Finite element methods for parabolic problems with emphasis on the singularly perturbed case

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In the main part of the talk, we consider the linear parabolic advection-diffusion-reaction model

$$\partial_t u - \epsilon \nabla \cdot (a \nabla u) + (b \cdot \nabla) u + cu = f \quad \text{in} \;\; (0,T) \times \Omega \subset \mathbf{R} \times \mathbf{R}^d \tag{1}$$

with $\epsilon \in (0,1]$. The singularly perturbed case $0 < \epsilon \ll 1$ is often of special interest in applications. A plain FEM-semidiscretization in space leads to a very large stiff and dissipative ODE-system. A proper implicit discretization in time gives a large set of algebraic equations to be solved within each time step, see [1]. Typical requirements are stability of the time discretization (A-stability, eventually B-stability) , high accuracy in time and space, adaptive time step control (hopefully with adaptive control of the spatial mesh) and efficient solvability of the algebraic systems.
First we discuss these aspects for the standard θ-scheme as an example of a low-order scheme in time, see [2]. Then we consider higher-order schemes in time, in particular B-stable Runge-Kutta methods and discontinuous Galerkin methods in time [3]. From the viewpoint of adaptivity, we propose to consider the time discretization in an outer loop, see also [4]. In the singularly perturbed case, one is also interested in robustness of a-priori and a-posteriori estimates with respect to the parameter ϵ. We will discuss difficulties which appear if standard residual-based stabilization techniques (like streamline upwinding) are applied. Recent stabilization methods, e.g. the edge-stabilization or local projection schemes, avoid some of these problems.
In applications, the model problem (1) will appear only as an auxiliary problem. In the final part of the talk, we will briefly discuss a suitable approach to time-dependent, thermally-coupled, incompressible and turbulent flow problems [5].

References

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[4] J. Lang: Adaptive Multilevel Solution of Nonlinear Parabolic PDE systems, Springer 2001

[5] T. Knopp, G. Lube, R. Gritzki, M. Rösler: A near-wall strategy for buoyancy-affected turbulent flows using stabilized FEM with applications to indoor air flow simulation, Comput. Methods Appl. Mech. Engr. 194 (2005), 3797-3816