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On the preconditioning in the domain decomposition technique for the p-version finite element method. Part II

Abstract

P-version finite element method for the second order elliptic equation in an arbitrary sufficiently smooth domain is studied in the frame of *DD* method. Two types square reference elements are used with the products of the integrated Legendre's polynomials for the coordinate functions. There are considered the estimates for the condition numbers, preconditioning of the problems arising on subdomains and the Schur complement, the derivation of the *DD* preconditioner. For the result we obtain the *DD* preconditioner to which corresponds the generalized condition number of order $(\log p)^2$.

The paper consists of two parts. In part I there are given some preliminary results for $1D$ case, condition number estimates and some inequalities for $2D$ reference element.

Part II is devoted to the derivation of the Schur complement preconditioner and conditionality number estimates for the p-version finite element matrixes. Also *DD* preconditioning is considered.

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Introduction

In the part I there have been obtained condition number estimates for the stiffness and mass matrices of the reference elements. Also subsidiary inequalities for some quadratic forms specified on the square $\Pi = (-1,1) \times (-1,1)$ have been derived. In this part we describe the p–version finite element method for the Dirichlet second order elliptic boundary value problem in an arbitrary sufficiently smooth domain. Then we derive the Schur complement and *DD* preconditioner and the estimates for the corresponding generalized condition numbers.

The numeration of the sections is continued. The notations introduced in the part I are used without comments.

3 Preconditioning of the Schur complement for the square reference element

For the construction of the effective *DD*-methods it is necessary to have good preconditioner for the Schur complement to the finite element stiffness matrix obtained after eliminating of the internal unknowns. Derivation of the preconditioner may be modelled on the reference element and then easily expanded on the finite elements of the numerical method.

Let us present matrix A_1 in the form

$$A_1 = \begin{pmatrix} A^{(1)} & A^{(1)}_{III} \\ \left(A^{(1)}_{III}\right)^T & A_{III} \end{pmatrix}, \qquad A^{(1)} = \begin{pmatrix} A_I & A_{I,II} \\ A_{II,I} & A_{II} \end{pmatrix}, \tag{3.1}$$

where blocks A_I, A_{II}, A_{III} correspond to the internal, side, and vertex functions. Introducing matrix

$$A_{1,d} = \begin{pmatrix} A^{(1)} & 0 \\ 0 & A_{III} \end{pmatrix}$$

we rewrite (2.7) in the form

$$\frac{c}{1+\log p} A_{1,d} \le A_1 \le 2A_{1,d}. \tag{3.2}$$

Thus we shall have good preconditioner if we are able to obtain good preconditioner for the Schur complement to $A^{(2)}$ arising at the factorization

$$\begin{gathered} A^{(1)} = \begin{pmatrix} I_I & 0 \\ A_{II,I}A_I^{-1} & I_{II} \end{pmatrix} \begin{pmatrix} A_I & 0 \\ 0 & S_{II} \end{pmatrix} \begin{pmatrix} I_I & A_I^{-1}A_{I,II} \\ 0 & I_{II} \end{pmatrix}, \\ S_{II} = A_{II} - A_{I,II}A_I^{-1}A_{I,II} \ . \end{gathered} \tag{3.3}$$

We shall seek for the preconditoner of the form

$$\hat{S}_{II} = \begin{pmatrix} S^{(1)} & & & 0 \\ & S^{(2)} & & \\ & & S^{(3)} & \\ 0 & & & S^{(4)} \end{pmatrix}$$

every block of which corresponds to one of the sides γ_i. Since these blocks are identical, it is sufficient to define one. For this purpose let us consider polynomial $u \in \mathcal{P}_{p,x}$ on I, its representations

$$u(x) = \sum_{i=0}^{p} a_i \hat{L}_i\,,\ \ u(x) = \sum_{i=0}^{p} b_i x^i,\ \ u(\cos\phi) = \sum_{i=0}^{p} c_i \cos i\phi, \tag{3.4}$$

corresponding vectors of coefficients $\bar{a} = \{a_i\}$, $\bar{b} = \{b_i\}$, $\bar{c} = \{c_i\}$ and transforms

$$B\bar{a} = \bar{b}, \quad C\bar{b} = \bar{c}. \tag{3.5}$$

By Λ we denote matrix $\Lambda = \mathrm{diag}\,[1, 2, \ldots, p+1]$.

Writing down matrix

$$\hat{S} = B^T C^T \Lambda^{1/2} C B \tag{3.6}$$

and crossing out two rows and two columns corresponding to a_0, a_1 in (3.4) we define $S^{(i)}$. Let us note that matrices $\hat{S}_{II}$ are defined in the same way for the both matrices $A^{(1)}$ as corresponding to the reference element $\hat{\mathcal{E}}$ so the reference element $\hat{\mathcal{E}}_0$. It is worth to note also that matrices B, C are triangular.

Theorem 3.1 *Let the Schur complement S_{II} corresponds to one of the reference elements $\hat{\mathcal{E}}, \hat{\mathcal{E}}_0$ and $\hat{S}_{II}$ is defined as it is described above. Then*

$$\hat{S}_{II} \prec S_{II} \prec (1+\log p)^2 \hat{S}_{II}.$$

Proof. First of all we shall give some results of [1], which will be used below. Let us consider function ψ, $x \in \partial\Pi$, which is continuous on $\partial\Pi$ and on each side is polynomial of the order not higher than p. According to Theorems 7.4, 7.5 in [1] there exists prolongations $u \in \mathcal{P}_x^{(p)}$ and $u \in \mathcal{P}_x^{[p]}$ on Π such that $u|_{\partial\Pi} = \psi$ and

$$\|u\|_{1,\Pi} \prec \|\psi\|_{1/2,\partial\Pi}. \tag{3.7}$$

For all $\psi \in \mathcal{P}_{p,x}$, $x \in I$, according to Lemma 6.1

$$\|\psi(x)\|_{1/2,I} \asymp \|\psi(\cos\phi)\|_{1/2,I^*}. \tag{3.8}$$

On the subspace of polynomials $\psi \in \mathcal{P}_{p,x}$, $x \in I$, , with zero values at $x = \pm 1$ there is valid inequality

$$_0\|\psi\|_{1/2,I} \le c(1+\log p)\|\psi\|_{1/2,I}, \tag{3.9}$$

see in [1] Lemma 6.5.

We denote by $\mathcal{P}_{p,L}$, $L = I, II, III$, the subspaces of $\mathcal{P}_x^{(p)}$ or $\mathcal{P}_x^{[p]}$, which are spanned over systems $\mathcal{M}_I$, $\mathcal{M}_{II}$, $\mathcal{M}_{III}$ or $\mathcal{M}_{I,p}$, $\mathcal{M}_{II}$, $\mathcal{M}_{III}$ and by $\mathcal{P}^{II} = \mathcal{P}_{(II)}$, $\mathcal{P}_{(II,p)}$, the subspaces spanned over $\mathcal{M}_{(II)}$, $\mathcal{M}_{(II,p)}$ respectively. To each $\hat{u}_{II} \in \mathcal{P}_{p,II}$ there are uniquely corresponding vector $\bar{u}_{II}$ of coefficients $u_{i,j}$, see (2.2), and $\hat{\hat{u}}_{II} \in \mathcal{P}^{II}$ obtained by the orthogonalization of $\hat{u}_{II}$ to the space $\mathcal{P}_{p,I}$. Now by the definition of the Schur complement

$$\bar{u}_{II}^T S_{II} \bar{u}_{II} = |\hat{\hat{u}}|^2_{i,\Pi}. \tag{3.10}$$

We can see also that $|\cdot|_{1,\Pi}$, $\|\cdot\|_{1,\Pi}$ are equivalent for any $\hat{u} \in \mathcal{P}_{p,I} \cup \mathcal{P}_{p,II}$. Indeed from Lemmas 2.1, 2.2 it follows that $A_{0,I} \prec A_{1,I}$, where by $A_{0,I}$ and $A_{1,I}$ the matrices are denoted, which are obtained from A_0 and A_1 by crossing out four rows and four columns corresponding to the vertex functions. From this inequality it follows, that

$$\|\hat{u}\|_{0,\Pi} \prec |\hat{u}|_{1,\Pi}, \quad \text{for any} \quad \hat{u} \in \mathcal{P}_{p,I} \cup \mathcal{P}_{p,II} \tag{3.11}$$

and consequently

$$\|\hat{u}\|_{1,\Pi} \asymp |\hat{u}|_{1,\Pi}, \quad \text{for any} \quad \hat{u} \in \mathcal{P}_{p,I} \cup \mathcal{P}_{p,II}. \tag{3.12}$$

Taking (3.11) into account we can write

$$|\hat{\hat{u}}_{II}|_{1,\Pi} \asymp |\tilde{\hat{u}}_{II}|_{1,\Pi}$$

with such $\tilde{\hat{u}}$, that

$$|\tilde{\hat{u}}_{II}|_{1,\Pi} = \inf_{v \in \mathcal{P}_{p,I}} \|\hat{u}_{II} + v\|_{1,\Pi}.$$

Thus, see also (3.10),

$$\bar{u}_{II}^T S_{II} \bar{u}_{II} \asymp \|\tilde{\hat{u}}\|_{1,\Pi}^2 \tag{3.13}$$

and by the trace theorem, (3.7) and equalities $\hat{u}|_{\partial\Pi} = \tilde{\hat{u}}_{II} = \hat{\hat{u}}_{II}$ we have

$$\bar{u}_{II}^T S_{II} \bar{u}_{II} \asymp \|\hat{u}_{II}\|_{1/2,\partial\Pi}^2 \,. \tag{3.14}$$

Now it is left to show that

$$(1 + \log p) \|\hat{u}_{II}\|_{1/2,\partial\Pi}^2 \prec \bar{u}_{II}^T S_{II} \bar{u}_{II} \prec \|\hat{u}_{II}\|_{1/2,\partial\Pi}^2 \,. \tag{3.15}$$

For this purpose we note that by the definition of $\hat{S}_{II}$, interpolation between $\|\cdot\|_{0,I^*}$ and $\|\cdot\|_{1,I^*}$ and by (3.8), (3.9) for $u \in \mathcal{P}_{p,x}$, $x \in I$, and $\bar{a}$, $\bar{c}$ related by (3.4) we have

$$\bar{a}^T \bar{S} \bar{a} = \bar{c}^T \Lambda^{1/2} \bar{c} \asymp \|u(\cos\phi)\|_{1/2,I^*}^2 \asymp \|u\|_{1/2,I}^2,$$

$$(1 + \log p)^{-1} {}_0\|u\|_{1/2,I} \prec \|u\|_{1/2,I} \prec {}_0\|u\|_{1/2,I}.$$

But on the other hand

$$\|\hat{u}_{II}\|_{1/2,\partial\Pi}^2 = \sum_{i=1}^4 {}_0\|\hat{u}\|_{1/2,\gamma_i}^2$$

from where it follows (3.15). Theorem has been proved.

Remark 3.1 *In order to solve system $\hat{S}_{II}\bar{u}_{II} = \bar{f}_{II}$ it is necessary to make not more than $\mathcal{O}(p^2)$ arithmetic operations. In the DD-method this would not worsen the asymptotic estimate of the computational work, since solving of system $A_{1,0}\bar{u}_{1,0} = \bar{f}_{1,0}$ in any case demand not less than $\mathcal{O}(p^2)$ operations.*

4 Conditionality of the p-version with the curvilinear finite elements

In this section we consider the p-version with, in general, curvilinear finite elements for the second order elliptic equation in an arbitrary sufficiently smooth domain. For the reference element we use the square elements $\hat{\mathcal{E}}_0$, $\hat{\mathcal{E}}$ described in Sec.2. In order not to touch the well studied question of the approximation of the curvilinear boundary and boundary condition, and its influence on the error there will be used the f.e.m., in which the boundary is taken into account exactly. Such f.e.m. are obtained by the technique suggested in [10], see also [11]. However, following [10] all the results can be also easily expanded on the case, when the boundary is approximated with the needed accuracy by piecewise polynomials.

Let us consider the Dirichlet problem for the elliptic equation of the second order in the arbitrary sufficiently smooth domain. The generalized formulation of the problem is

$$a_\Omega(u,v)-(f,v)=0 \quad \text{for any} \quad v \in \overset{o}{W}{}_2^1(\Omega), \tag{4.1}$$

where the bilinear form for simplicity is supposed to be symmetric and satisfy inequalities

$$\begin{gathered} a_\Omega(w,v) \le \mu_2 \|w\|_{1,\Omega}\|v\|_{1,\Omega}, \\ a_\Omega(v,v) \ge \mu_1 \|v\|_{1,\Omega}^2, \ \text{for any} \quad w,v \in \overset{o}{W}{}_2^1(\Omega), \end{gathered} \tag{4.2}$$

with some positive constants μ_1, μ_1.

We denote by $x = \bar{x}^{(r)}(y) : \bar{\Pi} \mapsto \bar{\Pi}_r$ a nonnegative mapping of $\bar{\Pi}$ upon some, generally speaking, curvilinear quadrangle $\bar{\Pi}_r$ and by $x = \bar{y}^{(r)}(x) : \bar{\Pi}_r \mapsto \bar{\Pi}$ the inverse mapping. These mapping are characterized by the Lame coefficients

$$H_k^{(r)}(y) = \left(\sum_{l=1,2} \left(\frac{\partial \bar{x}_l^{(r)}}{\partial y_k} \right)^2 \right)^{1/2}.$$

And by $\theta^{(r)}$ we denote the angle between the lines $\bar{y}_k^{(r)}(x) = \text{const}$, $k = 1,2$, on the plane x. By the vertices and sides of Π_r we understand below the images of vertices and sides of Π.

First of all we formulate the result, from which it follows that for any sufficiently small mesh parameter $h \le h_0$ the domain Ω can be subdivided in $\mathcal{O}(h^{-2})$ curvilinear quadrangles satisfying definite condition of regularity.

By $C^{(t)}$ it will be denoted the class of boundaries each of which consists of finite number of t times continuously differentiable curves the angles between which at the crossing points on $\partial\Omega$ are distinct from zero and 2π.

Lemma 4.1 *Let us assume that $\partial\Omega \in C^{(t)}$, $t \ge 2$, and $h > 0$ is sufficiently small. Then the domain Ω can be represented by the union S_h of such in general curvilinear quadrangles Π_r that*

$$\bar{\Omega} = \cup_r \bar{\Pi}_r, \quad r = 1,2,\dots,\mathcal{R}, \quad \mathcal{R} = \mathcal{O}(h^{-2}),$$

and

- *a) $\Pi_{r_1} \cap \Pi_{r_2}$ for any $r_1 \ne r_2$ is empty or a common for $\Pi_{r_1} \cap \Pi_{r_2}$ vertex or side,*
- *b) the separating points between t–times continuously differentiable parts of $\partial\Omega$ are contained in the set of vertices of quadrangles,*
- *c) there exist nondegenerate $(t-1)$–times continuously differentiable mappings $x = \bar{x}^{(r)}(y) : \bar{\Pi} \mapsto \bar{\Pi}_r$, for which*

$$\begin{gathered} \alpha^{(1)} h \le H_k^{(r)} \le h, \quad \theta \le \theta^{(r)} \le \Pi - \theta, \quad 0 < \alpha^{(1)}, \theta = const, \\ |D_y^q \bar{x}^{(r)}(y)| \le c h^{|q|}, \quad 2 \le |q| \le \bar{t}, \quad for\ any \quad y \in \Pi, \end{gathered} \tag{4.3}$$

with positive constants $\alpha^{(1)}$, θ and c depending only on $\partial\Omega$ and $\bar{t} := t - 1$.

Proof. In essential it is a consequence of results of [10], see also [11] and, thus, it is given briefly. According to the results of [10, 11] (see also Lemma 1 and the construction of the finite element space $H_{h,p}(\Omega)$ on p.p. 1216–1217 in [12]) there exists subdivision of Ω in triangles δ_r, $r = 1, 2, \ldots, R$, $R = \mathcal{O}(h^{-2})$, which are curvilinear near to the boundary and satisfy analogous to a)–c) conditions. In particular, there are defined mappings $x = \tilde{x}^{(r)}(\xi) : \bar{\Delta} \mapsto \bar{\delta}_r$, where $\Delta = \{\xi |\ 0 < \xi_1,\ \xi_2,\ \xi_1 + \xi_2 < 1\}$ is the unit triangle, of the simple form (see p.1216 in [12]) and satisfying analogous to (4.1) conditions. Each triangle δ_r is easily subdivided in three quadrangles δ_r^k, $k = 1, 2, 3$. For that purpose Δ is divided in three quadrangles $\Delta^{(k)}$ by the segments of the straight lines connecting the center of gravity of Δ with the middles of sides. Then by the definition we set $\delta_r^k = \tilde{x}^{(r)}(\Delta^{(k)})$. Redenoting quadrangles δ_r^k, $r = 1, 2, \ldots, R$, $k = 1, 2, 3$ via Π_r, $r = 1, 2, \ldots, \mathcal{R}$, $\mathcal{R} = 3R$, we get the subdivision S_h. It is left to define mappings $x = \bar{x}^{(r)}(y) : \bar{\Pi} \mapsto \bar{\Pi}_r$.

Let us suppose, that $\xi = \bar{\xi}^{(k)}(y) : \bar{\Pi} \mapsto \bar{\delta}^{(k)}$ is the bilinear mapping. Then for such r, $\bar{r}$, k, that $\Pi_r = \delta_{\bar{r}}^{(k)}$ we adopt $\bar{x}^{(r)}(y) = \tilde{x}^{(r)}\left(\bar{\xi}^{(k)}(y)\right)$. Now c) is fulfilled since it is fulfilled in a little strengthened form for $\tilde{x}^{(r)}(y)$ (see (7.37), (7.38) in [11]) and mappings $\xi = \bar{\xi}^{(k)}(y)$ are bilinear, i.e. fixed and smooth. Lemma is proved.

Remark 4.1 *Conditions (3.1), which we call the generalized condition of quasiuniformity, have a simple meaning, see in [11] section 7.2. If $h \to 0$ and $\bar{t} \geq 2$ then the second line of (3.3) expresses the fact, that mappings $x = \bar{x}^{(r)}(y)$ are getting closer to linear. The first line with the accuracy up to the higher order values is equivalent to the restriction on the sides and the angles of quadrangles. These conditions in the asymptotical sence may be also expressed in terms of inscribed and circumscribed circles.*

Remark 4.2 *For our purpose it is sufficient to prove the existence of subdivision S_h and it was chosen the easiest way to do this. For practical purposes more direct ways of obtaining S_h oftenly may be used and lead to partitions of a better structure and quasiuniformity characteristics. Sometimes they allow also to replace inequalities $2 \leq |q| < t$ in the second line of (3.3) by $2 \leq |q| \leq t$. But the consideration of such ways is not the aim of this paper.*

On each quadrangle Π_r we define finite element $\mathcal{E}_r$ associated by mapping $x = \bar{x}^{(r)}(y) : \bar{\Pi} \mapsto \bar{\Pi}_r$ with the reference element, for which it is used $\hat{\mathcal{E}}_0$ or $\hat{\mathcal{E}}$. In this way two finite element spaces are defined

$$H(\Omega) = \{\tilde{u}|\ \tilde{u} \in C(\bar{\Omega}),\ \tilde{u}(\bar{x}^{(r)}(y)) \in \mathcal{P}_y^{[p]},\ y \in \Pi, r = 1, 2, \ldots, \mathcal{R}\}$$

or

$$H(\Omega) = \{\tilde{u}|\ \tilde{u} \in C(\bar{\Omega}),\ \tilde{u}(\bar{x}^{(r)}(y)) \in \mathcal{P}_y^{(p)},\ y \in \Pi, r = 1, 2, \ldots, \mathcal{R}\}.$$

As well we can use reference element $\hat{\mathcal{E}}_0$ for definition of some elements $\mathcal{E}_r$ and $\hat{\mathcal{E}}$ for definition of others. We also introduce subspaces

$$H_0(\Omega) = H(\Omega) \cap \overset{o}{W}{}_2^1(\Omega).$$

Matrix $\bar{K}$ and vector $\bar{f}$ of the finite element system of algebraic equations for problem (4.1)

$$\bar{K}\bar{u} = \bar{f} \tag{4.5}$$

satisfy the identities

$$
\begin{aligned}
(\bar{v}, \bar{K}\bar{w}) &= a_\Omega(\tilde{v}, \tilde{w}), \\
(\bar{f}, \bar{w}) &= (f, \tilde{w})_\Omega, \quad \text{for any} \quad \tilde{v}, \tilde{w} \in H_0(\Omega).
\end{aligned}
\tag{4.6}
$$

Here and in the following isomorphism $\bar{v} \leftrightarrow \tilde{u}$, $\bar{w} \leftrightarrow \tilde{w}$ is established by the choice of the basis in $H(\Omega)$, which will be denoted $\{\phi^{(i)}\}$, $i = 1, 2, \ldots, \mathcal{N}$, $\mathcal{R}(p-1)^2 \le \mathcal{N} \le \mathcal{R}(p+1)^2$. In our case, as usually, it is defined by the choice of the reference element, i.e. $\phi^{(i)}$ is such that for any r function $\phi^{(i)}\left(\bar{x}^{(r)}(y)\right)$ at $y \in \Pi$ equals or zero or one of the functions from set $\mathcal{M}_p$ or $\mathcal{M}_\Pi$ depending on the choice of the reference element. Again we can subdivide set $\Phi := \{\phi^{(i)}\}$ in three subsets Φ_I, Φ_{II}, Φ_{III} consisting respectively of the internal functions, each distinct from zero on one finite element, the side functions, each distinct from zero not more than on a couple of adjacent elements, and the vertex or nodal functions. The vertex function may be distinct from zero not more than on a star of the elements with a common vertex and is piecewise bilinear.

Theorem 4.1 *Let bilinear form $a_\Omega(\cdot,\cdot)$ satisfy inequalities (4.2), the partition S_h be as in Lemma 4.1. Then*

$$
\begin{aligned}
\lambda_{\min}(\bar{K}) &\ge c_1\mu_1(\alpha^{(1)}\sin\theta)(1+\log p)^{-1}\ \min[(\alpha^{(1)})^3(\sin\theta)^2\ h^2, 1] \\
\lambda_{max}(\bar{K}) &\le c_2\mu_2(\alpha^{(1)}\sin\theta)^{-1}p^2,
\end{aligned}
\tag{4.7}
$$

with constants c_1, c_2 depending only on $\partial\Omega$.

Proof. These estimates follow from (4.2), Lemma 2.2, inequality (2.7) and Lemma 7.1 from [11]. Since the derivation of the similar estimates on the basis of the estimates for the reference elements and inequalities of the type (3.3) is well known, we shall give the proof only for the case, when finite elements are rectangular. This assumes that Ω is composed of rectangles and S_h is described by a part of a rectangular mesh. In this case $H_k^{(r)}$ are simply the lengths of the sides of Π_r and $\mathcal{D}_y^q\bar{x}^{(r)}(y) = 0$ for $|q| \ge 2$. For the simplicity we suppose that $\|\cdot\|_{1,\Omega}$ in (4.2) may be replaced by $|\cdot|_{1,\Omega}$.

Every $\tilde{u} \in H(\Omega)$ may be represented as $\tilde{u} = \tilde{u}_{III} + \tilde{u}_e$, where $\tilde{u}_{III}$ belongs to the space spanned over Φ_{III}. Paying attention to (4.2), (2.7), (4.6), we have

$$
(\bar{u}, \bar{K}\bar{u}) \ge \mu_1|\tilde{u}|_{1,\Omega} \ge c\frac{\mu_1 d^{(1)}}{1+\log p}\sum_r \left(|\tilde{u}_{III}(\bar{x}^{(r)}(y))|^2_{1,\Pi} + |\tilde{u}_e(\bar{x}^{(r)}(y))|^2_{1,\Pi}\right), \tag{4.8}
$$

From Lemma 2.2 it follows

$$
\sum_r |\tilde{u}_e(\bar{x}^{(r)}(y))|^2_{1,\Pi} \ge c(\bar{u}_e,\ \bar{u}_e), \tag{4.9}
$$

where function $\tilde{u}_e$ and vector $\bar{u}_e$ correspond each other. For the first sum in (4.8) the use of mappings $\Pi \to \Pi_r$, inequality $|v|^2_{1,\Omega} \ge C_\Omega\|v\|^2_{0,\Omega}$ with any $v \in \overset{o}{W}{}_2^1(\Omega)$ and mappings $\Pi_r \to \Pi$ allow to write

$$
\begin{aligned}
\sum_r |\tilde{u}_{III}(\bar{x}^{(r)}(y))|^2_{1,\Pi} &\ge \alpha^{(1)}|\tilde{u}_{III}|^2_{1,\Omega} \ge c_\Omega\alpha^{(1)}\|\tilde{u}_{III}\|^2_{0,\Omega} \ge \\
&\ge c_\Omega(\alpha^{(1)})^3h^2\sum_r \|\tilde{u}_{III}(\bar{x}^{(r)}(y))\|^2_{0,\Pi} \ge \\
&\ge c_\Omega(\alpha^{(1)})^3h^2\ (\bar{u}_{III},\ \bar{u}_{III}),
\end{aligned}
\tag{4.10}
$$

and vector $\bar{u}_{III}$ has nonzero components only in the places corresponding to the vertex unknowns.

Now, see (4.8), (4.9), (4.10),

$$(\bar{u}, \bar{K}\bar{u}) \geq c\mu_1\alpha^{(1)}\frac{1}{1+\log p}\min\left[c_\Omega(\alpha^{(1)})^3h^2, 1\right](\bar{u},\ \bar{u})$$

Let us suppose that the least rectangle with the sides parallel to axis covering Ω contains $l_1 \times l_2$ rectangular nests of the mesh. Then the inequality may be rewritten in the form

$$(\bar{u}, \bar{K}\bar{u}) \geq c\frac{\mu_1\alpha^{(1)}}{1+\log p}\left(\frac{1}{l_1^2}+\frac{1}{l_2^2}\right)(\bar{u},\ \bar{u})$$

with the slighter dependence on the nonuniformity of the mesh.

Since evidently $\lambda_{max}(A_1) \asymp \mathcal{O}(N^2)$ the estimation of $\lambda_{max}(\bar{K})$ is done by the usual for the finite element method way. Theorem has been proved.

5 *DD* preconditioning

The construction of *DD* preconditioner will be the following. First we construct much more simple than $\bar{K}$ matrix $\Lambda_{p,h}$, which is equivalent in the spectrum to $\bar{K}$ and then *DD* preconditioner is obtained on the basis of this matrix.

Let us describe the first step. Stiffness matrix A_1 of the reference element $\hat{\mathcal{E}}$ may be represented in a form

$$A_1 = K_1 \times (\Delta_{+,-} + D) + (\Delta_{+,-} + D) \times K_1, \tag{5.1}$$

where it is assumed that $K_0 = \Delta_{+,-} + D$ and $\Delta_{+,-}$, D are analogous to $\Delta_{0,+}, D_{0,+}$ in the representation $\bar{K}_{0,+} = \Delta_{0,+} + D_{0,+}$. In other words we use for K_0 the same representation as for $\bar{K}_{0,+}$, which is one block of K_0 corresponding to even coordinate functions and the first boundary condition on $\partial\Pi$. Now in the above expression we replace $\Delta_{+,-}$ by the more simple matrix denoted by $\bar{\Delta}_{(1)}$. Matrix $\bar{\Delta}_{(1)}$ by it's definition contains only two nonzero blocks of the form (1.12), which are placed on diagonal, have the form (1.12) and correspond to even and odd coordinate functions. For the result we obtain matrix

$$\hat{A}_1 = K_1 \times (\bar{\Delta}_{(1)} + D) + (\bar{\Delta}_{(1)} + D) \times K_1. \tag{5.2}$$

For each finite element $\mathcal{E}_r$ we define a subsidiary "stiffness" matrix $\hat{A}_{1,r}$ which in a local numeration of the finite element coordinate functions coincides with $\hat{A}_1$, i.e. $\hat{A}_{1,r} \equiv \hat{A}_1$. Via $\Lambda_{p,h}$ it is denoted matrix which is result of the usual for the finite element method procedure of assembling of the "stiffness" matrices $\hat{A}_{1,r}$, $r = 1, 2, .., \mathcal{R}$, according to the configuration of the partition S_h.

Let us order the unknowns and Galerkin coordinate functions of the finite element method and let $\omega_0 = \{i = 1, 2, .., \mathcal{N}\}$ be the set of the corresponding numbers. For each element $\mathcal{E}_r$ subsets $\omega_{I,r}$, $\omega_{II,r}$, $\omega_{III,r}$ of ω_0 corresponding to the integral, side, and vertex unknowns may be introduced alongside with their unions

$$\omega_L = \sum_r \omega_{L,r}, \quad L = I, II, III.$$

Representing $\Lambda_{p,h}$ in the block form

$$\Lambda_{p,h} = \begin{pmatrix} \Lambda_I & \Lambda_{I,II} & \Lambda_{I,III} \\ \Lambda_{II,I} & \Lambda_{II} & \Lambda_{II,III} \\ \Lambda_{III,I} & \Lambda_{III,II} & \Lambda_{III} \end{pmatrix} \tag{5.3}$$

we may define also matrix

$$\tilde{\Lambda}_{p,h} := \begin{pmatrix} \begin{pmatrix} \Lambda_I & \Lambda_{I,II} \\ \Lambda_{II,I} & \Lambda_{II} \end{pmatrix} & 0 \\ 0 & \Lambda_{III} \end{pmatrix} =: \begin{pmatrix} \Lambda^{(I)} & 0 \\ & \\ 0 & \Lambda_{III} \end{pmatrix}, \tag{5.4}$$

Lemma 5.1 *Let (4.2), (4.3) be fulfilled and $\bar{K}$ is the finite element matrix for the case of Neumann boundary condition. Then*

$$\begin{gathered} c_1\mu_1\Lambda_{p,h} \leq \bar{K} \leq c_2\mu_2\Lambda_{p,h}, \\ \frac{\hat{c}_1}{1+\log p}\tilde{\Lambda}_{p,h} \leq \bar{K} \leq \hat{c}_2\tilde{\Lambda}_{p,h}, \end{gathered} \tag{5.5}$$

with positive constants $c_k, \hat{c}_k$ depending only on $\alpha^{(1)}, \theta$.

Proof. Let us note first of all that in Lemma 5.1 we used the same notation $\bar{K}$ for the finite element matrix obtained at the Neumann boundary condition as for the matrix defined by (4.6) at the Dirichlet boundary condition. For the proof we represent $\hat{A}_1$ in analogous (5.3) form

$$\hat{A}_1 = \begin{pmatrix} \hat{A}_I & \hat{A}_{I,II} & \hat{A}_{I,III} \\ \hat{A}_{II,I} & \hat{A}_{II} & \hat{A}_{II,III} \\ \hat{A}_{III,I} & \hat{A}_{III,II} & \hat{A}_{III} \end{pmatrix}$$

and introduce similarly to (5.4) matrix

$$\tilde{A}_1 := \begin{pmatrix} \begin{pmatrix} \hat{A}_I & \hat{A}_{I,II} \\ \hat{A}_{II,I} & \hat{A}_{II} \end{pmatrix} & 0 \\ 0 & A_{III} \end{pmatrix} =: \begin{pmatrix} A^{(I)} & 0 \\ 0 & \hat{A}_{III} \end{pmatrix}.$$

The estimates for the reference element

$$\hat{A}_1 \prec A_1 \prec \hat{A}_1, \quad \frac{1}{1+\log p}\tilde{A}_1 \prec A_1 \prec \tilde{A}_1, \tag{5.6}$$

follow immediately from (1.13), (2.7). After that, the proof became customary for the energy inequalities based on the energy inequalities for the reference element, see, for instance theorem 5.1 of [13]. Constants $c_k, \hat{c}_k$ depend on $\alpha^{(1)}, \theta$ in the same manner as $c^{(k)}$ in (5.10) of [13]. Lemma is proved.

Remark 5.1 *Matrix $\Lambda_{p,h}$ is a good preconditioner for $\bar{K}$. In general $\bar{K}$ is much more complicated than $\Lambda_{p,h}$ and the latter matrix is rather simple. It is evident also the analogy of the blocks of $\Lambda_{p,h}$, look at (5.1), (5.2), (1.12), with the finite difference operators of the second order on the uniform grid. These operators at the "points" corresponding to $i \in \omega_0 \backslash \omega_{III}$ have five point stencils. This analogy allows, for instance, to apply for the solution of systems of algebraic equations of the form $\Lambda_{p,h}x = y$ the method of the nested dissection [7]. The computational work is estimated as $\mathcal{O}(\mathcal{N}^{1.5})$ where $\mathcal{N}$ is the dimension of $\Lambda_{p,h}$. If the number $\mathcal{R}$ of finite elements is not large and $\bar{K}$ is such that multiplication $\bar{K}z$ of this matrix by vector can't be done faster than for $\mathcal{O}(p^3)$ operations, then no other preconditioner may improve estimate $\mathcal{O}(p^3)$ of the total computational work in order. For many problems with variable coefficients stiffness matrices of the finite* *elements will be completely filled in. If to calculate these matrices of finite elements and use them for multiplication $\bar{K}z$, then the cost of such multiplication will be $\mathcal{O}(p^4)$. However, if each multiplication $\bar{K}z$ demands less than $\mathcal{O}(p^3)$ arithmetic operation then the more effective methods for solution of systems $\Lambda_{p,h}x = y$ will improve the estimate of the total computational work. We have such a situation when the number of coefficients of $\bar{K}$ is $\mathcal{O}(p^2)$, as in the case of the Poisson equation. One of the ways to diminish in such situation the computational work is to construct preconditioner of $\bar{K}$ as DD preconditioner for $\Lambda_{p,h}$.*

Now for the sake of definiteness we turn back again to the case of the first boundary condition assuming that $\Lambda_{p,h}$ also corresponds to this boundary condition.

DD preconditioner, which we denote via Λ_D, is defined thorough its inverse by topological sum

$$\Lambda_D^{-1} := \Lambda_I^{-1} + \tilde{\Lambda}_{II}^{+} + \tilde{\Lambda}_{III}^{+}, \tag{5.7}$$

summands of which are explained below. Matrix Λ_I was introduced in (5.3) and has the form

$$\Lambda_I = \begin{pmatrix} \hat{A}_{1,0} & & \cdots & 0 \\ & \hat{A}_{1,0} & & \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & \hat{A}_{1,0} \end{pmatrix}$$

where $\hat{A}_{1,0}$ is the submatrix of $\hat{A}_1$ corresponding to the first boundary condition on $\partial\Pi$. Matrix $\tilde{\Lambda}_{III}$ is defined by means of matrix Λ_{III} which was also introduced in (5.3). It is clear that Λ_{III} may be assembled from element matrices every of which is identical to $\hat{A}_1$ for $p = 1$. Let us denote $\mathcal{N}_L := \text{card}\ \ \omega_L, L = I, II, III$. If $\mathcal{N}_{III}$ is not large, solution of systems $\Lambda_{III}x = y$ does not cause any difficulties and thus we may set $\tilde{\Lambda}_{III} = \Lambda_{III}$. For the large $\mathcal{N}_{III}$, when the finite element method has to be considered as h–p–version, there are known a variety of iterative techniques, which are optimal in the order of arithmetical operations. Thus we are able to define $\tilde{\Lambda}_{III}^{+}$ implicitly by one of such iterative techniques as a rude approximation of Λ_{III}^{-1}. By such a rude approximation it is meant that $\tilde{\Lambda}_{III}^{+}$ satisfies inequalities

$$c_{1,III}\tilde{\Lambda}_{III} \leq \Lambda_{III} \leq c_{2,III}\tilde{\Lambda}_{III}, \tag{5.8}$$

with the sufficiently good constants $c_{k,III} > 0$ independent of h. Consequently, in the following we may assume that operation $\tilde{\Lambda}^{+}y$ is defined, demands not more than $\mathcal{O}(h^{-2})$ arithmetic operations and $\tilde{\Lambda}_{III}$ satisfies (5.8).

DD factorization of $\Lambda^{(I)}$ looks as

$$\Lambda^{(I)} = \begin{pmatrix} I_I & 0 \\ \Lambda_{II,I}\Lambda_I^{-1} & I_{II} \end{pmatrix} \begin{pmatrix} \Lambda_I & 0 \\ 0 & S_\Lambda \end{pmatrix} \begin{pmatrix} I_I & \Lambda_I^{-1}\Lambda_{I,II} \\ 0 & I_{II} \end{pmatrix}$$

The Schur complement S_Λ is the topological sum of matrices given for each finite element and equal, each, to S_{II}, see (3.3). In other words, if to introduce finite elements with only side unknowns and the corresponding element stiffness matrices $S_{II,r}$ defined in the local ordering of unknowns by equality $S_{II,r} = S_{II}$ then S_Λ is the global finite element matrix assembled of $S_{II,r}$. Taking into account theorem 3.1 we may use for the preconditioning of S_Λ matrix $\hat{S}_\Lambda$:

$$\hat{S}_\Lambda = \frac{2}{1+\log p}\begin{pmatrix} \hat{S} & & \cdots & 0 \\ & \hat{S} & & \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & \hat{S} \end{pmatrix},$$

where S is the same as $S^{(1)} = ... = S^{(4)}$ in $\hat{S}_{II}$ and corresponds to the common of some pair of the adjacent elements side. From theorem 3.1 it follows that

$$\frac{1}{1+\log p}\hat{S}_\Lambda \prec S_\Lambda \prec (1+\log p)\hat{S}_\Lambda, \tag{5.9}$$

and so $\hat{S}_\Lambda$ is sufficiently good preconditioner. Now we put

$$\tilde{\Lambda}^+_{II} = P\hat{S}_\Lambda^{-1}P^T, \tag{5.10}$$

where P is a properly chosen and sufficiently cheap prolongation operator, the construction of which is clear from below. Operator P maps any vector from $\mathbb{R}^{\mathcal{N}_{II}}$ containing side coefficients on the space $\mathbb{R}^{\mathcal{N}_I+\mathcal{N}_{II}}$ of vectors containing side and vertex coefficients.

Let $\mathcal{T}$ be the union of all closed sides of elements and $H^0_{II}(\mathcal{T})$ be space of traces on $\mathcal{T}$ of functions $\tilde{u} \in H^0_{I,II}(\Omega)$:=span $(\Phi_I, \Phi_{II}) \bigcap H^0(\Omega)$ supplied with the norm $\|\cdot\|_{1/2,\mathcal{T}}$. We won't write down this norm and only note that $\|\cdot\|^2_{1/2,\mathcal{T}}$ may be defined as the sum of $\|\cdot\|^2_{1/2,\partial\Pi_r}$, $r = 1,2,\ldots,\mathcal{R}$, and for each r the last norm is defined analogously with $\|\cdot\|^2_{1/2,\partial\Pi}$. Moreover for $u\,:\, u(\bar{x}^{(r)}(y)) = v(y)$ norms $\|u\|^2_{1/2,\partial\Pi_r}$, $\|v\|^2_{1/2,\partial\Pi}$ are equivalent due to the generalized qusiuniformity conditions. The construction of P is equivalent to the construction of the isomorphic prolongation operator $\tilde{P} : H^0_{II}(\mathcal{T}) \to H^0_{I,II}(\Omega)$. As it may be shown for providing better preconditioning operator $\tilde{P}$ should be such that if $\tilde{u}_\mathcal{T}$ is an arbitrary function from $H_{II}(\mathcal{T})$ and $\tilde{v} = \tilde{P}\tilde{u}_\mathcal{T}$ then

$$\|\tilde{v}(\bar{x}^{(r)}(y))\|_{1,\Pi} \le c\|\tilde{u}_\mathcal{T}(\bar{x}^{(r)}(y))\|_{1/2,\partial\Pi} \tag{5.11}$$

with the constant independent of $\tilde{u}_\mathcal{T}$, p, r, h. This conclusion is the consequence of the known results on prolongation operators at the Schur complement preconditioning in *DD* methods an the following. First, according to lemma 5.1 at constructing of the precondioner for $\bar{K}$ it is sufficient to construct the preconditioner for $\Lambda_{p,h}$. Second, matrix $\Lambda_{p,h}$ is assembled of "stiffness" matrices $\hat{A}_{1,r} \equiv \hat{A}_1$, while $\hat{A}_1$ is equivalent in the spectrum to matrix A_1 defined by the quadratic form, specified on Π, see (2.3) in part I and (5.11).

Let us note by the way that if (5.11) is fulfilled then

$$\|\tilde{v}\|_{1,\Omega} \le c\|\tilde{u}\|_{1/2,\mathcal{T}}$$

with the constant depending on $\alpha^{(3)}, \theta$. But in fact we need only the simpler inequality (5.11). From (5.11) it follows that operator P may be defined element by element in such a way that on every element it has the same form and is isomorphic to prolongation the operator which is defined for the reference element and denoted below $\hat{P}_\mathcal{E}$. If $H_{II}(\partial\Pi)$ is the space of traces on $\partial\Pi$ of polynomials from $H_{I,II}(\hat{\mathcal{E}})$:= span $(\mathcal{M}_I \bigcup \mathcal{M}_{II})$ then according to (5.11) it should be satisfied inequality

$$\|\hat{P}_\mathcal{E}\hat{u}_{\partial\Pi}\|_{1,\Pi} \le c\|\hat{u}_{\partial\Pi}\|_{1/2,\partial\Pi}, \quad \forall\ \hat{u}_{\partial\Pi} \in H_{II}(\partial\Pi)$$

with the constant independent of p.

Prolongation operators $\hat{P}_{\mathcal{E}}$ having described properties are known in the literature. They can be found, for instance, in [1] as for the space $H(\hat{\mathcal{E}})$ so for the space $H(\hat{\mathcal{E}}_0)$. Consequently, the prolongation operator P in (5.10) can be defined in such a way that inequality (5.11) will be fulfilled with the constant depending only on the type of the reference element and $\alpha^{(1)}, \theta$ but not on p and h.

Theorem 5.1 *Let $\bar{K}$ be the finite element matrix defined in (4.6) for the boundary value problem (4.1), (4.2) and Λ_D^{-1} is defined by (5.7). Then*

$$c_{1,D}\frac{1}{1+\log p}\Lambda_D \leq \bar{K} \leq c_{2,D}\frac{1}{1+\log p}\Lambda_D$$

with the positive constants independent of p and h.

Proof. Proof follows from (5.5),(5.7)–(5.11) and the results on the *DD* methods obtained in [14].

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