stone–Geary: ► [5, p. 227, FN 2] $q = \min\{(K-\bar{K})/a_k, (l-\bar{L})/a_L\}$, with q the output, K capital and L labour.

Product set: ► [16, p. 12] Let X_a be a set for each $a \in A$. Then, $\times_{a\in A} X_a = \{x = (x_a : a \in A) : x_a \in X_a$ for each $a \in A\}$ is the product set.

Program, competitive: ► [10, Def. 2.54] A feasible program (x^*, y^*, c^*) is competitive for $(f, \bar{y})$ iff there exists a $p \in \mathbb{R}^\infty_{++}$ such that for t, $(t = 0,1,2,\ldots)$, and for all $(x,y) \in \mathbb{R}^2_+$ such that $y \leq f_{t+1}(x)$ it is true that $p_{t+1}f_{t+1}(x^*_t) - p_t x^*_t \geq p_{t+1}y - p_t x$.

Power set of A: ► [10, p. 6] Is the family of subsets of A.

Principle of Mathematical Induction: ► [10, p. 16]If $A \subset \mathbb{N}$ satisfies (a) $1 \in A$ and (b) for each $k \in A$ we have also $k+1 \in A$, then $A = N$.

Product, inner: ► [10, Def. 1.42] For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ the inner product is defined as $\mathbf{x}\cdot\mathbf{y} = \sum_{i=1}^n x_i y_i$. ► [10, Prop. 1.43] *Properties*: if $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^n$ and $a \in \mathbb{R}$ then (1) $\mathbf{x}\cdot\mathbf{x} \geq 0$ and $\mathbf{x}\cdot\mathbf{x} = 0 \Rightarrow \mathbf{x} = 0$, (2) $\mathbf{x}\cdot\mathbf{y} = \mathbf{y}\cdot\mathbf{x}$, (3) $\mathbf{x}\cdot(a\mathbf{y}) = a(\mathbf{x}\cdot\mathbf{y}) = (a\mathbf{x})\cdot\mathbf{y})$, (4) $\mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) = \mathbf{x}\cdot\mathbf{y}+\mathbf{x}\cdot\mathbf{z}$.

Product order: ► [8] Let $x, z \in \mathbb{R}^n$ then $x \geq y$ iff $x_i \geq y_i$ for $i = 1,\ldots,n$.

Projection: ► [15] If S is a subset of $X \times T$, then the projection of S at $t \in T$ is $\Pi_t S = \{t : S_t \neq \emptyset\}$. ► *see also Section*

Q

Quasi–supermodularity: ► [17, p. 29] Let X be a lattice, the function $F : X \to \mathbb{R}$ is quasi–supermodular if for $x, y \in X$ $f(x) \geq f(\inf(x,y))$ implies that $f(\sup(x,y)) \geq f(y)$. ► [17, p. 29] Sufficient condition: f is quasi–supermodular if there is a strictly increasing function h such that $h \circ f$ is supermodular. ► [17, p. 29] Let $S \subset X$. $\phi(t,S) \equiv \arg\max_{x\in S} g(x,t)$ is increasing in (t,S) iff $g : X \times T \to \mathbb{R}$ is quasi–supermodular in x and satisfies the single crossing property in (x,t). If S is a sublattice of X then $\phi(t,S)$ is a sublattice of S. In in addition the constraint $S(t)$ is increasing in t and g satisfies the strict single crossing property in (x,t), then $\phi(t,S(t))$ is strongly increasing in t.

R

Rademacher's Theorem: ► [12, ch. 3, p. 9] Any Lipschitz continuous function $f : \mathbb{R} \to \mathbb{R}$ is differentiable almost everywhere.

Reaction correspondence: ► [1, p. 3] Sufficient conditions for the reaction correspondence of firm 1 to be globally non–increasing in p_2 is that either the profit function is log–supermodular or that the profit function satisfies the single–crossing property.

Relation, binary: ► Let G be a binary relation on a set X, then G is (1) total iff $(\forall x, y \in X) : xGy$ or yGx or $x = y$, (2) reflexive iff $(\forall x \in X) : xGx$, (3) irreflexive iff $(\forall x \in X) : \neg xGx$, (4) symmetric iff $(\forall x, y \in X) : xGy \Rightarrow yGx$, (5) asymmetric iff $(\forall x \in X) : xGy \Rightarrow \neg zGx$, (6) antisymmetric iff $(\forall x, y \in X) : [xGy \wedge yGx] \Rightarrow x = y$, (7) transitive iff $(\forall x, y, z \in X) : [xGy \wedge yGz] \Rightarrow xGz$ (8) [10, Def. 1.28] negative transitive $(\forall x, y, z \in X)$ if xPz then either xPy or yPz.

S

Section: ► [15] If S is a subset of $X \times T$, then the section of S at $t \in T$ is $S_t = \{x : (x,t) \in S\}$.[21] ► [15] If S_{zt} is ascending in t on T for each z in an arbitrary set Z and if $\cap_{z\in Z} S_{zt}$ is nonempty for each $t \in T$, then $\cap_{z\in Z} S_{zt}$ is ascending in $t \in T$. ► [15, Lemma 6.1] Let S^*_t let be the set of optimal solutions for given $t \in T$ and let $T^* = \{t : S^*_t \neq \emptyset\}$. If S is a lattice, T a partially ordered set, $S_t \subseteq S$ is ascending in t on T, and $f(x \wedge y, t) + f(x \vee y, b) \leq f(x,t) + f(y,b)$ for all $t, b \in T$ with $t \leq b$, $x \in S_t$, and $y \in S_b$, then S^*_t is ascending in t on T^*. ► [15, Theorem 6.1] Let S^*_t let be the set of optimal solutions for given $t \in T$ and let $T^* = \{t : S^*_t \neq \emptyset\}$. If S is a lattice, T a partially ordered set, $S_t \subseteq S$ is ascending in t on T, $f(x,t)$ is

[21] Hence, a section is the a subset of X so that $(x,t) \in S$ for a particular value of t.

submodular in x on S for each $t \in T$, and $f(x,t)$ has antitone differences in (x,t) on $S \times T$, then S_t^* is ascending in t on T^*. ▶ [15, Theorem 6.2] If, in addition to the hypotheses of Theorem 6.1, each S_t is compact in a topology finer that the interval topology and $f(x,t)$ is lower semi–continuous in x on S_t for each $t \in T$, then each S_t^* has a least element, s_t, and a greatest element, $\bar{s}_t$, and s_t and $\bar{s}_t$ are both isotone in t on T. ▶ [15, Theorem 6.3] If S is a lattice, T a partially ordered set, $S_t \subseteq S$ is ascending in t on T, $f(x,t)$ is submodular in x on S for each $t \in T$, and $f(x,t)$ has strictly antitone differences in (x,t) on $S \times T$, then S_t^* is strongly ascending in t on T^*.

Selection: ▶ [16, p. 35] If X, T are sets, S_t is a subset of X for each $t \in T$, and $x_t \in S_t$ for each $t \in T$, then the function $x_t : T \mapsto X$ is a selection from S_t.

Selection, increasing (decreasing): ▶ [16, p. 35] If X, T are posets, $S_t \subseteq X$ for each $t \in T$, and x_t is a selection from S_t that is an increasing (decreasing) function (correspondence) of t from T into X, then x_t is an increasing (decreasing) selection.

Selection, increasing (decreasing) optimal: ▶ [16, p. 35] For a collection of optimisation problems, maximise $f(x,t)$ subject to $x \in S_t$, with each problem determined by a parameter t, an increasing (decreasing) selection from $\text{argmax}_{x \in S_t} f(x,t)$ is an increasing (decreasing) optimal selection.

Semi–metric: ▶ [12, ch. 2, p.2] If d satisfies conditions (ii) and (iii) of a metric and $d(x,x) = 0$ for all $x \in X$, then d is called a semi–metric on X.[22]

Semi–metric space: ▶ [12, ch. 2, p.2] If d is a semi–metric on X, (X,d) is called a semi–metric space.

Sequence: ▶ [10, Def. 2.1] A sequence of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$, denoted by $f(n) = x_n$ for $n = 1, 2, \ldots$ or $\langle x_n \rangle$. ▶ [10, Def. 2.4] A sequence $\langle x_n \rangle$ is said to have the limit $+\infty$ $(-\infty)$ or diverge to $+\infty$ iff for every $\alpha \in \mathbb{R}$, there exists $m \in \mathbb{N}$ such that $(\forall n \geq m) : x_n > \alpha$ $((\forall n \geq m) : x_n < \alpha)$.

Sequence, bounded: ▶ [10, Def. 2.6] A sequence, $\langle x_n \rangle$ is said to be bounded below (respectively above) iff there exists $\alpha \in \mathbb{R}$ satisfying $x_n \geq \alpha$ $(x_n \leq \alpha)$ for $n = 1, 2, \ldots$. The sequence $\langle x_n \rangle$ is said to be bounded iff it is bounded both above and below. ▶ [10, Prop. 2.7] A sequence $\langle x_n \rangle$ is bounded if and only if there exists $\alpha \in \mathbb{R}$ satisfying $|x_n| < \alpha$ for $n = 1, 2, \ldots$. ▶ [10, Prop 2.29] If $\langle x_n \rangle$ is a Cauchy sequence, then $\langle x_n \rangle$ is bounded. ▶ [10, Prop. 2.31] Let $\langle x_n \rangle$ be a bounded sequence, and define the sequences $\langle y_m \rangle$, $\langle z_q \rangle$ by $y_m = \inf_{n \geq m} x_n$ for $m = 1, 2, \ldots$ and $z_q = \sup_{n \geq q} x_n$ for $q = 1, 2, \ldots$. Then, the sequences $\langle y_m \rangle$ and $\langle z_q \rangle$ both converge and $\lim_{q \to \infty} z_q \geq \lim_{m \to \infty} y_m$. (There is a largest and a smallest cluster point). ▶ [10, Prop 2.34] If $\langle x_n \rangle$ is a bounded sequence, then (1) $\langle x_n \rangle$ converges iff $\liminf x_n = \limsup x_n$ in which case $\lim_{n \to \infty} x_n = \limsup x_n$ and (2) if z is the limit of a subsequence of $\langle x_n \rangle$ then $\liminf x_n \leq z \leq \limsup x_n$.

Sequence, convergent: ▶ [10, Def. 2.2] Let $\langle x_n \rangle$ be a sequence and let $x^* \in \mathbb{R}$. The sequence $\langle x_n \rangle$ is said to be converge to x^* iff $(\forall \epsilon > 0)(\exists m \in \mathbb{N})(\forall n \geq m) : |x_n - x^*| < \epsilon$. ▶ [10, Prop. 2.8] If the sequence $\langle x_n \rangle$ converges then it is bounded. ▶ [10, Theorem. 2.9] Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be convergent sequences and define $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$ then: (1) $\lim_{n \to \infty} (x_n + y_n) = x + y$, (2) $(\forall \alpha \in \mathbb{R}) : \alpha x_n \to \alpha x$, (3) $\lim_{n \to \infty} x_n \cdot y_n = x \cdot y$. ▶ [10, Cor. 2.10] If $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences such that $x_n \to 0$ and $\langle y_n \rangle$ is bounded, then $x_n y_n \to 0$. ▶ [10, Prop. 2.11] Suppose there are m convergent sequences $\langle x_{in} \rangle$, $i = 1, \ldots, m$, and that $x_{in} \to \bar{x}_i$ for $i = 1, \ldots, m$. Given any real number α_i define the sequence $\langle z_n \rangle$ by $z_n = \sum_{i=1}^m \alpha_i x_{in}$, then $z_n \to \sum_{i=1}^m \alpha_i \bar{x}_i$. ▶ [10, Lem. 2.12] If $\langle x_n \rangle$ having no terms equal to zero, and $x_n \to x^* \neq 0$ then $1/x_n \to 1/x^*$. ▶ [10, Theorem 2.13] If $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences satisfying (1) $(\forall n \in \mathbb{N}) : x_n \neq 0$, (2) $x_n \to x^* \neq 0$, and (3) $y_n \to y^*$, then $y_n / x_n \to y^* / x^*$. ▶ [10, Prop 2.14] If $\langle x_n \rangle$ is a sequence and x^* is a real number, $x_n \to x^*$ if and only if $|x_n / x^*| \to 0$. ▶ [10, Prop. 2.15] Suppose $\langle x_n \rangle$ is a sequence satisfying $x_n \geq 0$ for $n = 1, 2, \ldots$, then if $\langle x_n \rangle$ converges, $\lim_{n \to \infty} x_n \geq 0$. ▶ [10, Prop. 2.16] If $\langle x_n \rangle$ and $\langle y_n \rangle$ satisfy $x_n \to x^*$, $y_n \to y^*$, and $x_n \geq y_n$ for $n = 1, 2, \ldots$, then $x^* \geq y^*$. ▶ [10, Prop. 2.17] If $\langle x_n \rangle$ and $\langle y_n \rangle$ satisfy $0 \leq x_n \leq |y_n|$ for $n = 1, 2, \ldots$, and $y_n \to 0$, then $x_n \to 0$ as well. ▶ [10, Prop. 2.18] If $\langle x_n \rangle$, $\langle y_n \rangle$, and $\langle z_n \rangle$ satisfy $x_n \leq y_n \leq z_n$ for $n = 1, 2, \ldots$, $x_n \to x^*$, and $z_n \to x^*$, then $y_n \to x^*$ as well. ▶ [10, Theorem 2.30] A sequence $\langle x_n \rangle$ is convergent iff it is a Cauchy sequence.

Sequence of partial sums of a series: ▶ [10, Def. 2.35] If $\langle a_n \rangle$ is a sequence, the sequence $\langle s_n \rangle$ defined by $s_n = \sum_{i=1}^n a_i$ is called the sequence of partial sums of the series.

Sequence, monotone: ▶ [10, Def. 2.19] A sequence $\langle x_n \rangle$ is said to be monotone iff either (1) $x_n \leq x_{n+1}$ for all $n = 1, 2, \ldots$ or (2) $x_n \geq x_{n+1}$ for all $n = 1, 2, \ldots$.

[22] Note: the difference between a metric and a semi–metric is that for the latter $d(x,y)$ may be zero for $x \neq y$.

► [10, Theorem 2.20] Every bounded monotone sequence converges.

Separating hyperplane theorem: ► [13, p. 12, FN 12] The separating hyperplane theorem states that two compact, convex sets which intersect at a unique point have a common tangent through that point.

Series, bounded: ► [10, Def. 2.41] A series $\sum_{n=1}^{\infty} a_n$ is bounded iff the sequence of partial sums $\langle s_n \rangle$ is bounded.

Series, geometric: ► [10, p. 84] Let $a_n = \alpha^n$. Then $\sum_{n=1}^{\infty} \alpha^n$ is the geometric series. ► [10, p. 84] Note: (1) For $|\alpha| \geq 1$ the series does not converge as $\lim_{n\to\infty} \alpha^n \neq 0$. (2) If $|\alpha| < 1$ then $\sum_{n=0}^{\infty} \alpha^n = 1/(1-\alpha)$ and $\sum_{m=1}^{\infty} \alpha^m = \alpha/(1-\alpha)$.

Series, harmonic: ► [10, p. 85] $\sum_{n=1}^{\infty} 1/n$ is the harmonic series.

Series, infinite: ► [10, Def. 2.35] If $\langle a_n \rangle$ is a sequence, the expression $\sum_{n=1}^{\infty} a_n$ is called an infinite series. ► [10, Theorem 2.36] The infinite series $\sum_{n=1}^{\infty} a_n$ converges iff for each $\epsilon > 0$ there exists a $n \in \mathbb{N}$ such that for all $p \in \mathbb{N}$ $|S_{n,p}| < \epsilon$, with $S_{n,p} = \sum_{m=n+1}^{n+p} a_m = s_{n+p} - sn$. ► [10, p. 83] Note: If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n \to 0$. ► [10, Theorem 2.38] Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent. If we form new series by letting (1) $c_n = a_n + b_n$ for $n = 1, 2, \ldots$, then $\sum_{n=1}^{\infty} c_n$ also converges and we have $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$. (2) $d_n = \alpha a_n$ for $n = 1, 2, \ldots$, where α is any real number, then $\sum_{n=1}^{\infty} d_n$ also converges and $\sum_{n=1}^{\infty} d_n = \alpha \sum_{n=1}^{\infty} a_n$. ► [10, Def. 2.39] If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent*. If $\sum_{n=1}^{\infty} a_n$ is convergent, but $\sum_{n=1}^{\infty} |a_n|$ is not, the series is *conditionally convergent*. ► [10, Prop. 2.40] If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent. ► *see also Comparison test, D'Alembert's ratio test.*

Series, non–negative: ► [10, Def. 2.41] A series $\sum_{n=1}^{\infty} a_n$, is non–negative iff $a_n \geq 0$, $n = 1, 2, \ldots$. ► [10, Prop. 2.42] A non–negative series $\sum_{n=1}^{\infty} a_n$ is convergent iff it is bounded. ► *see also Comparison test, D'Alembert's ratio text*

set, bimonotone: ► [16, p. 20] Let X_1, X_2 be posets, $S \subseteq X_1 \times X_2$. S is bimonotone if either $[x_1, \infty) \times (-\infty, x_2] \in S$ or $(-\infty, x_1] \times [x_2, \infty) \in S$ for each $(x_1, x_2) \in S$. Hence, S is bimonotone if it equals one of its bimonotone hulls. ► [16, p. 20] The bimonotone hulls $H_1(S), H_2(S)$ are bimonotone sets. ► [16, p. 20] Any bimonotone set containing S must also contain at least one of the two bimonotone hulls. ► [16, p. 20] If X_1, X_2 are chains, then any bimonotone subset of $X_1 \times X_2$ and hence any bimonotone hull of any subset of $X_1 \times X_2$ is a sublattice of $X_1 \times x_2$.[23] ► [16, p. 20] If $X_1, \ldots X_n$ are posets, S is a bivariate subset of $\times_{i=1}^{n} X_i$, a subset L of $X_{i'} \times X_{i''}$ for $i' \neq i''$ is the $i'i''$–generator of S, and L is bimonotone, then S is bimonotone.

Set, bounded: ► *see also Supremum and Infimum* ► [10, Def. 1.50] A set $A \in \mathbb{R}^n$ is bounded iff there exists a positive real number α satisfying $(\forall \mathbf{x}, \mathbf{y} \in A) : d(x, y) < \alpha$. ► [10, Prop. 1.52] A subset A of $\mathbb{R}^n$ is bounded iff there exists an $\epsilon \in \mathbb{R}_{++}$ such that $A \subseteq N(0, \epsilon)$.

Set, closed: ► [10, Def. 1.61] A subset of $\mathbb{R}^n$ is said to be closed iff its complement is open. ► [10, Theorem 1.62] *Properties*: In $\mathbb{R}^n$ (1) $\emptyset$ and $\mathbb{R}^n$ are closed sets, (2) if C_i is a closed subset of $\mathbb{R}^n$, $i = 1, \ldots, m$, then $C := \cup_{i=1}^{m} C_i$ is a closed set, (3) if $\{D_a | a \in A\}$ is a family of closed subsets of $\mathbb{R}^n$, then $D := \cap_{a \in A} D_a$ is a closed set. ► [10, Theorem 1.66] A subset of A o $\mathbb{R}^n$ is closed iff $A = \bar{A}$, i.e it is closed if it is equal to its own closure.

Set, closed relative: ► [10, Def. 1.74] Let X be a subset of $\mathbb{R}^n$. A subset C of X is closed relative to X iff $A \equiv X \backslash C$ is open relative to X. ► [10, Theorem 1.75] *Properties*: If X is a subset of $\mathbb{R}^n$ then (1) $\emptyset$ and X are closed relative to X, (2) if $C_i \subseteq X$ is closed relative to X for $i = 1, \ldots, m$ then $C := \cup_{i=1}^{m} C_i$ is closed relative to X, (3) if $C_\alpha \subseteq X$ is closed relative to X for each $\alpha \in A$, then $C := \cap_{\alpha \in A}$ is closed relative to X. ► [10, Prop 1.76] Let $A \subseteq X \subseteq \mathbb{R}^n$. Then A is closed relative to X if and only if there exists a closed set C (closed in $\mathbb{R}^n$) such that $A = X \cap C$. ► [10, Theorem 1.77] If $A \subseteq X \subseteq \mathbb{R}^n$, then A is closed relative to X if and only if $A = \bar{A} \cap X$.

Set, compact: ► [2, p. 3] A set $K \subset X$ is compact if every open cover of K has a finite sub–cover, i.e. whenever $K \subset \cup_{\alpha \in J} O_\alpha$, where J is an arbitrary index set and O_α is open for each $\alpha \in J$, then $K \subset \cup_{i=1}^{n} O_{\alpha_i}$ for some $\alpha_1, \ldots, \alpha_n \in J$. ► [12, ch. 3, p. 15] If $f : X \to Y$ is a continuous function, then $f(S)$ is compact in Y whenever S is compact in X. ► [2, Theorem 11.3] Let K be a compact set: (1) Every infinite sequence in K has a cluster point. (2) If $f : K \to \mathbb{R}$ is continuous then f attains its supremum. (3) If A is a closed subset of K then A is compact.

Set, complete: ► [4] A set X is complete if $(\forall A \subset$

[23] A bimonotone subset of a direct product need not to be a sublattice of $X_1 \times X_2$.

$X)$: $\inf A, \sup A \in X$.

Set, increasing: ▶ [16, p. 15] Let $(X, \preceq)$ be a poset, $X' \subset X$, and $X \cap [x, \infty) \subset X'$ for each $x \in X'$. Then, X' is an increasing set. ▶ [16, p. 15] Let $(X, \preceq)$ be a poset. A set $X' \subset X$ is an increasing set if the indicator function $f : X' \cap [x, \infty) \to \{0, 1\}$ is an increasing function, i.e. if $x, x' \in X' \subset X$ with $x \preceq x'$ implies $f(x) \preceq f(x')$.

Set, level: ▶ [16, p. 15] Let X be a set, $(Y, \preceq)$ be a poset and $f : X \to Y$. Then, the level sets of $f(x)$ on X are the sets $\{x : x \in X, y \preceq f(x)\}$ for $y \in Y$. ▶ [16, p. 23] Each level set of a bimonotone function is bimonotone.

Set, lower than: ▶ [15] Suppose a lattice S with a relation $\leq$ is given. For X and Y in the power set $P(S)$, X is lower than Y ($X \leq^p Y$) if $x \in X$ and $y \in Y$ imply that $x \wedge y \in X$ and $x \vee y \in Y$. ▶ [15] For a lattice S, the relation $\leq^p$ is antisymmetric and transitive in $P(S)$.

Set of bounded functions: ▶ [12, ch. 2, p. 4] Let T be a non–empty set. The set of bounded real functions is defined by $B(T) := \{f \in \mathbb{R}^T : \sup\{|f(t)| : t \in T\} < \infty\}$. The usual metric of this space is the sup–metric d_∞.

Set of continuous functions: ▶ [12, ch. 3, p. 39] Let T be a metric space. If T is compact, then $C(T)$ is separable.

Set of continuous bounded functions: ▶ [12, ch. 3, p. 32] For any metric space, T the set of continuous bounded functions, $CB(T)$ is a closed and complete metric subspace of the set of bounded functions $B(T)$.

Set of indices, increasable: ▶ [16, p. 27] Suppose $X_1, \ldots, X_n$ are chains, S is a sublattice of $\times_{i=1}^n X_i$. A subset I of $\{1, \ldots, n\}$ is an increasable set of indices for $x' \in S$ if there exists $x'' \in S$ with $x' \preceq x''$ and with $x'_i \prec x''_i$ if and only iff $i \in I$.

Set, open: ▶ [10, Def. 1.58] A subset X of $\mathbb{R}^n$ is open if int$X = X$. ▶ [10, Prop. 1.59] *Properties*: in $\mathbb{R}^n$ (1) $\emptyset$ and $\mathbb{R}^n$ are open sets, (2) if $U = \{U_a | a \in A\}$ is a family of subsets of $\mathbb{R}^n$ satisfying $(\forall a \in A) : U_a$ is an open subset of $\mathbb{R}^n$ then $U \equiv \cup_{a \in A} U_a$ is an open subset, (3) if V_i is an open subset of $\mathbb{R}^n$ for $i = 1, \ldots, n$ then $V := \cap_{i=1}^n V_i$ is an open subset of $\mathbb{R}^n$.

Set, open relative: ▶ [10, Def. 1.71] Let X be a non–empty subset of $\mathbb{R}^n$. A subset A of X is open relative to X iff there exists an open subset of $\mathbb{R}^n$, U, such that $A = X \cap U$. ▶ [10, Theorem 1.72] Let X be a subset of $\mathbb{R}^n$ and let A be a subset of X. Then A is open relative to X iff for each $\mathbf{x} \in A$, there exists $\epsilon \in \mathbb{R}_{++}$ such that $N(\mathbf{x}, \epsilon) \cap X \subseteq A$. ▶ [10, Theorem 1.73] *Properties*: If X is a subset of $\mathbb{R}^n$ then (1) $\emptyset$ and X are open relative to X, (2) if $A_i \subseteq X$ is open relative to X for $i = 1, \ldots, m$ then $A := \cap_{i=1}^m A_i$ is open relative to X, (3) if $U_\beta \subseteq X$ is open relative to X for each $\beta \in B$, then $U := \cup_{\beta \in B} B_\beta$ is open relative to X.

Set ordering, induced $\sqsubseteq$: ▶ [16, p. 32] Suppose that X is a lattice with ordering relation $\preceq$. The induced set ordering, $\sqsubseteq$, is defined on a collection of non–empty members of the power set $\mathcal{P}(X) \setminus \{\emptyset\}$ such that $X' \sqsubseteq X'' \in \mathcal{P}(X) \setminus \{\emptyset\}$ if $x' \in X'$ and $x'' \in X''$ imply that $x' \wedge x'' \in X'$ and $x' \vee x'' \in X''$. ▶ [16, p. 32, L. 2.4.1.] If X is a lattice, then the binary relation $\sqsubseteq$ is anti–symmetric and transitive on $\mathcal{P}(X) \setminus \{\emptyset\}$.[24] ▶ [16, p. 33] $\mathcal{L}(X)$ is the greatest subset of $\mathcal{P}(X) \setminus \{\emptyset\}$ on which $\sqsubseteq$ is reflexive. ▶ [16, p. 33, T. 2.4.1.] If X is a lattice, then $\mathcal{L}(X)$ is a poset with the ordering relation $\sqsubseteq$. Furthermore, any subset of $\mathcal{P}(X) \setminus \{\emptyset\}$ that is a poset with ordering relation $\sqsubseteq$ is a subset of $\mathcal{L}(X)$.

Set, partially ordered ▶ [15] $(X, \leq)$ is a set X on which there is a binary relation $\leq$ that it is reflexive, antisymmetric and transitive. ▶ [15] If S_α is a partially ordered set with relation $\leq_\alpha$ for each $\alpha \in A$, then the direct product of these partially ordered sets is the partially ordered set consisting of the set $\times_{\alpha \in A} S_\alpha = \{x = (x_\alpha) : x_\alpha \in S_\alpha, \forall \alpha \in A\}$ with the relation $\leq$ where $x \leq y$ if $x_\alpha \leq_\alpha y_\alpha$ for each $\alpha \in A$. (note: relation is defined componentwise). ▶ [15] If S is a lattice with relation $\leq$, then the set of non–empty sublattices $L(S)$ is a partially ordered set with the relation $\leq^p$. ▶ *see also Dual.* ▶ *see also Cover* ▶ *see also Elements, ordered pair of* ▶ *see also Elements, unordered pair of*

Set, subcomplete: ▶ [4] A non–empty set $A \subset X$ is subcomplete if $B \subset A$, $B \neq \emptyset$ implies $\inf_X B, \sup_X B \in A$is complete if $(\forall A \subseteq X)$: $\inf A, \sup A \in X$. $\inf_X B, \sup_X B$ means that the supremum and infimum is taken as a subset of X.

Set–theoretic operations: ▶ [10, Prop 1.2] If A, B, C are subsets of S then (1) $A \subseteq B \Leftrightarrow B^C \subseteq A^C$, (2) $\emptyset^C = S$, $S^C = \emptyset$, (3) $A \cup A^C = S$, $A \cap A^C = \emptyset$, (4) $(A^C)^C = A$, (5) $A \setminus B = A \cap B^C$, (6) $A \cap B = \emptyset \Leftrightarrow A \subseteq B^C \Leftrightarrow B \subseteq A^C$, (7) $A \cap B \subseteq A$ and $A \cap B \subseteq B$, (8) $A \cap B = A \Leftrightarrow A \subseteq B$, (9) $A \subseteq A \cup B$ and $B \subseteq A \cup B$, (10) $A \cup B = A \Leftrightarrow B \subseteq A$, (11) $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$, (12) $A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$, (13) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, (14) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, (15) $A = (A \cap B) \cup (A \cap B^C) = (A \cap B) \cup (A \setminus B)$. ▶ [10, Prop. 1.6] Or instead of (11)–

[24] Both need not hold if $\emptyset$ is included.

(13): (11') $A \cap (\cap_{\lambda\in\Lambda} B_\lambda) = \cap_{\lambda\in\Lambda}(A \cap B_\lambda)$ and $A \cup (\cup_{\lambda\in\Lambda} B_\lambda) = \cup_{\lambda\in\Lambda}(A \cup B_\lambda)$, (12') $A \cap (\cup_{\lambda\in\Lambda} B_\lambda) = \cup_{\lambda\in\Lambda}(A \cap B_\lambda)$, (13') $A \cup (\cap_{\lambda\in\Lambda} B_\lambda) = \cap_{\lambda\in\Lambda}(A \cup B_\lambda)$.

Single Crossing Property: ► [17, p. 29] Let X be a lattice, T a poset and $g : X \times T \to \mathbb{R}$. Then g satisfies the single crossing property if for $x' > x$ and $t' > t$, $g(x', t) \geq g(x, t)$ implies that $g(x', t') \geq g(x, t')$, and $g(x', t) > g(x, t)$ implies that $g(x', t') > g(x, t')$ ► [17, p. 29] Let $S \subset X$. $\phi(t, S) \equiv \arg\max_{x\in S} g(x, t)$ is increasing in (t, S) iff $g : X \times T \to \mathbb{R}$ is quasi–supermodular in x and satisfies the single crossing property in (x, t). If S is a sublattice of X then $\phi(t, S)$ is a sublattice of S. In in addition the constraint $S(t)$ is increasing in t and g satisfies the strict single crossing property in (x, t), then $\phi(t, S(t))$ is strongly increasing in t.

Solution, increasing (decreasing) optimal: ► [16, p. 33] A collection of optimisation problems maximise $f(x, t)$ subject to $x \in S_t$, with each problem determined by a parameter t in a poset T and with each constraint set S_t contained in a lattice X has increasing (decreasing) optimal solutions if the correspondence $\text{argmax}_{x\in S_t} f(x, t)$ is increasing (decreasing) as a function of t from $\{t : t \in T, \text{argmax}_{x\in S_t} f(x, t) \neq \emptyset\}$ into $\mathcal{L}(X)$ with the induced ordering set $\sqsubseteq$.

Space, measurable: ► [2, p. 9] A pair $(\Omega, \mathfrak{F})$, where Ω is a set and $\mathfrak{F}$ is a σ–field of subsets of Ω, is called a measurable space.

Space, probability: ► [2, p. 10] The triple $(\Omega, \mathfrak{F}, P)$, where Ω is a set, $\mathfrak{F}$ is a σ–field of subsets of Ω and P a probability measure on $(\Omega, \mathfrak{F})$ is called a probability space.

Space, topological: ► [2, p. 2] A topological space is a set X together with a collection $\mathcal{T}$ of subsets called open sets that have the following properties: (1) $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$; (2) If $A, B \in \mathcal{T}$ then $A \cap B \in \mathcal{T}$; (3) If $A_\alpha \in \mathcal{T}$ for all $\alpha \in J$, where J is an arbitrary index set, then $\cup_{\alpha\in J} A_\alpha \in \mathcal{T}$.

Stone–Geary technology: ► *see Production function, Stone–Geary*

Stone–Weierstrass Theorem: ► [12, ch. 3, p. 38] Let T be a compact metric space, and let $\mathcal{P}$ be a set in $C(T)$ with the following properties: (i) $\alpha f + \beta g \in \mathcal{P}$ for all $f, g \in \mathcal{P}$ and $\alpha, \beta \in \mathbb{R}$; (ii) $fg \in \mathcal{P}$ for all $f, g \in \mathcal{P}$; (iii) all constant functions belong to $\mathcal{P}$; (iv) for any distinct $x, y \in T$, there exists an $f \in \mathcal{P}$ such that $f(x) \neq g(y)$. Then $\mathcal{P}$ must be in $C(T)$.[25]

Stochastic process: ► *see Process, stochastic*.

Strategy set, rationalisable: ► [8] The sets of strategies $\bar{S}_m \subset S_m$, $m = 1, \ldots, N$ are rationalisable if for each n and $x_n \in \bar{S}_n$, x_n maximises $E[f_n(\cdot, x_{-n})]$ for some probability distribution on s_{-n} with support in $\bar{S}_{-n}$.

Strategy set of undominated responses: ► [8] Given a product set $\hat{S}$ the set of n's undominated responses to $\hat{S}$ is defined by $U(\hat{S}) = \{x_n \in S_n | (\forall x'_n \in S_n)(\exists \hat{x} \in \hat{S}) : f_n(x_n, \hat{x}_{-n}) \geq f_n(x'_n, \hat{x}_{-n})\}$.

Strategy, strongly dominated: ► [8] A pure strategy x_n for player n is strongly dominated by another pure strategy $\hat{x}_n$ if it is the case that for all x_{-n}, $f_n(x_n, x_{-n}) < f_n(\hat{x}_n, x_{-n})$.

Subadditivity: ► [10, Def. 5.8.1] If $f : K \to \mathbb{R}$, where K is a convex cone, then f is sub–additive iff, for all $x, y \in K$ $f(x + y) \leq f(x) + f(y)$. ► [10, Prop. 5.82] Suppose $f : K \to \mathbb{R}$ is positively homogeneous of degree one on K, where K is a convex cone. Then f is concave (convex) on K if, and only if, f is super- (sub-) additive on K.

Sublattice: ► [4] $A \subset X$ is a sublattice if $(\forall x, y \in A) : x \wedge_X y, x \vee_X y \in A$, where $x \wedge_X y, x \vee_X y$ are obtained taking the infimum and supremum as elements of X as opposed to using the relative order on A. ► [14] $S \subset \mathbb{R}^2$ is a sublattice of $\mathbb{R}^2$ if $(x'_1, x'_2) \in S$ and $(x''_1, x''_2) \in S \Rightarrow (\max\{x'_1, x''_1\}, \max\{x'_2, x''_2\}) \in S$ and $(\min\{x'_1, x''_1\}, \min\{x'_2, x''_2\}) \in S$. ► [15] If T is a subset of a lattice S and T contains the join and meet (with respect to S) of each pair of elements of T then T is a sublattice of S. ► [16, p. 13] If X' is a sublattice of the lattice X, then X' is itself a lattice and in X' the join and meet of any two elements are the same as the join and meet of those same two elements in X. ► [16, p. 17, L. 2.2.2] If X is a lattice and X_a is a sublattice of X for each $a \in A$, then $\cap_{a\in A} X_a$ is a sublattice of X.[26] ► [16, p. 17, L. 2.2.3] Suppose that X, T are lattices and S is a sublattice of $X \times T$. (a) the section S_t of S at each $t \in T$ is a sublattice of X (b) the projection $\Pi_T S$ of S on T is a sublattice of T. ► [16, p. 18] If $X_1, \ldots, X_n$ are non–empty lattices, S is a bivariate subset of $\times_{i=1}^n X_i$ and L the $i'i''$–generator of S then S is a sublattice of $\times_{i=1}^n X_i$ iff L is a sublattice of $X_{i'} \times X_{i''}$. ► [16, p. 18, L. 2.2.4] If $n \geq 2$ and $X_1, \ldots, X_n$ are lattices, then a set is a sublattice of $\times_{i=1}^n X_i$ iff it is the intersection of $n(n-1)/2$ bivariate sublattices of $\times_{i=1}^n X_i$. ► [16, p. 20] If X_1, x_2 are lattices and S is a sublattice of $X_1 \times X_2$, then the bimonotone hulls $H_1(S), H_2(S)$ are sublattices. ► [16, p. 20] If X_1, X_2

[25] This is brick for separability of $C(T)$.

[26] Hence, the intersection of sublattices are sublattices.

are chains, then any bimonotone subset of $X_1 \times X_2$ and hence any bimonotone hull of any subset of $X_1 \times X_2$ is a sublattice of $X_1 \times x_2$. ▶ [16, p. 20, T. 2.2.1.] If $X_1, \ldots, X_n$ are lattices, then a set S is a sublattice of $\times_{i=1}^{n} X_i$ iff it is the intersection of $n(n-1)$ bimonotone sublattices of $\times_{i=1}^{n} X_i$ together with the direct product of the projection of S on each X_i where the projection of S on each X_i is a sublattice of X_i.

Sublattice, collection of non–empty $\mathcal{L}(X)$: ▶ [16, p. 32] Suppose X is a lattice. Then $\mathcal{L}(X)$ is the collection of all non–empty sublattices of X. ▶ [16, p. 33] $\mathcal{L}(X)$ is the greatest subset of $\mathcal{P}(X) \setminus \{\emptyset\}$ on which $\sqsubseteq$ is reflexive. ▶ [16, p. 33, T. 2.4.1.] If X is a lattice, then $\mathcal{L}(X)$ is a poset with the ordering relation $\sqsubseteq$. Furthermore, any subset of $\mathcal{P}(X) \setminus \{\emptyset\}$ that is a poset with ordering relation $\sqsubseteq$ is a subset of $\mathcal{L}(X)$.

Sublattice, subcomplete:[27] ▶ [14] X is a subcomplete sublattice of S if for each non–empty subset $X' \subseteq X$ the least upper bound and the greatest lower bound exist and are contained in X. ▶ [16, p. 29] If X' is a sublattice of a lattice X and if, for each non–empty subset X'' of X', $\sup_X X''$ and $\inf_X X''$ exist and are contained in X', then X' is a subcomplete sublattice of X. ▶ [p. 29]topkis98b By L. 2.2.1, any finite sublattice of a lattice is subcomplete. Hence, any sublattice of a finite lattice of X is subcomplete. ▶ [16, T. 2.3.1] A sublattice of $\mathbb{R}^n$ is subcomplete iff it is compact.

Submodularity: ▶ [14] If $-f(x)$ is supermodular then $f(x)$ is submodular. ▶ [15] If S_i is a lattice for $i = 1, \ldots, n$, S is a sublattice of $\times_{i=1}^{n} S_i$ and f is (strictly) submodular on S, then f has (strictly) antitone differences on S. ▶ [15] If S_i is a chain for $i = 1, \ldots, n$ and f has (strictly) antitone differences on $\times_{i=1}^{n} S_i$, then f is (strictly) submodular on $\times_{i=1}^{n} S_i$. (note, this result is not valid for the product of a countable collection of chains!). ▶ [15] *Implications*: Let u^i be the ith unit vector in E^n. A function f is submodular iff (1) $f(x + \epsilon u^i) - f(x)$ is antitone in x_j $\forall i \neq j$, $\epsilon \in \mathbb{R}_{++}$ and x or iff (2) $\partial f(x)/\partial x_i$ is antitone in x_i, $\forall i \neq j$ and x or iff (3) $\partial^2 f(x)/\partial x_i \partial x_j \leq 0$ $\forall i \neq j$ and x. ▶ [15, Lemma 3.1] If S is a lattice and $g_i(x)$ is non–positive, isotone (antitone), and submodular on S for $i = 1, \ldots, k$, then $f(x) = (-1)^{k-1} g_1(x) g_2(x) \ldots g_k(x)$ is also non–positive, isotone (antitone), and submodular on S. ▶ [15, Theorem 4.1] If f is submodular on a lattice S, then the set S^* of points at which f attains its minimum on S is a sublattice of S. ▶ [15, Theorem 4.2] If f is strictly submodular on a lattice S, then the set S^* of points at which f attains its minimum on S in a chain. ▶ [15, Theorem 4.3] If X, T are lattices, S is a sublattice on $X \times T$, f is submodular on S, S_t is the section of S at $t \in T$, and $g(t) = \inf_{x \in S_t} f(x,t)$ is finite on the projection $\Pi_t S$ then $g(t)$ is submodular on $\Pi_t S$.

Subsequence ▶ [10, Def. 2.21] If $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences, $\langle y_n \rangle$ is said to be a subsequence of $\langle x_n \rangle$ iff there exists a sequence $\langle n_k \rangle$ of positive integers satisfying: (1) $(\forall i, j \in \mathbb{N}) : i > j \Rightarrow n_i > n_j$ and (2) $(\forall k \in \mathbb{N}) : y_i = x_{n_i}$. ▶ [10, Prop. 2.23] If $\langle x_n \rangle$ is a sequence such that $x_n \to x^*$, and $\langle x_{n_i} \rangle$ is any subsequence of $\langle x_n \rangle$, then $x_{n_i} \to x^*$.

subset, bivariate: ▶ [16, p. 18] Suppose that $X_1, \ldots, X_n$ are sets and $x = (x_1, \ldots, x_n) \in \times_{i=1}^{n} X_i$ with $x_i \in X_i$. If L is a subset of $X_{i'} \times X_{i''}$ for $i' \neq i''$ and $S = \{x : x \in \times_{i=1}^{n} X_i, (x_{i'}, x_{i''}) \in L\}$. Then S is the bivariate subset of $\times_{i=1}^{n}$. ▶ *see also* $i'i''$*–Generator* ▶ [16, p. 20] If $X_1, \ldots X_n$ are posets, S is a bivariate subset of $\times_{i=1}^{n} X_i$, a subset L of $X_{i'} \times X_{i''}$ for $i' \neq i''$ is the $i'i''$–generator of S, and L is bimonotone, then S is bimonotone.

Supermodularity: ▶ [1] $\Pi^1(p_1, p_2) = (p_1 - c_1) D^1(p_1, p_2)$ is supermodular in (p_1, p_2) if $\Delta^C \triangleq D^1_{p_2} + (p_1 - c_1) D^1_{p_1 p_2} \leq 0$ for all $(p_1, p_2) \in P_1 \times P_2$. ▶ [14] A real valued function $f(x)$ on a lattice X is called supermodular on X if $(\forall x', x'' \in X) : f(x') + f(x'') \leq f(x' \wedge x'') + f(x' \vee x'')$. ▶ [7] *Equivalent representation*: A function $f : \mathbb{R}^n \to \mathbb{R}$ is supermodular if for all $x, x' \in \mathbb{R}^n$, $f(x) + f(x') \leq f(\min\{x, x'\}) + f(\max\{x, x'\})$. This inequality is equivalent to $[f(x) - f(\min\{x, x'\})] + [f(x') - f(\min\{x, x'\})] \leq f(\min\{x, x'\}) + f(\max\{x, x'\})$, i.e. the sum of the changes in the function when several arguments are increased separately is less than the change resulting from increasing all the arguments together; or $f(\max\{x, x'\}) - f(x') \geq f(x) - f(\min\{x, x'\})$, i.e. increasing one or more variables raises the return to increasing the other variables. ▶ [8] *Topkis's Characterisation Theorem*: Let $I = [\underline{\mathbf{x}}, \bar{\mathbf{x}}]$ be an interval in $\mathbb{R}^n$. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable on some open set containing I. Then, f is supermodular iff for all $\mathbf{x} \in I$ and all $i \neq j$, $\partial^2 f / \partial x_i \partial x_j \geq 0$. ▶ [8] If f is an order upper semi–continuous, supermodular function from a complete lattice S to $\mathbb{R} \cup \{-\infty\}$, then f has a maximum in S. ▶ [8] If f is an order upper semi–continuous, supermodular function from a complete lattice S to $\mathbb{R} \cup \{-\infty\}$, then the set of maximisers of f is a complete sublat-

[27] [17, p. 18]: Note that a lattice need not to be a sublattice of a larger lattice.

tice of S. ► [17, Remark 6] If g is (strictly) supermodular on $X \times T$, then it has (strictly) increasing differences on $X \times T$ and g is (strictly) supermodular on X for any $t \in T$. ► [17, Theorem 2.3] Let $g : X \times T \to \mathbb{R}$ be supermodular on a lattice X for each t in the poset T. Let S be a sublattice of X and $\phi(t) = \arg\max_{x \in S}\{g(x,t)\}$. (i) Then $\phi(t)$ is a sublattice of S for all t. (ii) If g has increasing (decreasing) differences in (x,t), then, ϕ is increasing (decreasing). (iii) If g is strictly supermodular on X for each $t \in T$, then $\phi(t)$ is ordered for all t. (iv) If g has strictly increasing differences in (x,t), then ϕ is strongly increasing. ► [17, p. 28] Sufficient condition for $g(x,t)$ to be supermodular on $X \times T$: The off–diagonal elements of H_x, the Jacobian of $\nabla_x g$ and $\nabla_{xt} g$ are nonnegative. Note: The off–diagonal elements of H_x are nonnegative iff $g(x,t)$ is supermodular on X for a given $t \in T$. ► [17, Theorem 2.4] *Existence of a solution*: Let $g : X \times T \to \mathbb{R}$ be supermodular on S, a compact sublattice of the lattice X for each $t \in T$. If g is upper semi–continuous on X for any t, then $\phi(t) \equiv \arg\max g$ is a non–empty compact sublattice of S for all t.

Supremum of A: ► [10, Def. 1.10] $\alpha \in \mathbb{R}$ is a supremum of $A \subset \mathbb{R}$, $\sup A$, iff (1) α is an upper bound for A and (2) if β is an upper bound for A, then $\alpha \leq \beta$. ► [10, p. 12] Any non–empty set of real numbers that is bounded above has a supremum. ► [10, Theorem 1.14] $\alpha = \sup A$ iff for every positive real number ϵ: (1) $(\forall x \in A) : x < \alpha + \epsilon$ and (2) $(\exists x' \in A) : x' > \alpha - \epsilon$. ► [10, Prop. 1.15] Suppose $A, B \subset \mathbb{R}$ and non–empty satisfying $(\forall a \in A)(\forall b \in B) : a \geq b$ then $\inf A$ and $\sup B$ both exist and $\inf A \geq \sup B$. ► [10, Prop. 1.17] Let $\lambda A = \{x \in \mathbb{R} | (\exists x' \in A) : x = \lambda x'\}$. If $A \subset \mathbb{R}$ and non–empty, which is bounded above, and $\lambda \in \mathbb{R}_+$, then $\sup(\lambda A)$ exists and $\sup(\lambda A) = \lambda \sup A$. ► [10, Prop. 1.18] Let $A + B = \{x \in \mathbb{R} | (\exists a \in A, b \in B) : x = a + b\}$. If $A, B \subset \mathbb{R}$ and non–empty which are bounded from above, then $A + B$ is bounded above and $\sup(A + B) = \sup A + \sup B$. ► [15] If S is a complete lattice X, Y are in $P(S)$ and $X \leq^p Y$, then $\inf X \leq \inf Y$ and $\sup X \leq \sup Y$.

T

Tarski's Fixed Point Theorem: ► *see Fixed Point Theorem*

Tatônnement, simultaneous discrete: ► [17, p. 51] A simultaneous discrete tatônnement with a^0 as initial point, $a^0 \in A$, is a strategy $\{a^t\}$, $t = 0, 1, \ldots$, such that $a^t = r(a^{t-1})$.

Tietze's Extension Theorem: ► [12, ch. 3, p. 38] Let T be a non–empty closed set in a metric space X. Any function $f \in CB(T)$ can be extended to a function $F \in CB(X)$, that is, there exists a continuous and bounded function $F \in \mathbb{R}^X$ such that $F_{|_T} = f$.

Topkis's Monotonicity Theorem: ► *see Monotonicity Theorem*

Topological Space: ► *see Space, topological*

Topology, metric: ► [2, p. 4] The metric topology on X is defined as follows: a set O is open if and only if $x \in O$ implies $B_\epsilon(x) \subset O$ for some $\epsilon > 0$, where $B_\epsilon(x)$ is the ϵ–ball.

Topology, usual: ► [2, p. 2] The usual topology in $\mathbb{R}$ is defined as follows: a set A is open if and only if for every $x \in A$ there exists a $\epsilon > 0$ such that $]x-\epsilon, x+\epsilon[\subset A$.

U

upper semi–continuous: see also *Function, upper semi–continuous* and *binary relation, upper semi–continuous*

Urysohn's Lemma: ► [12, ch. 3, p. 40] Let X be any metric space and let A and B be two non–empty closed sets in X with $A \cap B = \emptyset$. For any $-\infty < a < b < \infty$, there exists a continuous function $g : X \to [a, b]$ such that $g(x) = a$ for all $x \in A$ and $g(x) = b$ for all $x \in B$.

Utility function, properties: ► [17, Theorem 3.1] Consider a utility function in the class U^n which satisfies the regularity conditions, the uniform Inada property, and the curvature property. Let prices for any good be in a compact and positive interval. (i) The demand for any good and the marginal utility of income are uniformly bounded above and away from zero.[28] (ii) The order of magnitude of the (Euclidean) norm of the income derivative of demand of any good is $1/\sqrt{n}$. If preferences are representable by additive sparable or homothetic utility functions, then the order of magnitude of the income derivative of demand of any good is $1/n$.[29] (iii) The associated Slutsky matrix is non–degenerate.[30]

Utility function, regularity conditions: ► [17, p. 80] For any number of goods n the assumed properties will

[28] Implies that the expenditure share on any good is small (and positive)

[29] Implies that the income effects are small.

[30] Implies that the substitution effects are non–degenerate.

define a class of utility functions U^n. *Regularity condition*: Let $U : \mathbb{R}^n_{++} \rightarrow \mathbb{R}$ be smooth (twice continuously differentiable) with strict positive gradient, $\nabla U(q) \gg 0$, and a negative definite Hessian, $H_u(q)$, for all $q \in \mathbb{R}^n_{++}$. *Uniform Inada Condition*: There exist decreasing positive functions $\underline{\phi} : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ and $\bar{\phi} : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$, with $\underline{\phi} \leq \bar{\phi}$, $\underline{\phi}(z) \longrightarrow_{z \to 0} \infty$ and $\bar{\phi}(z) \longrightarrow_{z \to \infty} 0$, such that $\underline{\phi}(q_i) \leq \partial U(q)/\partial q_i \leq \bar{\phi}(q_i)$ for all $q \in \mathbb{R}^n_{++}$ and for all i.[31] *Curvature property*: The absolute values of the eigenvalues of H_u are bounded above and away from zero (uniformly in n) provided that the consumption of each good lies in a compact set bounded away from zero.[32]

V

Valuation: ► [15] A function that is both submodular and supermodular is a valuation.

W

Weierstrass' Theorem: ► [12, ch. 3, p. 16] If X is a compact metric space and $f : X \rightarrow \mathbb{R}$ is a continuous function, then there exists $x, y \in X$ with $f(x) = \sup f(X)$ and $f(y) = \inf f(X)$.[33] ► *Generalisation* [12, ch. 3 p. 18] Let X be a compact metric space, and let $f : X \rightarrow \mathbb{R}$ be any function. If f is upper semi–continuous, then there exists an $x \in X$ with $f(x) = \sup f(X)$. If f is lower semi–continuous, then there exists a y with $f(y) = \inf f(X)$.

Well–Ordering Principle for Positive Integers: ► [10, p. 17] Every non–empty subset of $\mathbb{N}$ has a least element; that is if $A \subseteq \mathbb{N}$ is nonempty, then there exists $k^* \in A$ satisfying $(\forall n \in A) : k^* \leq n$.

[31] Implies that utility function is not too asymmetric and that there are no two goods which are close to perfect substitutes. Implies also, that the expenditure shares become small as the number of goods increase; the demands will be uniformly bounded above and away from zero provided that the prices are not of different orders and magnitude.

[32] Allows us to obtain bounds on the income derivative of demand in terms of the umber of goods; implies that the slopes of the Hicksian demand curves are non–degenerate.

[33] Is used to show that a maximum or a minimum of optimisation problem exists.

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