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Exact discretizations of two-point boundary value problems

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Exact discretizations of two-point boundary value problems

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Summary. In the paper we construct exact three-point discretizations of linear and nonlinear two-point boundary value problems with boundary conditions of the first kind. The finite element approach uses basis functions defined by the coefficients of the differential equations. All the discretized boundary value problems are of inverse isotone type and so are its exact discretizations which involve tridiagonal M-matrices in the linear case and M-functions in the nonlinear case.

1. Introduction

Let $u(x)$ be a solution of a two-point boundary value problem

$$\begin{aligned} &Lu = f(x), \quad 0 < x < 1, \\ &u(0) = u_0, \quad u(1) = u_1. \end{aligned} \tag{1.1}$$

Using a finite set of grid points $\bar{\omega}_h = \{x_i\}_{i=0}^{n+1} \subset [0,1]$, a discretization method is said to be "exact" at $u(x)$ if it generates a system of equations

$$L_h u_h = f_h, \tag{1.2}$$

whose solution is given by $\{u(x_i)\}_{i=0}^{n+1}$.

We shall illustrate this by a simple example. Consider

$$\begin{aligned} &-u'' = 0, \quad 0 < x < 1, \\ &u(0) = u_0, \quad u(1) = u_1, \end{aligned} \tag{1.3}$$

which has the unique solution $u(x) = u_0 + x\,(u_1 - u_0)$. Then, the standard finite difference method on the uniform grid $\bar{\omega}_h = \{x_i = ih,\ i = 0, \ldots, n+1,\ h = \frac{1}{n+1}\}$ defined by

$$\begin{array}{rcll} y_0 & = & u_0, & \\ -\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} & = & 0, & i = 1, \ldots, n, \\ y_{n+1} & = & u_1, & \end{array} \tag{1.4}$$

is exact because $y_i = u(x_i) = u_0 + x_i(u_1 - u_0),\ i = 0, \ldots, n+1$ is the unique solution of the system of linear equations (1.4).

2. Exact discretization of differential equations with constant coefficients

In this section we consider the two-point boundary value problem

$$\begin{aligned} &Lu = -u'' + bu' + cu = f(x), \quad 0 < x < 1, \\ &u(0) = u_0, \quad u(1) = u_1, \end{aligned} \tag{2.1}$$

where b and c are real constants. Furthermore, we assume that $f(x)$ is at least continuous. It is known that the solution $u(x)$ of problem (2.1) may exhibit boundary layers, see [2], [3].

2.1 Exact discretization of $Lu = -u'' = f(x)$

The solution of

$$\begin{aligned} &Lu = -u'' = f(x), \quad 0 < x < 1, \\ &u(0) = u_0, \quad u(1) = u_1, \end{aligned} \tag{2.2}$$

is given by

$$u(x) = u_0(1-x) + u_1 x + \int_0^1 G(x,\xi) f(\xi) d\xi, \tag{2.3}$$

where

$$G(x,\xi) = \begin{cases} \xi(1-x), & \xi < x, \\ \\ x(1-\xi), & \xi \geq x, \end{cases} \tag{2.4}$$

is the Green's function of the differential operator of problem (2.2). It is obvious that the Green's function is symmetric and nonnegative, i.e. $G(x,\xi) = G(\xi,x)$ and $G(x,\xi) \geq 0$ for all $(x,\xi) \in [0,1]^2$.

Let $\bar{\omega}_h = \{0 = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = 1\}$ be a grid with step sizes $h_i = x_i - x_{i-1} > 0$ for $i = 1, \ldots, n+1$. Then, the three–point discretization of problem (2.2)

$$\begin{aligned} &-\tfrac{1}{h_i}\, y_{i-1} + \left(\tfrac{1}{h_i} + \tfrac{1}{h_{i+1}}\right)\, y_i - \tfrac{1}{h_{i+1}}\, y_{i+1} = \int_0^1 \ \phi_i(\xi) f(\xi)\ d\xi, \\ \\ &i = 1, \ldots, n, \end{aligned} \tag{2.5}$$

with $y_0 = u_0, \;\; y_{n+1} = u_1$ and

$$\phi_i(\xi) = \begin{cases} \frac{\xi - x_{i-1}}{h_i}, & x_{i-1} \le \xi \le x_i, \\ \frac{x_{i+1}-\xi}{h_{i+1}}, & x_i < \xi \le x_{i+1}, \\ 0, & else, \end{cases} \qquad i = 1, \dots, n, \tag{2.6}$$

is exact for any $n \ge 1$.

The latter statement will be proved by showing that

$$y_i = u(x_i) = u_0(1 - x_i) + u_1 x_i + \textstyle\int_0^1 G(x_i, \xi) f(\xi) d\xi, \quad i = 1, \dots, n, \tag{2.7}$$

satisfy all of the equations (2.5). The proof is quite technical and it will be therefore omitted here.

Next, we rewrite the exact discretization (2.5) of problem (2.2) in matrix form. For this purpose define

$$\begin{aligned} A &= tridiag\left(-\frac{1}{h_i} \, , \, \frac{1}{h_i} + \frac{1}{h_{i+1}} \, , \, -\frac{1}{h_{i+1}} \right)_{n \times n}, \\ y &= (y_1, \dots, y_n)^T, \\ b &= (b_1, \dots, b_n)^T, \\ b_1 &= \frac{u_0}{h_1} + \int_0^1 \phi_1(\xi) f(\xi) \; d\xi, \\ b_i &= \int_0^1 \phi_i(\xi) f(\xi) \; d\xi, \qquad i = 2, \dots, n-1, \\ b_n &= \frac{u_1}{h_{n+1}} + \int_0^1 \phi_n(\xi) f(\xi) \; d\xi. \end{aligned}$$

Then, the linear equation system (2.5) has short form

$$Ay = b, \tag{2.8}$$

where the tridiagonal matrix $A = A^T$ is an irreducible, weakly diagonally dominant M-matrix with

$$\det A = \frac{h_1 + \cdots + h_{n+1}}{h_1 \cdots h_{n+1}} > 0$$

and $A^{-1} > 0$. Thus, the solution (2.7) of the system of equations (2.8) is unique.

We shall show now that the system of linear equations (2.5) also results from a finite element discretization of problem (2.2). For this purpose, we start with weak formulation of problem (2.2).

Define

$$
\begin{aligned}
U &= \{u(x) \in W_2^1(0,1),\ u(0) = u_0,\ u(1) = u_1\}, \\
V &= \overset{\circ}{W}{}_2^1(0,1), \\
a(u,v) &= \int_0^1 u'v'\,dx.
\end{aligned}
$$

Then, we seek a function $u \in U$ such that

$$
a(u,v) = \textstyle\int_0^1 u'v'\,dx = \int_0^1 vf\,dx, \qquad \forall\, v \in V. \tag{2.9}
$$

We define the finite dimensional approximation of problem (2.9) as follows. First, we supplement the set of functions (2.6) with the two functions

$$
\phi_0(\xi) = \begin{cases} \frac{x_1-\xi}{h_1}, & 0 \le \xi \le x_1, \\ 0, & else, \end{cases} \qquad \phi_{n+1}(\xi) = \begin{cases} \frac{\xi-x_n}{h_{n+1}}, & x_n \le \xi \le 1 \\ 0, & else, \end{cases}
$$

and introduce

$$
U_h = \{u_h(x) = u_0\phi_0(x) + \sum_{i=1}^{n} y_i\phi_i(x) + u_1\phi_{n+1}(x), \quad y_i \in \mathbb{R},\ i = 1,\ldots,n\},
$$

$$
V_h = span\{\phi_i(x)\}_{i=1}^n.
$$

Second, seek $u_h \in U_h$ such that

$$
a(u_h,\phi_i) = \textstyle\int_0^1 \phi_i f\,dx \qquad i = 1,\ldots,n. \tag{2.10}
$$

Now, it has been found that the finite element approximation (2.10) is equivalent to the exact discretization (2.8) because of

$$
A = tridiag\left(a(\phi_{i-1},\phi_i)\ ,\ a(\phi_i,\phi_i)\ ,\ a(\phi_{i+1},\phi_i)\right)_{n\times n},
$$

where

$$
\begin{array}{lllll}
a(\phi_{i-1},\phi_i) & = & \int_0^1 \ \phi'_{i-1}\phi'_i \ dx & = & -\frac{1}{h_i}, \\
a(\phi_i,\phi_i) & = & \int_0^1 \ (\phi'_i)^2 \ dx & = & \frac{1}{h_i}+\frac{1}{h_{i+1}}, \\
a(\phi_{i+1},\phi_i) & = & \int_0^1 \ \phi'_{i+1}\phi'_i \ dx & = & -\frac{1}{h_{i+1}}.
\end{array}
$$

2.2 Exact discretization of $Lu = -u'' + cu = f(x), \quad c > 0$

For any constant $c > 0$, the unique solution of

$$
\begin{array}{l}
Lu = -u'' + cu = f(x), \quad 0 < x < 1, \\
u(0) = u_0, \quad u(1) = u_1,
\end{array} \tag{2.11}
$$

is given by

$$
u(x) = u_0 \frac{\sinh(\sqrt{c}(1-x))}{\sinh(\sqrt{c})} + u_1 \frac{\sinh(\sqrt{c}x)}{\sinh(\sqrt{c})} + \int_0^1 G_c(x,\xi) f(\xi) d\xi, \tag{2.12}
$$

where $G_c(x,\xi)$ is the Green's function of the differential operator of problem (2.11) defined dy

$$
G_c(x,\xi) = \begin{cases} \frac{\sinh(\sqrt{c}(1-x))\sinh(\sqrt{c}\xi)}{\sqrt{c}\sinh(\sqrt{c})}, & \xi < x, \\ \\ \frac{\sinh(\sqrt{c}x)\sinh(\sqrt{c}(1-\xi))}{\sqrt{c}\sinh(\sqrt{c})}, & \xi \geq x. \end{cases} \tag{2.13}
$$

It is thus clear that $G_c(x,\xi)$ is symmetric and nonnegative. This means that $G_c(x,\xi) = G_c(\xi,x)$ and $G_c(x,\xi) \geq 0$ for all $(x,\xi) \in [0,1]^2$.

Now let us consider the exact discretization of problem (2.11) on the uniform grid $\bar{\omega}_h = \{x_i = ih, \ i = 0,\ldots,n+1, \ h = \frac{1}{n+1}\}$.

It turns out that the three-point discretization

$$
\begin{array}{l}
-y_{i-1} + 2\cosh(\sqrt{c}h) \ y_i - y_{i+1} = \frac{\sinh(\sqrt{c}h)}{\sqrt{c}} \ \int_0^1 \ \psi_i(\xi) f(\xi) \ d\xi, \\
\\
i = 1,\ldots,n,
\end{array} \tag{2.14}
$$

with $y_0 = u_0$, $y_{n+1} = u_1$ and

$$\psi_i(\xi) = \begin{cases} \frac{\sinh(\sqrt{c}(\xi - x_{i-1}))}{\sinh(\sqrt{c}h)}, & x_{i-1} \le \xi \le x_i, \\ \frac{\sinh(\sqrt{c}(x_{i+1} - \xi))}{\sinh(\sqrt{c}h)}, & x_i < \xi \le x_{i+1}, \qquad i = 1, \ldots, n, \\ 0, & else, \end{cases} \tag{2.15}$$

is exact for any $n \ge 1$.

In order to prove this statement, we have to show that

$$y_i = u(x_i) = u_0 \frac{\sinh(\sqrt{c}(1-x_i))}{\sinh(\sqrt{c})} + u_1 \frac{\sinh(\sqrt{c}x_i)}{\sinh(\sqrt{c})} + \int_0^1 G_c(x_i, \xi) f(\xi) d\xi, \tag{2.16}$$
$$i = 1, \ldots, n,$$

is the solution of the system of equations (2.14). The proof will be omitted here because it only needs technical details.

To get an impression how the functions $\psi_i(\xi)$ differ from the pieceweise linear functions $\phi_i(\xi)$ defined by (2.6) we illustrate the behaviour of $\psi_i(\xi)$ for $c = 16$, see Figure 1.

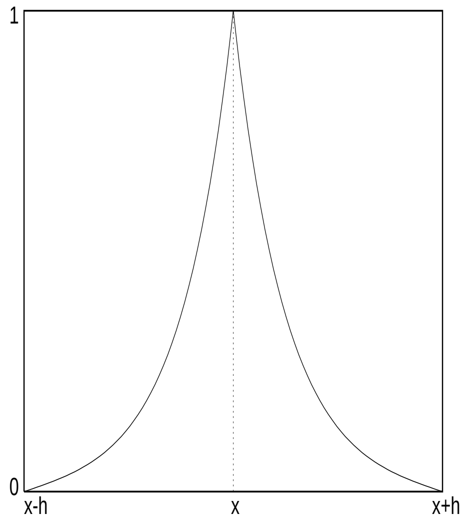

Figure 1

The left-hand side expression of the exact discetization (2.14) can be deduced directly from the Taylor series expansion

$$-v(x-h) + 2\cosh(\sqrt{c}h)\ v(x) - v(x+h) = \sum_{k=1}^{\infty} r_{2k} \left(-v^{(2k)}(x) + c^k v(x)\right) h^{2k},$$

which holds true for any function $v \in C^\infty$, where $r_{2k} \in \mathbb{R},\ k = 1, 2, \ldots$, are well-defined coefficients. Exploiting this for the solution $u(x)$ of the homogeneous differential equation, we get

$$-u(x-h)+2\cosh(\sqrt{c}h)u(x)-u(x+h)=0,$$

because $-u''+cu=0$ implies $-u^{(2k)}(x)+c^k u(x)=0$ for $k=1,2,\ldots$.

Next, we introduce a matrix formulation of the exact discretization (2.14). For this let

$$\begin{aligned}
A &= \frac{\sqrt{c}}{\sinh(\sqrt{c}h)}\; tridiag\Big(-1\ ,\ 2\cosh(\sqrt{c}h)\ ,\ -1\Big)_{n\times n},\\
y &= (y_1,\ldots,y_n)^T,\\
b &= (b_1,\ldots,b_n)^T,\\
b_1 &= \frac{\sqrt{c}}{\sinh(\sqrt{c}h)}u_0+\int_0^1 \psi_1(\xi)f(\xi)\ d\xi,\\
b_i &= \int_0^1 \psi_i(\xi)f(\xi)\ d\xi, \qquad\qquad i=2,\ldots,n-1,\\
b_n &= \frac{\sqrt{c}}{\sinh(\sqrt{c}h)}u_1+\int_0^1 \psi_n(\xi)f(\xi)\ d\xi.
\end{aligned}$$

Thus, (2.14) is equivalent to

$$Ay=b, \tag{2.17}$$

where the tridiagonal matrix $A=A^T$ is an irreducible, strictly diagonally dominant M-matrix with $A^{-1}>0$.

The exact discretization (2.17) of problem (2.11) is also deducable by a finite element method which uses the function system $\{\psi_i(x)\}_{i=1}^n$ defined by (2.15).

With the same U, V as in Section 2.1 we now define the bilinear form

$$a(u,v)=\int_0^1 (u'v'+cuv)\ dx.$$

Then, the weak form of problem (2.11) is defined as follows: Seek $u(x)\in U$ such that

$$a(u,v)=\textstyle\int_0^1 (u'v'+cuv)\ dx\ =\int_0^1\ vf\ dx, \qquad \forall\, v\in V. \tag{2.18}$$

For a finite dimensional approximation of problem (2.18), we supplement the function system $\{\psi_i(\xi)\}_{i=1}^n$ with the two functions

$$\psi_0(\xi) = \begin{cases} \frac{\sinh(\sqrt{c}(x_1-\xi))}{\sinh(\sqrt{c}h)}, & 0 \le \xi \le x_1, \\ \\ 0, & else, \end{cases}$$

$$\psi_{n+1}(\xi) = \begin{cases} \frac{\sinh(\sqrt{c}(\xi-x_n))}{\sinh(\sqrt{c}h)}, & x_n \le \xi \le 1, \\ \\ 0, & else, \end{cases}$$

and define

$$U_h = \{u_h(x) = u_0\psi_0(x) + \sum_{i=1}^n y_i\psi_i(x) + u_1\psi_{n+1}(x), \quad y_i \in \mathbb{R},\, i = 1, \ldots, n\},$$

$$V_h = span\{\psi_i(x)\}_{i=1}^n.$$

Thus, seek $u_h \in U_h$ such that

$$a(u_h, \psi_i) = \int_0^1 (u_h'\psi_i' + c\, u_h\psi_i)\, dx = \int_0^1 \psi_i f\, dx, \qquad i = 1, \ldots, n. \quad (2.19)$$

To show now that the finite element approach (2.19) leads to the exact discretization (2.17), we have to compute $a(\psi_k, \psi_i)$ for $k = i-1, i, i+1$. In fact, we find the entries of the matrix

$$A = tridiag\,(a(\psi_{i-1}, \psi_i), a(\psi_i, \psi_i), a(\psi_{i+1}, \psi_i))_{n\times n}\,,$$

as

$$\begin{aligned} a(\psi_{i-1}, \psi_i) &= \int_0^1 (\psi_{i-1}'\psi_i' + c\, \psi_{i-1}\psi_i)\, dx &&= -\frac{\sqrt{c}}{\sinh(\sqrt{c}h)}, \\ a(\psi_i, \psi_i) &= \int_0^1 ((\psi_i')^2 + c\,(\psi_i)^2)\, dx &&= \frac{2\sqrt{c}\cosh(\sqrt{c}h)}{\sinh(\sqrt{c}h)}, \\ a(\psi_{i+1}, \psi_i) &= \int_0^1 (\psi_{i+1}'\psi_i' + c\, \psi_{i+1}\psi_i)\, dx &&= -\frac{\sqrt{c}}{\sinh(\sqrt{c}h)}. \end{aligned}$$

We leave the verification of the latter three relations to the reader because it only needs a certain amount of rather technical integrations.

Remarks

1. Consider the standard finite difference discretization
$$\begin{aligned} y_0 &= u_0, \\ -y_{i-1} + (2 + h^2 c) y_i - y_{i+1} &= h^2 f(x_i), \quad i = 1, \ldots, n, \\ y_{n+1} &= u_1, \end{aligned}$$
of problem (2.11) on the uniform grid $\bar{\omega}_h$. For $c > 0$ and $f(x) \not\equiv 0$ it can never be an exact discretization because the coefficient $(2 + h^2 c)$ of y_i represents only the first two terms of the Taylor series expansion
$$2\cosh(\sqrt{c}h) = 2 + ch^2 + \frac{2c^2h^4}{4!} + \cdots$$
and the right-hand side is approximated by
$$\frac{\sinh(\sqrt{c}h)}{\sqrt{c}} \int_0^1 \psi_i(\xi) f(\xi)\, d\xi \;\approx\; h^2 f(x_i).$$

2. If we let $c \to +0$ in the exact discretization (2.14), we immediately derive the exact discretization (2.5) of problem (2.2).

2.3 Exact discretization of $\quad Lu = -u'' + bu' + cu = f(x), \quad b, c \in \mathrm{IR},\ c \geq 0$

We turn our attention now to the construction of an exact discretization of the boundary value problem
$$\begin{aligned} &Lu = -u'' + bu' + cu = f(x), \quad 0 < x < 1, \\ &u(0) = u_0, \quad u(1) = u_1. \end{aligned} \tag{2.20}$$
Suppose now that
$$b, c \in \mathrm{IR} \qquad \text{with} \qquad c \geq 0, \qquad \max\{c, |b|\} > 0. \tag{2.21}$$
Then problem (2.20) has a unique solution $u(x)$.

We start with the representation of $u(x)$. From assumptions (2.21) we get that the characteristic equation
$$-\lambda^2 + b\lambda + c = 0$$

of the homogeneous differential equation $Lu = 0$ has two different real roots

$$\lambda_2 = \frac{b - \sqrt{b^2+4c}}{2} \quad < \quad \lambda_1 = \frac{b + \sqrt{b^2+4c}}{2}.$$

It holds

$$\lambda_1 + \lambda_2 = b, \qquad \lambda_1\lambda_2 = -c,$$

$$e^{\lambda_1} - e^{\lambda_2} > 0, \qquad (\lambda_1 - \lambda_2)(e^{\lambda_1} - e^{\lambda_2}) > 0.$$

Then the solution of problem (2.20) is given by

$$u(x) = u_0 \tfrac{e^{\lambda_1+\lambda_2 x} - e^{\lambda_1 x + \lambda_2}}{e^{\lambda_1} - e^{\lambda_2}} + u_1 \tfrac{e^{\lambda_1 x} - e^{\lambda_2 x}}{e^{\lambda_1} - e^{\lambda_2}} + \textstyle\int_0^1 \; G_{c,b}(x,\xi) f(\xi)\; d\xi, \tag{2.22}$$

where $G_{c,b}(x,\xi)$ is Green's function of the differential operator of problem (2.20). For $(x,\xi) \in [0,1]^2$, we have

$$G_{c,b}(x,\xi) = \begin{cases} \frac{(e^{\lambda_1+\lambda_2 x} - e^{\lambda_1 x+\lambda_2})(e^{-\lambda_2\xi} - e^{-\lambda_1\xi})}{(\lambda_1-\lambda_2)(e^{\lambda_1} - e^{\lambda_2})}, & \xi < x, \\ \frac{(e^{\lambda_1 x} - e^{\lambda_2 x})(e^{\lambda_1(1-\xi)} - e^{\lambda_2(1-\xi)})}{(\lambda_1-\lambda_2)(e^{\lambda_1} - e^{\lambda_2})}, & \xi \geq x. \end{cases} \tag{2.23}$$

In any case it holds that $G_{c,b}(x,\xi) \geq 0$ for $(x,\xi) \in [0,1]^2$.

We remark that $b = 0,\ c > 0$ implies $\lambda_1 = \sqrt{c} = -\lambda_2$, hence $G_{c,0}(x,\xi) = G_c(x,\xi)$, see (2.13).

If $b \neq 0,\ c \geq 0$, then $G_{c,b}(x,\xi) \neq G_{c,b}(\xi,x)$ for all $x \neq \xi$.

We now shall next turn to the exact three-point discretization of problem (2.20) on the uniform grid $\bar{\omega}_h$ with step size $h = \frac{1}{n+1}$. One can show that

$$\begin{aligned} &-(e^{\lambda_1 h} - e^{\lambda_2 h})\, y_{i-1} + 2\sinh((\lambda_1 - \lambda_2)h)\, y_i - (e^{-\lambda_2 h} - e^{-\lambda_1 h})\, y_{i+1} = \\ &\tfrac{2(\cosh((\lambda_1-\lambda_2)h)-1)}{\lambda_1-\lambda_2} \textstyle\int_0^1 \; \chi_i(\xi) f(\xi)\; d\xi, \qquad i = 1, \ldots, n\,, \end{aligned} \tag{2.24}$$

where $y_0 = u_0,\ y_{n+1} = u_1$ and

$$\chi_i(\xi) = \begin{cases} \frac{e^{-\lambda_2(\xi - x_{i-1})} - e^{-\lambda_1(\xi - x_{i-1})}}{e^{-\lambda_2 h} - e^{-\lambda_1 h}}, & x_{i-1} \leq \xi \leq x_i, \\ \frac{e^{\lambda_1(x_{i+1}-\xi)} - e^{\lambda_2(x_{i+1}-\xi)}}{e^{\lambda_1 h} - e^{\lambda_2 h}}, & x_i < \xi \leq x_{i+1}, \qquad i = 1, \ldots, n, \\ 0, & else, \end{cases} \tag{2.25}$$

is an exact discretization of problem (2.20).

For the proof, a calculation reveals that

$$y_i = u(x_i) = u_0 \frac{e^{\lambda_1 + \lambda_2 x_i} - e^{\lambda_1 x_i + \lambda_2}}{e^{\lambda_1} - e^{\lambda_2}} + u_1 \frac{e^{\lambda_1 x_i} - e^{\lambda_2 x_i}}{e^{\lambda_1} - e^{\lambda_2}} + \int_0^1 G_{c,b}(x_i, \xi) f(\xi)\, d\xi,$$

$$i = 1, \ldots, n,$$

is solution of the system of linear equations (2.24). It will be seen later on that this solution is unique.

We have depicted one such function $\chi_i(x)$ just defined for the values $b = 8,\ c = 12$ in Figure 2.

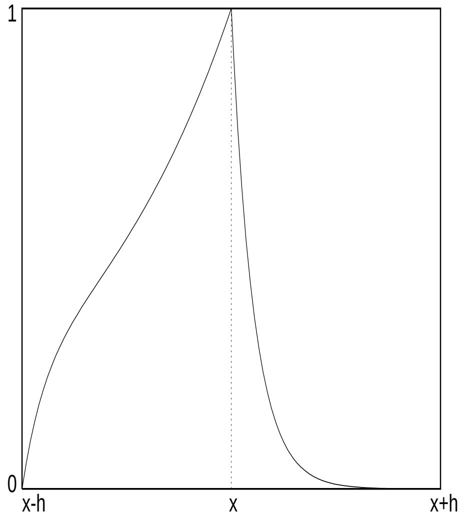

Figure 2

The idea for the left-hand side expression of exact discretization (2.24) is to use a finite difference scheme for $Lu = 0$ of type

$$-\frac{u(x-h) - 2u(x) + u(x+h)}{\alpha(h)^2} + b\frac{u(x+h) - u(x-h)}{2\beta(h)} + cu(x) = 0, \tag{2.26}$$

where $\alpha(h)^2, \beta(h)$ have to be chosen such that (2.26) is exact discretization on the uniform grid $\bar{\omega}_h$.

Assuming temporarily $b \neq 0, c > 0$, we are able to determine unique $\alpha(h)^2$ and $\beta(h)$. Substituting expression (2.22) for $f(x) \equiv 0$ in (2.26) gives

$$\begin{aligned}
\alpha(h)^2 &= \tfrac{2}{c}\, \tfrac{\sinh((\lambda_1-\lambda_2)h)+\sinh(\lambda_2 h)-\sinh(\lambda_1 h)}{\sinh(\lambda_1 h)-\sinh(\lambda_2 h)}\,, \\
\beta(h) &= \tfrac{b}{c}\, \tfrac{\sinh((\lambda_1-\lambda_2)h)+\sinh(\lambda_2 h)-\sinh(\lambda_1 h)}{\cosh(\lambda_1 h)-\cosh(\lambda_2 h)}\,.
\end{aligned}$$

Making use of the latter two expressions in (2.26) and multiplying then (2.26) through by $(\sinh((\lambda_1 - \lambda_2)h) + \sinh(\lambda_2 h) - \sinh(\lambda_1 h))/c$ gives the left-hand side expression of (2.24) after rearranging the coefficients.

Next we describe the exact discretization (2.24) in matrix form. Define

$$\begin{aligned}
\rho &= \frac{\lambda_1 - \lambda_2}{2(\cosh((\lambda_1 - \lambda_2)h) - 1)}, \\
A &= \rho\ tridiag\left(-(e^{\lambda_1 h} - e^{\lambda_2 h}), 2\sinh((\lambda_1 - \lambda_2)h), -(e^{-\lambda_2 h} - e^{-\lambda_1 h})\right)_{n\times n}, \\
y &= (y_1, \ldots, y_n)^T, \\
b &= (b_1, \ldots, b_n)^T, \\
b_1 &= \rho\,(e^{\lambda_1 h} - e^{\lambda_2 h})\, u_0 + \int_0^1 \chi_1(\xi) f(\xi)\, d\xi, \\
b_i &= \int_0^1 \chi_i(\xi) f(\xi)\, d\xi, \qquad i = 2, \ldots, n-1, \\
b_n &= \rho\,(e^{-\lambda_2 h} - e^{-\lambda_1 h})\, u_1 + \int_0^1 \chi_n(\xi) f(\xi)\, d\xi.
\end{aligned}$$

Thus, we may write the exact discretization (2.24) as

$$Ay = b. \tag{2.27}$$

The matrix A is an irreducible tridiagonal M-matrix.

To see this, we remark that $\lambda_2 \leq 0 < \lambda_1$ implies

$$\rho > 0, \quad e^{\lambda_1 h} - e^{\lambda_2 h} > 0, \quad e^{-\lambda_2 h} - e^{-\lambda_1 h} > 0, \quad 2\sinh((\lambda_1 - \lambda_2)h) > 0,$$

so that A is irreducible, off-diagonally nonpositive and diagonally positive. Hence, A is an irreducible L-matrix. The proof is complete if we can show that A has diagonal dominance property.

We get that A is weakly row diagonally dominant in the case $c = 0,\ b \neq 0$, and, that A is strictly row diagonally dominant if $c > 0,\ b \in \mathbb{R}$.

For this, let r_i, $i = 1, \ldots, n$, denote the row sums of A. In any case we have $r_1 > 0$ and $r_n > 0$. For $i = 2, \ldots, n-1$ we get

$$\begin{aligned} r_i &= 2\sinh((\lambda_1 - \lambda_2)h) - (e^{\lambda_1 h} - e^{\lambda_2 h}) - (e^{-\lambda_2 h} - e^{-\lambda_1 h}) \\ &= -8\sinh\left(\tfrac{\lambda_1 h}{2}\right)\ \sinh\left(\tfrac{\lambda_2 h}{2}\right)\ \sinh\left(\tfrac{(\lambda_1 - \lambda_2)h}{2}\right). \end{aligned}$$

The case $c = 0,\ b \neq 0$ implies $\lambda_2 = 0$ and hence $r_i = 0$ for $i = 2, \ldots, n-1$. Then the matrix A is an irreducible and weakly row diagonally dominant L-matrix.

Assuming $c > 0$, we get $\lambda_2 < 0 < \lambda_1$. Thus, $r_i > 0$ for $i = 2, \ldots, n-1$ and the matrix A is a strictly row diagonally dominant L-matrix.

Therefore, under assumption (2.21) the matrix A is an irreducible tridiagonal M-matrix with $A^{-1} > 0$. This proves that the given solution $y_i = u(x_i)$ for $i = 0, \ldots, n+1$, of the exact discretization (2.24) is unique.

The M-matrix A is symmetic if and only if $b = 0$ holds.

To see this remember that $b = 0$ implies $0 < \sqrt{c} = \lambda_1 = -\lambda_2$. Putting this in the definition of the matrix A yields

$$A = 2\rho \quad tridiag(-\sinh(\sqrt{c}h), \sinh(2\sqrt{c}h), -\sinh(\sqrt{c}h))_{n\times n} = A^T$$

For $b \neq 0$ it follows that $\lambda_1 + \lambda_2 = b \neq 0$. Hence $A \neq A^T$ because of

$$e^{-\lambda_2 h} - e^{-\lambda_1 h} = \frac{e^{\lambda_1 h} - e^{\lambda_2 h}}{e^{(\lambda_1 + \lambda_2)h}} = \frac{e^{\lambda_1 h} - e^{\lambda_2 h}}{e^{bh}} \neq e^{\lambda_1 h} - e^{\lambda_2 h}.$$

Let the case $b \neq 0$, $c = 0$ briefly catch our attention. This assumption implies

$$\lambda_1 = \frac{b + |b|}{2} = b^+ \geq 0, \quad \lambda_2 = \frac{b - |b|}{2} = b^- \leq 0, \quad \lambda_1 - \lambda_2 = |b| > 0.$$

Then the matrix A has the form

$$A = \rho \quad tridiag(-(e^{b^+h} - e^{b^-h}), 2\sinh(|b|h), -(e^{-b^-h} - e^{-b^+h}))_{n\times n} \neq A^T,$$

where $\rho = \frac{|b|}{2(\cosh(|b|h)-1)}$.

Our next goal is to show that the exact discretization (2.27) of problem (2.20) also results from the application of Galerkin's method. The crucial question, of course, is to adapt the basic function system to the boundary value problem (2.20). The hint how to choose the best basic function system comes directly from the right-hand side terms of the exact discretization (2.24).

We start with a weak formulation of problem (2.20). Let U, V be the function sets defined in Section 2.1. Then, seek $u(x) \in U$ such that

$$a(u,v) = \int_0^1 (u'v' + b\ u'v + c\ uv)\ dx = \int_0^1 vf\ dx, \quad \forall\ v \in V.$$

To derive a finite dimensional approximation of the just stated weak formulation of problem (2.20), we first supplement the set of function (2.25) with

$$\chi_0(\xi) = \begin{cases} \frac{e^{\lambda_1(x_1-\xi)} - e^{\lambda_2(x_1-\xi)}}{e^{\lambda_1 h} - e^{\lambda_2 h}}, & 0 \le \xi \le x_1, \\ \\ 0, & else, \end{cases}$$

$$\chi_{n+1}(\xi) = \begin{cases} \frac{e^{-\lambda_2(\xi-x_n)} - e^{-\lambda_1(\xi-x_n)}}{e^{-\lambda_2 h} - e^{-\lambda_1 h}}, & x_n \le \xi \le 1, \\ \\ 0, & else. \end{cases}$$

We now define

$$U_h = \{u_h(x) = u_0\chi_0(x) + \sum_{i=1}^n\ y_i\chi_i(x) + u_1\chi_{n+1}(x),\ y_i \in \mathrm{IR},\ i = 1,\ldots,n\},$$

$$V_h = span\{\chi_i(x)\}_{i=1}^n.$$

Then, seek $u_h(x) \in U_h$ which satisfies

$$a(u_h, \chi_i) = \int_0^1 (u_h' \chi_i' + b\ u_h' \chi_i + c\ u_h \chi_i)\ dx = \int_0^1\ \chi_i f\ dx, \qquad i = 1, \ldots, n. \tag{2.28}$$

Now we can show that the system of linear equations (2.28) is equivalent to (2.27) which represents the exact discretization of problem (2.20). It holds true that

$$A = tridiag(a(\chi_{i-1}, \chi_i), a(\chi_i, \chi_i), a(\chi_{i+1}, \chi_i))_{n\times n}.$$

The computation of the integrals $a(\chi_k, \chi_i)$ for $k = i-1, i, i+1$, which define the nonzero entries of the matrix A, proves the assertion.

We get

$$\begin{aligned} a(\chi_{i-1}, \chi_i) &= \int_0^1 (\chi_{i-1}' \chi_i' + b\chi_{i-1}' \chi_i + c\ \chi_{i-1}\chi_i)\ dx &= -\ \tfrac{(\lambda_1-\lambda_2)(e^{\lambda_1 h} - e^{\lambda_2 h})}{2(\cosh((\lambda_1-\lambda_2)h)-1)}, \\ a(\chi_i, \chi_i) &= \int_0^1 ((\chi_i')^2 + b\ \chi_i'\chi_i + c\ (\chi_i)^2)\ dx &= \tfrac{2(\lambda_1-\lambda_2)\sinh((\lambda_1-\lambda_2)h)}{2(\cosh((\lambda_1-\lambda_2)h)-1)}, \\ a(\chi_{i+1}, \chi_i) &= \int_0^1 (\chi_{i+1}'\chi_i' + b\ \chi_{i+1}'\chi_i + c\ \chi_{i+1}\chi_i)\ dx &= -\ \tfrac{(\lambda_1-\lambda_2)(e^{-\lambda_2 h} - e^{-\lambda_1 h})}{2(\cosh((\lambda_1-\lambda_2)h)-1)}. \end{aligned}$$

As in all previous cases, the computation of the latter three terms only needs elementary integrations. The reader will find out no difficulty in doing this.

3. Exact discretization of $Lu = -(p(x)u')' = f(x)$

Assume $p(x) > 0$ for $x \in [0,1]$ with

$$0 < q = \int_0^1 \frac{dt}{p(t)} < \infty. \tag{3.1}$$

Then, the unique solution $u(x)$ of

$$\begin{aligned} &Lu = -(p(x)u')' = f(x), \quad 0 < x < 1, \\ &u(0) = u_0, \quad u(1) = u_1, \end{aligned} \tag{3.2}$$

has the form

$$u(x) = \frac{u_0}{q} \int_x^1 \frac{dt}{p(t)} + \frac{u_1}{q} \int_0^x \frac{dt}{p(t)} + \int_0^1 G_p(x,\xi) f(\xi) d\xi\ , \tag{3.3}$$

with Green's function

$$G_p(x,\xi) = \begin{cases} \frac{1}{q}\int_0^\xi \frac{dt}{p(t)} \int_x^1 \frac{dt}{p(t)}, & \xi < x, \\ \\ \frac{1}{q}\int_\xi^1 \frac{dt}{p(t)} \int_0^x \frac{dt}{p(t)}, & \xi \ge x. \end{cases} \tag{3.4}$$

We mention that Green's function $G_p(x,\xi)$ of the differential operator of problem (3.2) is symmetric and nonnegative on $[0,1]^2$.

Let $\bar{\omega}_h = \{0 = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = 1\}$ and define

$$p_i = \int_{x_{i-1}}^{x_i} \frac{dt}{p(t)}\,, \qquad i = 1,\dots,n+1\,. \tag{3.5}$$

It can be shown that the three-point discretization of problem (3.2)

$$\begin{aligned} &-\tfrac{1}{p_i}\; y_{i-1} + \left(\tfrac{1}{p_i} + \tfrac{1}{p_{i+1}}\right)\; y_i - \tfrac{1}{p_{i+1}}\; y_{i+1} = \int_0^1\; \eta_i(\xi) f(\xi)\; d\xi, \\ &i = 1,\dots,n, \end{aligned} \tag{3.6}$$

with $y_0 = u_0,\;\; y_{n+1} = u_1$ and

$$\eta_i(\xi) = \begin{cases} \frac{1}{p_i q}\left(\int_{x_{i-1}}^1 \frac{dt}{p(t)} \int_0^\xi \frac{dt}{p(t)} - \int_0^{x_{i-1}} \frac{dt}{p(t)} \int_\xi^1 \frac{dt}{p(t)}\right), & x_{i-1} \le \xi \le x_i, \\ \frac{1}{p_{i+1} q}\left(\int_0^{x_{i+1}} \frac{dt}{p(t)} \int_\xi^1 \frac{dt}{p(t)} - \int_{x_{i+1}}^1 \frac{dt}{p(t)} \int_0^\xi \frac{dt}{p(t)}\right), & x_i < \xi \le x_{i+1}, \\ 0, & else, \end{cases} \tag{3.7}$$

for $i = 1,\dots,n$ is exact for any $n \ge 1$.

All we need to prove is that

$$\begin{aligned} &y_i = u(x_i) = \tfrac{u_0}{q}\;\int_{x_i}^1 \tfrac{dt}{p(t)} + \tfrac{u_1}{q}\;\int_0^{x_i} \tfrac{dt}{p(t)} + \int_0^1 G_p(x_i,\xi) f(\xi) d\xi, \\ &i = 1,\dots,n, \end{aligned} \tag{3.8}$$

satisfy the system of linear equations (3.6). We leave the details of the proof to the reader because it requires

To analyse qualitative properties of the system of linear equations (3.6), we next rewrite it in matrix form. Setting

$$\begin{aligned}
A &= tridiag\left(-\frac{1}{p_i}\ ,\ \frac{1}{p_i}+\frac{1}{p_{i+1}}\ ,\ -\frac{1}{p_{i+1}}\right)_{n\times n},\\
y &= (y_1,\ldots,y_n)^T,\\
b &= (b_1,\ldots,b_n)^T,\\
b_1 &= \frac{u_0}{p_1}+\int_0^1 \eta_1(\xi)f(\xi)\ d\xi,\\
b_i &= \int_0^1 \eta_i(\xi)f(\xi)\ d\xi, \qquad\qquad i=2,\ldots,n-1,\\
b_n &= \frac{u_1}{p_{n+1}}+\int_0^1 \eta_n(\xi)f(\xi)\ d\xi.
\end{aligned}$$

we get

$$Ay = b \tag{3.9}$$

as short form of (3.6), where $A = A^T$ is an irreducible, weakly diagonally dominant M-matrix. Hence, the solution (3.8) is unique.

We shall now derive the just obtained exact discretization of problem (3.2) from the application of a finite element method which uses the function system $\{\eta_i(x)\}$ defined by (3.7).

Let U and V be the same as in Section 2.1 and define

$$a(u,v) = \textstyle\int_0^1\ p(x)u'v'dx.$$

Then we seek a function $u \in U$ such that

$$a(u,v) = \textstyle\int_0^1\ p(x)u'v'\ dx\ \ =\ \ \int_0^1\ vf\ dx, \qquad \forall v\in V. \tag{3.10}$$

Before we are going to formulate a finite dimensional approximation of the weak form (3.10) on the grid $\bar{\omega}_h$, we define

$$\eta_0(\xi) = \begin{cases} \frac{1}{p_1 q}\left(\int_0^{x_1}\frac{dt}{p(t)}\int_\xi^1\frac{dt}{p(t)} - \int_{x_1}^1\frac{dt}{p(t)}\int_0^\xi\frac{dt}{p(t)}\right), & 0\le\xi\le x_1,\\[2ex] 0, & else,\end{cases}$$

$$\eta_{n+1}(\xi) = \begin{cases} \frac{1}{p_{n+1} q}\left(\int_{x_n}^{1}\frac{dt}{p(t)}\int_0^\xi\frac{dt}{p(t)} - \int_0^{x_n}\frac{dt}{p(t)}\int_\xi^1\frac{dt}{p(t)}\right), & x_n\le\xi\le 1\\[2ex] 0, & else.\end{cases}$$

Now let

$$U_h = \{u_h(x) = u_0\eta_0(x) + \sum_{i=1}^{n} \ y_i\eta_i(x) + u_1\eta_{n+1}(x), \quad y_i \in \mathrm{IR},\ i = 1, \ldots, n\},$$

$$V_h = span\{\eta_i(x)\}_{i=1}^{n},$$

and seek $u_h \in U_h$ such that

$$a(u_h, \eta_i) = \int_0^1 \ p(x)u_h'\eta_i' dx = \int_0^1 \ \eta_i f \ dx, \qquad i = 1, \ldots, n. \tag{3.11}$$

From (3.11) now results the exact discretization (3.8) of problem (3.2). To see this, we prove that

$$A = tridiag\left(a(\eta_{i-1}, \eta_i),\ a(\eta_i, \eta_i),\ a(\eta_{i+1}, \eta_i)\right)_{n\times n}.$$

A straightforward calculation shows that

$$\begin{aligned} a(\eta_{i-1}, \eta_i) &= \int_0^1 \ p(x)\eta_{i-1}'\eta_i' \ dx &&= -\ \tfrac{1}{p_i}, \\ a(\eta_i, \eta_i) &= \int_0^1 \ p(x)(\eta_i')^2 \ dx &&= \tfrac{1}{p_i} + \tfrac{1}{p_{i+1}}, \\ a(\eta_{i+1}, \eta_i) &= \int_0^1 \ p(x)\eta_{i+1}'\eta_i' \ dx &&= -\ \tfrac{1}{p_{i+1}}\ . \end{aligned}$$

Remarks

1. Assuming $p(x) \equiv 1$, we get immediately $p_i = h_i = x_i - x_{i-1}$ and $\eta_i(x) \equiv \phi_i(x)$ for all indices i, see (3.5) and (3.7), respectively. In this case, the exact approximation (3.6) is identical with the exact approximation (2.5) of problem (2.2), see Section 2.1 .
2. Problem (3.2) also covers boundary layer problems. For example, consider problem (3.2) with $f(x) \equiv 0$ and $p(x) = e^{-bx}$. Then the differential equation reduces to $-u'' + bu' = 0$, where for $|b| \gg 1$ boundary layers may occure.

4. Exact discretization of $Lu = -(\Phi(u))'' = f(x)$

Consider the nonlinear boundary value problem

$$\begin{aligned} &Lu = -(\Phi(u))'' = f(x), \quad 0 < x < 1, \\ &u(0) = u_0, \quad u(1) = u_1, \end{aligned} \tag{4.1}$$

with

$$\Phi(u) \in C^1(\mathrm{I\!R}) \qquad \text{such that} \quad \Phi'(u) > 0, \, \forall u \in \mathrm{I\!R}\,. \tag{4.2}$$

Then there exists Φ^{-1} which we need in the sequel.

Remark that

$$-(k(u)u')' = f(x) \qquad \text{with} \quad k(u) > 0$$

can be transformed via $\Phi(u) = \int_0^u k(t)\; dt$ into the differential equation of problem (4.1).

One checks easily now that

$$u(x) = \Phi^{-1}\left((1-x)\Phi(u_0) + x\Phi(u_1) + \int_0^1 G(x,\xi)f(\xi)d\xi\right) = \tag{4.3}$$

$$\Phi^{-1}\left((1-x)\Phi(u_0) + x\Phi(u_1) + (1-x)\int_0^x \xi f(\xi)\; d\xi + x\int_x^1 (1-\xi)f(\xi)\; d\xi\right)$$

is the solution of problem (4.1), where $G(x,\xi)$ is Green's function of the differential operator of problem (2.2), see Section 2.1.

Let $\bar{\omega}_h = \{0 = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = 1\}$ with $h_i = x_i - x_{i-1} > 0$ for $i = 1, \ldots, n+1$. It turns out that the following three-point discretization of problem (4.1)

$$\begin{aligned} &-\tfrac{1}{h_i}\; \Phi(y_{i-1}) + \left(\tfrac{1}{h_i} + \tfrac{1}{h_{i+1}}\right)\; \Phi(y_i) - \tfrac{1}{h_{i+1}}\; \Phi(y_{i+1}) = \int_0^1 \; \phi_i(\xi)f(\xi)\; d\xi, \\ &i = 1, \ldots, n, \end{aligned} \tag{4.4}$$

with $y_0 = u_0,\;\; y_{n+1} = u_1$ and the system of basis functions $\{\phi_i(x)\}_{i=1}^n$ defined by (2.5) is an exact discretization of problem (4.1) for any $n \geq 1$.

Substituting

$$y_i = u(x_i) = \Phi^{-1}\left((1-x_i)\Phi(u_0) + x_i\Phi(u_1) + \textstyle\int_0^1 G(x_i,\xi)f(\xi)\ d\xi\ \right), \qquad i = 1,\ldots,n, \tag{4.5}$$

in (4.4) and applying the results of Section 2.1 gives the proof.

For $\Phi(u) \not\equiv u$ the exact discretization (4.4) of problem (4.1) is a system of nonlinear equations.

Next we shall rewrite the nonlinear equation system (4.4) in short form. For this purpose let

$$\begin{aligned}
F(y) &= (f_1(y),\ldots,f_n(y))^T\ :\ \mathrm{IR}^n \to\ \mathrm{IR}^n,\\
y &= (y_1,\ldots,y_n)^T,\\
b &= (b_1,\ldots,b_n)^T,\\
f_1(y) &= \left(\frac{1}{h_1}+\frac{1}{h_2}\right)\ \Phi(y_1) - \frac{1}{h_2}\ \Phi(y_2),\\
f_i(y) &= -\frac{1}{h_i}\ \Phi(y_{i-1}) + \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)\ \Phi(y_i) - \frac{1}{h_{i+1}}\ \Phi(y_{i+1}), \quad i = 2,\ldots,n,\\
f_n(y) &= -\frac{1}{h_n}\ \Phi(y_{n-1}) + \left(\frac{1}{h_n}+\frac{1}{h_{n+1}}\right)\ \Phi(y_n),\\
b_1 &= \frac{1}{h_1}\ \Phi(u_0) + \int_0^1\ \phi_1(\xi)f(\xi)\ d\xi,\\
b_i &= \int_0^1\ \phi_i(\xi)f(\xi)\ d\xi, \quad i = 2,\ldots,n,\\
b_n &= \frac{1}{h_{n+1}}\ \Phi(u_1) + \int_0^1\ \phi_n(\xi)f(\xi)\ d\xi.
\end{aligned}$$

The exact discretization (4.4) now becomes

$$F(y) = b. \tag{4.6}$$

The function $F(y)$ is an M-function on $\mathrm{I\!R}^n$ because

$$
\begin{aligned}
F'(y) &= tridiag\left(-\tfrac{1}{h_i}\ ,\ \tfrac{1}{h_i}+\tfrac{1}{h_{i+1}}\ ,\ -\tfrac{1}{h_{i+1}}\right)_{n\times n} diag\left(\Phi'(y_1),\dots,\Phi'(y_n)\right)_{n\times n} \\
&= A\,\phi(y)
\end{aligned}
$$

is an M-matrix for all $y \in R^n$. This follows directly from the fact that A is an M-matrix, see Section 2.1, and that

$$\phi(y) = diag\left(\Phi'(y_1),\dots,\Phi'(y_n)\right) \geq 0 \quad \text{with} \quad \det\,\phi(y) > 0 \quad \forall y \in \mathrm{I\!R}^n.$$

Thus $F(y) = b$ has at most one solution. Its unique solution is given by (4.5).

5. Exact discretization of some nonlinear differential operators

In this section we shall briefly describe exact discretizations of two nonlinear boundary value problems. To do this, it is necessary to have expressions of the solutions at which a discretization may be exact. In each case we assume a uniform grid $\bar{\omega}_h$ with step size $h = \frac{1}{n+1}$.

5.1 Exact discretization of $Lu = -u'' + \frac{3}{2}u^2 = 0$

Consider

$$
\begin{aligned}
&Lu = -u'' + \tfrac{3}{2}u^2 = 0, \quad 0 < x < 1, \\
&u(0) = 4, \quad u(1) = 1,
\end{aligned}
\tag{5.1}
$$

see [1]. Problem (5.1) has two solutions

$$u_1(x) = \frac{4}{(1+x)^2}, \qquad u_2(x) \text{ is an elliptic function.} \tag{5.2}$$

It can be seen now that

$$
\begin{aligned}
y_0 &= 4, \\
-\frac{y_{i-1}-2y_i+y_{i+1}}{h^2} + \tfrac{3}{2}y_{i-1}y_{i+1}\left(1-\tfrac{h^2}{12}y_i\right) &= 0, \qquad i = 1,\dots,n, \\
y_{n+1} &= 1,
\end{aligned}
\tag{5.3}
$$

is exact discretization of problem (5.1) at its solution $u_1(x)$ because

$$y_i = u(x_i) = \frac{4}{(1+x_i)^2}, \qquad i = 0, \ldots, n+1, \tag{5.4}$$

is a solution of the nonlinear equation system (5.3). The interesting fact is, that we have to approximate the term u^2 over three grid points. At the other solution $u_2(x)$ of problem (5.1) the given approximation (5.3) is not exact. Nevertheless, the system of nonlinear equations (5.3) has not only the solution (5.4).

5.2 Exact discretization of $\quad Lu = -\epsilon u'' - uu' = 0, \quad \epsilon > 0$

Consider the boundary value problem

$$\begin{aligned} &Lu = -\epsilon u'' - uu' = 0, \quad 0 < x < 1, \\ &u(0) = 0, \quad u(1) = \tanh\left(\tfrac{1}{2\epsilon}\right), \end{aligned} \tag{5.5}$$

which has the solution

$$u(x) = \tanh\left(\tfrac{x}{2\epsilon}\right). \tag{5.6}$$

The discretization of problem (5.5)

$$\begin{aligned} y_0 &= 0, \\ -y_{i-1} + 2y_i - y_{i+1} - y_i\left(y_{i+1} - y_{i-1}\right)s(h) &= 0, \qquad i = 1, \ldots, n, \\ y_{n+1} &= \tanh\left(\tfrac{1}{2\epsilon}\right), \end{aligned} \tag{5.7}$$

is exact at solution (5.6), where

$$s(h) = \frac{\cosh\left(\frac{h}{\epsilon}\right) - 1}{\sinh\left(\frac{h}{\epsilon}\right)} = \frac{1}{2}\frac{h}{\epsilon} - \frac{1}{24}\left(\frac{h}{\epsilon}\right)^3 + O\left(\frac{h}{\epsilon}\right)^5, \tag{5.8}$$

which satisfies

$$s(h) \in (-1, 1), \quad s'(h) = \frac{1}{\epsilon(\cosh(\frac{h}{\epsilon}) + 1)} > 0,$$

The solution of the system of nonlinear equation (5.7) is given by

$$y_i = u(x_i) = \tanh\left(\tfrac{x_i}{2\epsilon}\right), \qquad i = 0, \ldots, n+1. \tag{5.9}$$

6. Conclusions

We are now going to comment the results and point out some further problems.

All of the two–point boundary value problems stated in Sections 2 to 4 exhibit the property of inverse isotonicity (sind "von monotoner Art", see [1]). This means

$$Lu = f(x) \leq Lv = g(x), \qquad 0 < x < 1,$$

$$u(0) = u_0 \leq v(0) = v_0, \qquad u(1) = u_1 \leq v(1) = v_1,$$

implies

$$u(x) \leq v(x), \quad 0 \leq x \leq 1.$$

The representations of its solution (2.3), (2.12), (2.22), (3.3) and (4.3) confirm these assertions directly.

The presented exact three–point discretizations of the boundary value problems (2.2), (2.11), (2.20) and (3.2) yield in any case systems of linear equations $Ay = b$, where A is a tridiagonal M-matrix, see [5]. The system matrix A is symmetric if the Green's function of the corresponding differential operator is symmetric and it is not symmtric vice versa. Furthermore, the exact discretization of the nonlinear inverse isotone problem (4.1) yields a nonlinear system of equations $F(y) = b$, where $F(y)$ is an M-function, see [5].

In each case, the right-hand side vector $b = b(u_0, u_1, f(x))$ of the exact discretizations is isotone according to its arguments, i.e.

$$u_0 \leq v_0, \quad u_1 \leq v_1, \quad f(x) \leq g(x), \quad \forall x \in [0,1],$$

implies

$$b(u_0, u_1, f(x)) \leq b(v_0, v_1, g(x)).$$

Hence, we get for the solutions of the systems of equations $Ay = b$ that

$$A^{-1}b(u_0, u_1, f(x)) \leq A^{-1}b(v_0, v_1, g(x))$$

which reflects the inverse isotonicity in the exact discretizations. The same assertion holds true for $y = F^{-1}(b)$ because the inverse function F^{-1} of an M-function $F(y)$ exists and is isotone, i.e.

$$F^{-1}(b(u_0, u_1, f(x))) \leq F^{-1}(b(v_0, v_1, g(x))).$$

Similar results are possible if the boundary conditions of first kind are replaced by other types of boundary conditions, for instance, by boundary conditions of second or third type, or by mixed.

One of the further questions is how to apply this results if the differential equation contains variable coefficients.

We remark that it is also possible to construct exact $(2s+1)$–point discretizations with $s \geq 2$ over a uniform grid $\bar{\omega}_h$ for the stated two–point boundary value problems. For linear problems, the resulting system of linear equations $Ay = b$ involves a banded monotone matrix A which also guaranties the inverse isotonicity of the exact discretizations.

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