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Canonical Forms for Linear Descriptor Systems with Variable Coefficients*

Abstract

We study linear descriptor systems with rectangular variable coefficient matrices. Using local and global equivalence transformations we introduce normal and condensed forms and get sets of characteristic quantities. These quantities allow us to decide whether a linear descriptor system with variable coefficients is regularizable by derivative and/or proportional state feedback or not. Regularizable by feedback means for us that their exist a feedback which makes the closed loop system uniquely solvable for every consistent initial vector.

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1 Introduction

In this paper, we study descriptor systems with linear variable coefficient

$$E(t)\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1}$$

in the interval $[t_1, t_2] \subset \mathcal{R}$ together with an initial condition

$$x(t_0) = x_0. \tag{2}$$

Let $C^r([t_1, t_2], \mathcal{C}^{n,l})$ denote the set of r-times continuously differentiable functions from the interval $[t_1, t_2]$ to the vector space $\mathcal{C}^{n,l}$ of complex $n \times l$ matrices. We assume that

$$\begin{array}{l} E(t),\ A(t) \in C([t_1, t_2], \mathcal{C}^{n,l}), \\ B(t) \in C([t_1, t_2], \mathcal{C}^{n,m}), \\ x(t) \in C([t_1, t_2], \mathcal{C}^{l}), \\ u(t) \in C([t_1, t_2], \mathcal{C}^{m}) \end{array} \tag{3}$$

and $B(t)$ has full column rank for all $t \in [t_1, t_2]$. $x(t)$ is called the state and $u(t)$ the control of the system.

Descriptor systems of the form (1) arise naturally in a variety of circumstances, i.e. they are used in modelling of mechanical multibody systems [31, 32] and electrical circuits [19].

The constant coefficient case shows that one has to have first a good understanding of the behaviour of the corresponding differential algebraic equations (DAEs). For a square constant coefficient system ($n = l$)

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{4}$$

it is well known that the behaviour of the system (4),(2) (and the corresponding DAE) depends upon the properties of the matrix pencil

$$\alpha E - \beta B. \tag{5}$$

The system (4) and the corresponding pencil (5) are called regular if

$$\det(\alpha A - \beta B) \neq 0 \text{ for some } (\alpha, \beta) \in \mathcal{C}^2. \tag{6}$$

While regularity of the system (4) guarantees the existence and uniqueness of classical solutions [7, 1], this is not true for the system (1) with variable coefficients [18, 22].

The constant coefficient system (4) and the corresponding pencil (5) are said to have index at most one if the dimension of the largest nilpotent block in the Kronecker canonical form of the pencil (5) is less than or equal to one (see e.g. [1, 14, 33]). For higher index descriptor systems (4) impulses can arise if the control is not sufficiently smooth or the system can even lose causality (see [16, 17, 34]). Therefore, one is interested in a proportional and/or derivative feedback for which the closed loop system is regular and at most of index one to guarantee existence and uniqueness of the solution and to avoid impulsive modes [3, 5].

The main difficulty in understanding the DAE that corresponds to the descriptor systems (1) is that different generalizations of the concepts of solvability, index, etc from constant DAEs to variable coefficient DAEs are possible and have been discussed in the literature [1, 18, 20, 22]. These different concepts can be used as a basis for different results for linear descriptor systems with variable coefficients. Until today, only few results have been achieved in this direction. The results in [11, 12], for example, use the solvability concepts for DAEs as described in [1, 8, 9, 10].

In a series of articles, Kunkel and Mehrmann discussed a more general solvability concept and presented new canonical forms for linear DAEs with variable coefficients [22, 24]. Furthermore, they presented new numerical methods based on an index reduction process [23]. Recently, Rabier and Rheinboldt generalized this approach [27] and in [29] they showed that, as in the constant coefficient case, impulse modes can only occur for higher index systems.

We will briefly discuss the main results from [22, 24] in Section 2.

In Section 3 and 4 we show that analogous methods can be used to study linear descriptor systems with variable coefficients. First, we obtain local characteristic quantities and local canonical forms for the system (1) in Section 3. Then in Section 4, we show that this local quantities can be used to study the global properties of the system and we end up with global canonical forms from which we can read off system properties.

Finally in Section 5 we study under which conditions a linear descriptor system with variable coefficients is regularizable. That means we give necessary and sufficient conditions for the existence of derivative and/or proportional state feedback so that the closed loop system is uniquely solvable for all consistent initial values. Furthermore Section 5 shows how we can get in theory a closed loop system of index at most one.

2 Canonical forms for linear differential–equations with variable coefficients

We begin our analysis of the descriptor system (1), (2) with a short look at canonical forms for differential–algebraic equations (DAEs) with variable coefficients [22, 24]. These DAEs are of the form

$$E(t)\dot{x}(t) = A(t)x(t) + f(t),\ t \in [t_1, t_2] \subset \mathcal{R} \tag{7}$$

with initial condition (2), E, A as in (3) and $f \in C([t_1, t_2], \mathcal{C}^n)$.

The standard variable coefficient transformations that can be applied to linear DAEs are changes of bases, i.e. $x(t) = Q(t)y(t)$ and pre–multiplication of (7) by $P(t)$. Equation (7) then transforms to

$$P(t)E(t)Q(t)\dot{x}(t) = \left(P(t)A(t)Q(t) - P(t)E(t)\dot{Q}(t)\right)x(t) + P(t)x(t) \tag{8}$$

and we get the following definition of equivalence for pairs of matrix functions.

Definition 1 *Two pairs of matrix functions* $(E_i(t), A_i(t))$, $E_i, A_i \in C([t_1, t_2], \mathcal{C}^{n,l})$, $i = 1, 2$ *are called equivalent if there are* $P \in C([t_1, t_2], \mathcal{C}^{n,n})$ *and* $Q \in C^1([t_1, t_2], \mathcal{C}^{l,l})$ *with* $P(t), Q(t)$ *nonsingular for all* $t \in [t_1, t_2]$ *such that*

$$(E_2(t), A_2(t)) = P(t)(E_1(t), A_1(t)) \begin{bmatrix} Q(t) & -\dot{Q}(t) \\ 0 & Q(t) \end{bmatrix}. \tag{9}$$

This approach is very useful for an analysis of DAEs, but for a numerical solution the derivative $\dot{Q}(t)$ creates difficulties. Taking into account that at a fixed point $t \in [t_1, t_2]$ we can choose $Q(t)$ and $\dot{Q}(t)$ independently [15, 22] one obtains the following definition of equivalence for constant pencils.

Definition 2 *Two pairs of matrices* (E_i, A_i), $E_i, A_i \in \mathcal{C}^{n,l}$, $\quad i = 1, 2$ *are called equivalent if there are matrices* $P \in \mathcal{C}^{n,n}$, Q, $R \in \mathcal{C}^{l,l}$ *with* P, Q *nonsingular such that*

$$(E_2, A_2) = P(E_1, A_1) \begin{bmatrix} Q & -R \\ 0 & Q \end{bmatrix}. \tag{10}$$

For this local equivalence we find in [24] the following canonical form:

Theorem 3 *Let* $E, A \in \mathcal{C}^{n,l}$ *and*

$$\begin{array}{ll} (a) & T \text{ basis of } \operatorname{kernel} E \\ (b) & Z \text{ basis of } \operatorname{corange} E = \operatorname{kernel} E^* \\ (c) & T' \text{ basis of } \operatorname{cokernel} E = \operatorname{range} E^* \\ (d) & V \text{ basis of } \operatorname{corange}(Z^*AT). \end{array} \tag{11}$$

Then, the quantities (with the convention $\operatorname{rank} \emptyset = 0$*)*

$$\begin{array}{lll} (a) & r = \operatorname{rank} E & \text{(rank)} \\ (b) & a = \operatorname{rank}(Z^*AT) & \text{(algebraic part)} \\ (c) & s = \operatorname{rank}(V^*Z^*AT') & \text{(strangeness)} \\ (d) & d = r - s & \text{(differential part)} \\ (e) & u^l = n - r - a - s & \text{(left undetermined part)} \\ (f) & u^r = l - r - a - s & \text{(right undetermined part)} \end{array} \tag{12}$$

are invariant under (9) and (E, A) *is equivalent to the canonical form*

$$\left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{array}{l} s \\ d \\ a \\ s \\ u \end{array} \tag{13}$$

where the last block column in both matrices has width u^r.

Applying now the results for the local canonical form (13) to equation (7) one obtains functions $r, a, s, u^l, u^r : [t_1, t_2] \to \mathcal{N}_0$. Currently we do not know in general how to characterize points, where these quantities change their values with t. For these reasons, we exclude such phenomena by assuming

$$r(t) \equiv r,\ a(t) \equiv a,\ s(t) \equiv s,\ u^l(t) \equiv u^l,\ u^r(t) \equiv u^r. \tag{14}$$

For analytic matrix functions $E(t), A(t)$ the functions $r(t), a(t), s(t), u^l(t), u^r(t)$ change their values only at isolated points and for the theory such points do not cause any problems.[1] For nonanalytic matrix functions $E(t), A(t)$ a characterization of such points is still under investigation. Recently, Rabier and Rheinboldt [28, 29] generalized the approach of [22, 24] and studied generalized (weak) solutions of the DAE (7).

Applying equivalence (9) to the pair $(E(t), A(t))$ we obtain from [24] the following canonical form:

Theorem 4 *Let E, A as in (3) and let (14) hold. Then $(E(t), A(t))$ is equivalent to a pair of matrix functions of the form*

$$\begin{array}{c} \left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12}(t) & 0 & A_{14}(t) & A_{15}(t) \\ 0 & 0 & 0 & A_{24}(t) & A_{25}(t) \\ 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \end{array} \begin{array}{c} s \\ d \\ a \\ s \\ u^l \end{array} \tag{15}$$

where the last block column in both matrices has width u^r.

Writing down the system of differential–algebraic equations that corresponds to (15), we get

$$\begin{array}{ll} (a) & \dot{x}_1(t) = A_{12}(t)x_2(t) + A_{14}(t)x_4(t) + A_{15}(t)x_5(t) + f_1(t) \\ (b) & \dot{x}_2(t) = A_{24}(t)x_4(t) + A_{25}(t)x_5(t) + f_2(t) \\ (c) & 0 = x_3(t) + f_3(t) \\ (d) & 0 = x_1(t) + f_4(t) \\ (e) & 0 = f_5(t). \end{array} \tag{16}$$

Now we can differentiate equation (16d) and insert it in (16a). This corresponds to passing from (15) to

$$\begin{array}{c} \left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12}(t) & 0 & A_{14}(t) & A_{15}(t) \\ 0 & 0 & 0 & A_{24}(t) & A_{25}(t) \\ 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \end{array} \begin{array}{c} s \\ d \\ a \\ s \\ u^l \end{array} \tag{17}$$

[1] For analytic $E(t), A(t)$ we can use the analytic singular value decomposition [2, 26, 35] to compute a cononical form similar to (15) where we get $\Sigma(t)$'s instead of the identities. These $\Sigma(t)$'s are diagonal and can become singular only at isolated points.

for which we again compute characteristic values r, a, s, d, u^l, u^r.

The above procedure leads to an inductive definition of a sequence of pairs of matrix functions $(E_i(t), A_i(t)),\ i \in \mathcal{N}_0$, where $(E_0(t), A_0(t)) = (E(t), A(t))$ and $(E_{i+1}(t), A_{i+1}(t))$ is derived from $(E_i(t), A_i(t))$ by one step of this procedure.

Here we must assume (14) for every occurring pair of matrices. Connected with this sequence, we then have sequences $r_i, a_i, s_i, d_i, u^l_i, u^r_i,\ i \in \mathcal{N}_0$ of nonnegative integers. The sequences $r_i, a_i, s_i,\ i \in \mathcal{N}_0$ are characteristic for the given DAE, that is, they do not depend on the specific way they are obtained (recall that d_i, u^l_i, u^r_i are not independent of these). Furthermore, the sequences stop after finitely many (say μ) steps with $s_i = 0$. The quantity μ is called the *strangeness index* of the pencil $(E(t), A(t))$.

As last result from [22, 24] we cite an appropriate generalization of the Weierstraß–Kronecker canonical form for constant pencils (E, A) in the case of variable pencils:

Theorem 5 *Let the strangeness index μ be well-defined for the pair $(E(t), A(t))$ of smooth matrix functions. Let $r_i, a_i, s_i, d_i, u^l_i, u^r_i,\ i \in \mathcal{N}_0$ be the related characteristic values as above. Define*

$$\begin{array}{lll} (a) & b_0 = a_0, & b_i = \operatorname{rank}\left(\left[\begin{array}{cc} A_{14}^{(i-1)}(t) & A_{15}^{(i-1)}(t) \end{array}\right]\right), \\ (b) & c_0 = a_0 + s_0, & c_i = \operatorname{rank}\left(\left[\begin{array}{ccc} A_{12}^{(i-1)}(t) & A_{14}^{(i-1)}(t) & A_{15}^{(i-1)}(t) \end{array}\right]\right), \\ (c) & w_0 = u^l_0, & w_i = u^l_i - u^l_{i-1}, \qquad i = 1, \ldots, \mu. \end{array} \tag{18}$$

We then have

$$\begin{array}{lll} (a) & c_i = b_i + s_i, & i = 0, \ldots, \mu \\ (b) & w_i = s_{i-1} - c_i, & i = 1, \ldots, \mu \end{array} \tag{19}$$

and the pair $(E(t), A(t))$ is equivalent to a pair of matrix functions of the form (without arguments)

$$\left(\left[\begin{array}{ccccccccc} I & 0 & \ldots & 0 & 0 & * & \ldots & * \\ 0 & 0 & \ldots & 0 & 0 & F_m & & * \\ \vdots & \vdots & & \vdots & & \ddots & \ddots & \\ \vdots & \vdots & & \vdots & & & \ddots & F_1 \\ 0 & 0 & \ldots & 0 & & & & 0 \\ 0 & 0 & \ldots & 0 & 0 & G_m & & * \\ \vdots & \vdots & & \vdots & & \ddots & \ddots & \\ \vdots & \vdots & & \vdots & & & \ddots & G_1 \\ 0 & 0 & \ldots & 0 & & & & 0 \end{array} \right], \left[\begin{array}{cccccccc} * & * & \ldots & * & 0 & \ldots & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & \ldots & 0 & I & & & \\ \vdots & \vdots & & \vdots & & \ddots & & \\ \vdots & \vdots & & \vdots & & & \ddots & \\ 0 & 0 & \ldots & 0 & & & & I \end{array} \right] \right) \begin{array}{l} d_\mu \\ w_\mu \\ \vdots \\ \vdots \\ w_0 \\ c_\mu \\ \vdots \\ \vdots \\ c_0 \end{array} \tag{20}$$

where

$$\operatorname{rank}\left(\left[\begin{array}{c} F_i \\ G_i \end{array}\right]\right) = c_i + w_i = s_{i-1} \leq c_{i-1} \tag{21}$$

and the second block column in both matrices has width u^r_μ.

In the next two sections we will prove generalizations of these theorems for the descriptor system (1).

3 Local canonical forms

In this section we will generalize the local canonical form for linear DAEs with variable coefficients of Theorem 3 for the descriptor system (1).

For constant coefficient systems canonical and condensed forms have been studied for unitary transformations in [3, 4, 5] and for general transformations in [25].

Note that for a linear descriptor system with variable coefficients (1) we cannot apply directly the results of Section 2 since usually we cannot assume that the control $u(t)$ is sufficiently differentiable. In principle we can apply differentiation of components only in the uncontrollable subspace, i.e., the part of the system operating in the left nullspace of $B(t)$. Recently, a condensed form for unitary transformations has been studied in [6]. In the approach of [11, 12] it is assumed that the control is sufficiently smooth, which is a major difference to our approach.

The standard variable coefficient transformations that can be applied to the linear descriptor system (1) are pre–multiplication of (1) by a nonsingular matrix $P(t)$ and changes of the bases for the state $x(t)$ and control $u(t)$ of the system. Therefore, we use the following global equivalence transformations for a triple of matrix functions $(E(t), A(t), B(t))$.

Definition 6 *Two triples of matrix functions $(E_i(t), A_i(t), B_i(t))$, $B_i(t) \in C([t_1, t_2], \mathcal{C}^{n,m})$, $E_i(t), A_i(t) \in C([t_1, t_2], \mathcal{C}^{n,l})$, $i = 1, 2$ are called equivalent if there are $P(t) \in C([t_1, t_2], \mathcal{C}^{n,n})$, $Q(t) \in C([t_1, t_2], \mathcal{C}^{l,l})$ and $S(t) \in C([t_1, t_2], \mathcal{C}^{m,m})$ with $P(t), Q(t), S(t)$ nonsingular for all $t \in [t_1, t_2]$ such that*

$$(E_2(t), A_2(t), B_2(t)) = P(t)(E_1(t), A_1(t), B_1(t)) \begin{bmatrix} Q(t) & -\dot{Q}(t) & 0 \\ 0 & Q(t) & 0 \\ 0 & 0 & S(t) \end{bmatrix}. \tag{22}$$

Standard rules for differentiation show that this is indeed an equivalence relation.

As in the case of linear DAEs we get the responding local equivalence by choosing $\dot{Q}(t)$ independent of $Q(t)$ at a fixed point $t \in [t_1, t_2]$.

Definition 7 *Two triples of matrices (E_i, A_i, B_i), $E_i, A_i \in \mathcal{C}^{n,l}$, $B_i \in \mathcal{C}^{n,m}$, $i = 1, 2$ are called equivalent if there are matrices $P \in \mathcal{C}^{n,n}$, $Q, R \in \mathcal{C}^{l,l}$, $S \in \mathcal{C}^{m,m}$ with P, Q, S nonsingular such that*

$$(E_2, A_2, B_2) = P(E_1, A_1, B_1) \begin{bmatrix} Q & -R & 0 \\ 0 & Q & 0 \\ 0 & 0 & S \end{bmatrix}. \tag{23}$$

Again, it is easily checked that the local transformations describe an equivalence transformation.

Using the local equivalence transformations we obtain the following canonical form for a triple of matrices (E, A, B).

Theorem 8 *Let* $E, A \in \mathcal{C}^{n,l}$, $B \in \mathcal{C}^{n,m}$ *and*

$$\begin{array}{ll} (a) & T \text{ basis of } \operatorname{kernel} E \\ (b) & Z \text{ basis of } \operatorname{corange} E = \operatorname{kernel} E^* \\ (c) & T' \text{ basis of } \operatorname{cokernel} E = \operatorname{range} E^* \\ (d) & K \text{ basis of} \operatorname{corange}(Z^*B) \\ (e) & V \text{ basis of } \operatorname{corange}(K^*Z^*AT) \\ (f) & L \text{ basis of } \operatorname{kernel}(Z^*B) \\ (g) & Y \text{ basis of } \operatorname{kernel}(V^*K^*Z^*AT') \\ (h) & Y' \text{ basis of } \operatorname{cokernel}(V^*K^*Z^*AT') \\ (i) & N \text{ basis of} \operatorname{kernel}([I_s\ 0][Y'\ Y]^{-1}(Z'^*ET')^{-1}Z'^*BL). \end{array} \tag{24}$$

Then, the quantities

$$\begin{array}{lll} (a) & r = \operatorname{rank} E & \text{(rank)} \\ (b) & f = \operatorname{rank}(Z^*B) & \text{(feedback part)} \\ (c) & a = \operatorname{rank}(K^*Z^*AT) & \text{(algebraic part)} \\ (d) & s = \operatorname{rank}(V^*K^*Z^*AT') & \text{(strangeness)} \\ (e) & d = r - s & \text{(differential part)} \\ (f) & u^l = n - r - a - s - f & \text{(left undetermined part)} \\ (g) & u^r = l - r - a - s & \text{(right undetermined part)} \\ (h) & v = m - f & \\ (i) & s^c = \operatorname{rank}([I_s\ 0][Y'\ Y]^{-1}(Z'^*ET')^{-1}Z'^*BL & \\ (j) & s^u = s - s^c & \\ (k) & d^c = \operatorname{rank}([0\ I_d][Y'\ Y]^{-1}(Z'^*ET')^{-1}Z'^*BLN & \\ (l) & d^u = d - d^c. & \end{array} \tag{25}$$

are invariant under (23) and (E, A, B) is equivalent to the canonical form

$$\left(\begin{bmatrix} I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{d^c} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{d^u} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right. \tag{26}$$

$$\left. \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_a & 0 & 0 & 0 \\ I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_{s^c} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{d^c} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{array}{c} s^c \\ s^u \\ d^c \\ d^u \\ a \\ s^c \\ s^u \\ f \\ u^l \end{array}.$$

and the last column in the first and second matrix has width u^r.

Proof. Let (E_i, A_i, B_i), $i = 1, 2$, be equivalent. Since

$$\mathrm{rank}(E_2) = \mathrm{rank}(PE_1Q) = \mathrm{rank}(E_1),$$

r is invariant. For f, a, s, s^c and d^c we must first show that they are well–defined with respect to the choice of the bases. Each change of bases can be represented by

$$\tilde{T} = TM_T,\ \tilde{Z} = ZM_Z,\ \tilde{T'} = T'M_{T'},\ \tilde{Z'} = Z'M_{Z'},\ \tilde{K} = M_Z^{-1}KM_K,\ \tilde{V} = M_K^{-1}VM_V$$
$$\tilde{L} = LM_L,\ \tilde{Y'} = M_{T'}^{-1}Y'M_{Y'},\ \tilde{Y} = M_{T'}^{-1}YM_Y, \tilde{N} = M_L^{-1}NM_N$$

with nonsingular matrices $M_T,\ M_Z,\ M_{T'},\ M_K,\ M_V,\ M_L,\ M_{Y'},\ M_Y$ and M_N. The well–definiteness follows from

$$\mathrm{rank}(\tilde{Z}^*B) = \mathrm{rank}(M_Z^*Z^*B) = \mathrm{rank}(Z^*B),$$

$$\begin{aligned} &\mathrm{rank}\left([I_s\ 0][\tilde{Y'}\ \tilde{Y}]^{-1}(\tilde{Z'}^*E\tilde{T'})^{-1}\tilde{Z'}^*B\tilde{L}\right) \\ &= \ \mathrm{rank}\left([I_s\ 0][M_{T'}^{-1}Y'M_{Y'}\ M_{T'}^{-1}YM_Y]^{-1}\right. \\ &\qquad\qquad \left. \times\,(M_{Z'}^*Z'^*ET'M_{T'})^{-1}M_{Z'}^*Z'^*BLM_L\right) \end{aligned}$$

$$
\begin{aligned}
&= \operatorname{rank}\left([I_s\ 0]\left(\operatorname{diag}(M_{Y'}^{-1}\ M_Y^{-1})[Y'\ Y]^{-1}M_{T'}\right)\right.\\
&\qquad\qquad \left.\times\left(M_{T'}^{-1}(Z'^*ET')^{-1}M_{Z'}^{-*}\right)M_{Z'}^*Z'^*BLM_L\right)\\
&= \operatorname{rank}\left(M_{Y'}^{-1}[I_s\ 0][Y'\ Y]^{-1}(Z'^*ET')^{-1}Z'^*BLM_L\right)\\
&= \operatorname{rank}\left([I_s\ 0][Y'\ Y]^{-1}(Z'^*ET')^{-1}Z'^*BL\right)
\end{aligned}
$$

and similar calculations for the other values.

Let now bases $T_2, Z_2, Z_2', T_2', K_2, V_2, L_2, Y_2', Y_2, N_2$ be given for (E_2, A_2, B_2), i.e.

$$
\begin{array}{lll}
\operatorname{rank}(E_2T_2) = 0, & T_2^*T_2 \text{ nonsingular}, & \operatorname{rank}(T_2^*T_2) = n-r\\
\operatorname{rank}(Z_2^*E_2) = 0, & Z_2^*Z_2 \text{ nonsingular}, & \operatorname{rank}(Z_2^*Z_2) = n-r\\
\operatorname{rank}(E_2T_2') = r, & T'^*_2T_2' \text{ nonsingular}, & \operatorname{rank}(T_2'^*T_2') = r\\
\operatorname{rank}(Z_2'^*E_2) = r, & Z_2'^*Z_2' \text{ nonsingular}, & \operatorname{rank}(Z_2'^*Z_2') = r\\
\operatorname{rank}(K_2Z_2^*B_2) = 0, & K_2^*K_2 \text{ nonsingular}, & \operatorname{rank}(K_2^*K_2) = \hat{f}_2\\
\operatorname{rank}(V_2^*Z_2^*K_2^*A_2T_2) = 0, & V_2^*V_2 \text{ nonsingular}, & \operatorname{rank}(V_2^*V_2) = \hat{a}_2\\
\operatorname{rank}(Z_2^*B_2L_2) = 0, & L_2^*L_2 \text{ nonsingular}, & \operatorname{rank}(K_2^*K_2) = f_2\\
\operatorname{rank}(V_2^*Z_2^*A_2T'_2Y_2) = 0, & Y_2^*Y_2 \text{ nonsingular}, & \operatorname{rank}(Y_2^*Y_2) = \hat{s}_2\\
\operatorname{rank}(V_2^*Z_2^*A_2T_2'Y_2') = s_2, & Y'^*_2Y_2' \text{ nonsingular}, & \operatorname{rank}(Y_2'^*Y_2') = s_2\\
\multicolumn{3}{l}{\operatorname{rank}([I_s\ 0][Y_2'\ Y_2]^{-1}(Z_2'^*E_2T_2')^{-1}Z_2'^*B_2L_2N_2) = d_2^c,}\\
& N_2^*N_2 \text{ nonsingular}, & \operatorname{rank}(N_2^*N_2) = d_2^c
\end{array}
$$

with $\hat{f}_2 = \dim(\operatorname{corange}(Z_2^*B_2))$, $\hat{a}_2 = \dim(\operatorname{corange}(Z_2^*K_2^*A_2T_2))$ and $\hat{s}_2 = \dim(\operatorname{kernel}(V_2^*Z_2^*A_2T_2'))$. Using (23) and setting

$$
\begin{gathered}
T_1 = QT_2,\ Z_1^* = Z_2^*P,\ T_1' = QT_2',\ Z_1'^* = Z_2'^*P,\ K_1^* = K_2^*,\ V_1^* = V_2^*\\
L_1 = WL_2,\ Y_1 = Y_2,\ Y_1' = Y_2',\ N_1 = N_2
\end{gathered}
$$

we obtain the same relations for (E_1, A_1, B_1) and the above $T_1, Z_1, T_1', K_1, V_1, L_1, Y_1, Y_1', N_1$, i.e. they form bases according to (24). Since

$$
\begin{aligned}
f_2 &= \operatorname{rank}(Z_2^*B_2)\\
&= \operatorname{rank}(Z_2^*PB_1W)\\
&= \operatorname{rank}(Z_1^*B_1) = f_1
\end{aligned}
$$

we get the invariance of f. With the same technique, the invariance of a and s can be shown. s^c is invariant, since

$$
\begin{aligned}
s_2^c &= \operatorname{rank}([I_s\ 0][Y_2'\ Y_2]^{-1}(Z_2'^*E_2T_2')^{-1}Z_2'^*B_2L_2)\\
&= \operatorname{rank}([I_s\ 0][Y_1'\ Y_1]^{-1}(Z_2'^*PE_1QT_2')^{-1}Z_2'^*PB_1WL_2)\\
&= \operatorname{rank}([I_s\ 0][Y_1'\ Y_1]^{-1}(Z_1'^*E_1T_1')^{-1}Z_1'^*B_1L_1) = s_1^c,
\end{aligned}
$$

this also holds for N_1, i.e. d^c is invariant. Therefore, the invariance of the other values in (25) follows immediately.

For the derivation of the canonical form (26) we always use nonsingular transformation matrices, i.e. in the first step we take a basis Z' of range E and set $Q = [Z'\ Z]$, etc. As result we obtain the following sequence of equivalent ($\sim$) matrix pairs:

$$
\begin{aligned}
(E,A,B) &\sim \left(\begin{bmatrix} Z'^*ET' & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} Z'^*AT' & Z'^*AT \\ Z^*AT' & Z^*AT \end{bmatrix}, \begin{bmatrix} Z'^*B \\ Z^*B \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} Z'^*ET' & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * \\ K'^*Z^*AT' & K'^*Z^*AT \\ K^*Z^*AT' & K^*Z^*AT \end{bmatrix}, \begin{bmatrix} Z'^*BL' & Z'^*BL \\ K'^*Z^*BL' & 0 \\ 0 & 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * \\ K'^*Z^*AT' & K'^*Z^*AT \\ K^*Z^*AT' & K^*Z^*AT \end{bmatrix}, \right. \\
&\qquad\qquad \left. \begin{bmatrix} (Z'^*ET')^{-1}Z'^*BL' & (Z'^*ET')^{-1}Z'^*BL \\ K'^*Z^*BL' & 0 \\ 0 & 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * \\ K'^*Z^*AT' & K'^*Z^*AT \\ K^*Z^*AT' & K^*Z^*AT \end{bmatrix}, \begin{bmatrix} 0 & (Z'^*ET')^{-1}Z'^*BL \\ I_f & 0 \\ 0 & 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ K'^*Z^*AT' & K'^*Z^*AT \\ K^*Z^*AT' & K^*Z^*AT \end{bmatrix}, \begin{bmatrix} 0 & (Z'^*ET')^{-1}Z'^*BL \\ I_f & 0 \\ 0 & 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ V'^*K^*Z^*AT' & I_a & 0 \\ V^*K^*Z^*AT' & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & (Z'^*ET')^{-1}Z'^*BL \\ I_f & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ 0 & I_a & 0 \\ V^*K^*Z^*AT' & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & (Z'^*ET')^{-1}Z'^*BL \\ I_f & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & * \\ 0 & I_a & 0 \\ V^*K^*Z^*AT' & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & (Z'^*ET')^{-1}Z'^*BL \\ I_f & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * \\ 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ I_f & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right)
\end{aligned}
$$

where $B_{12} = [I_s\ 0][Y'\ Y]^{-1}(Z'^*ET')^{-1}Z'^*BL$ and
$B_{22} = [0\ I_d][Y'\ Y]^{-1}(Z'^*ET')^{-1}Z'^*BL$

$$\sim \left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ 0 & 0 \\ 0 & 0 \\ I_f & 0 \\ 0 & 0 \end{bmatrix} \right)$$

$$\sim \left(\begin{bmatrix} I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_a & 0 & 0 & 0 \\ I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0 & I_{s^c} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & [0\ I_d](Z'^*ET')^{-1}Z'^*BLN \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right),$$

which at last leads to (26) by a similar final transformation step. ■

If we do not split the d and s blocks of $B(t)$ in the proof of Theorem 8 we get the following condensed form.

Corollary 9 *Let $E, A \in \mathcal{C}^{n,l}$, $B \in \mathcal{C}^{n,m}$. Then (E, A, B) is equivalent to the form*

$$\left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * \\ 0 & * \\ 0 & 0 \\ 0 & 0 \\ I_f & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{matrix} s \\ d \\ a \\ s \\ f \\ u^l \end{matrix}. \tag{27}$$

where the last block column in the first and second matrix has width u^r and the last block column of the last matrix has width v. The quantities s, d, a, f, u^l, u_r and v are defined as in Theorem 8 and invariant under (23).

4 Global canonical forms

As in Section 2, we can apply the results for the local canonical form (26) to equation (1) and one obtains functions $r, f, a, s, s^c, d^c : [t_1, t_2] \to \mathcal{N}_0$. Note that the other values depend only on this invariants. Again, we do not know in general how to characterise points, where these quantities change their values with t. Therefore, we exclude such phenomena by assuming

$$r(t) \equiv r,\ \ f(t) \equiv f,\ \ a(t) \equiv a,\ \ s(t) \equiv s,\ \ s^c(t) \equiv s^c,\ \ d^c(t) \equiv d^c. \tag{28}$$

Applying transformation (22) to (1) we get the following canonical form:

Theorem 10 *Let E, A, B in (1) be sufficiently smooth and let (28) hold. Then the triple $(E(t), A(t), B(t))$ is equivalent to a triple of matrix functions of the form*

$$\left(\begin{bmatrix} I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{d^c} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{d^u} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 & 0 & A_{13}(t) & A_{14}(t) & 0 & A_{16}(t) & A_{17}(t) & A_{18}(t) \\ 0 & 0 & A_{23}(t) & A_{24}(t) & 0 & A_{26}(t) & A_{27}(t) & A_{28}(t) \\ 0 & 0 & 0 & A_{34}(t) & 0 & A_{36}(t) & A_{37}(t) & A_{38}(t) \\ 0 & 0 & A_{43}(t) & 0 & 0 & A_{46}(t) & A_{47}(t) & A_{48}(t) \\ 0 & 0 & 0 & 0 & I_a & 0 & 0 & 0 \\ I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{83}(t) & A_{84}(t) & 0 & A_{86}(t) & A_{87}(t) & A_{88}(t) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_{s^c} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{d^c} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{array}{c} s^c \\ s^u \\ d^c \\ d^u \\ a \\ s^c \\ s^u \\ f \\ u^l \end{array} . \tag{29}$$

The proof of Theorem 10 is given in Appendix A.

Again, as in Section 3 we get a condensed form if we do not split the d and s blocks of $B(t)$.

Corollary 11 *Let E, A, B in (1) be sufficiently smooth and let*

$$r(t) \equiv r,\ \ f(t) \equiv f,\ \ a(t) \equiv a,\ \ s(t) \equiv s$$

hold. Then $(E(t), A(t), B(t))$ is equivalent to a triple of matrix functions of the form

$$\left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12}(t) & 0 & A_{14}(t) & A_{15}(t) \\ 0 & 0 & 0 & A_{24}(t) & A_{25}(t) \\ 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & A_{52}(t) & 0 & A_{54}(t) & A_{55}(t) \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12}(t) \\ 0 & B_{22}(t) \\ 0 & 0 \\ 0 & 0 \\ I_f & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{array}{c} s \\ d \\ a \\ s \\ f \\ u^l \end{array}. \tag{30}$$

From the analysis of linear DAEs with variable coefficients we know that higher index problems, i.e., of the index greater than one, are indicated by a non–vanishing strangeness s (see [24]).

Our main goal is, to study the regularization of the descriptor system (1) by feedback. As the next lemma shows, Corollary 11 is a first step in this direction.

Lemma 12 *Let a quadratic descriptor system (1), i.e. $n = l$, be in the form (30) and assume that $s = 0$.*

If $u_l = 0$, then their exists a state feedback $u(t) = F(t)x(t) + w(t)$, such that the closed loop system

$$E(t)\dot{x}(t) = (A(t) + B(t)F(t))x(t) + B(t)w(t), \; x(t_0) = x_0$$

is uniquely solvable for every consistent initial value x_0 and any given control $w(t)$.

Proof. The descriptor system is of the form

$$\begin{bmatrix} I_d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & 0 & A_{13}(t) \\ 0 & I_a & 0 \\ A_{31}(t) & 0 & A_{33}(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 & B_{12}(t) \\ 0 & 0 \\ I_f & 0 \end{bmatrix} w(t).$$

Choosing $F(t) = \begin{bmatrix} -A_{31}(t) & 0 & I_f - A_{33}(t) \\ 0 & 0 & 0 \end{bmatrix}$, we get the closed loop system

$$\begin{bmatrix} I_d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & 0 & A_{13}(t) \\ 0 & I_a & 0 \\ 0 & 0 & I_f \end{bmatrix} x(t) + \begin{bmatrix} 0 & B_{12}(t) \\ 0 & 0 \\ I_f & 0 \end{bmatrix} w(t). \tag{31}$$

For any given control $w(t)$ (31) is a DAE with characteristic values $s_{DAE} = 0$, $d_{DAE} = d$, $a_{DAE} = a + f$ and $u^l_{DAE} = u^r_{DAE} = 0$. From [24, Corollary 20] we now get immediately that (31) is uniquely solvable for every consistent initial value x_0. ∎

Lemma 12 shows, that under certain assumptions the condensed form (30) allows us to construct a feedback which makes the closed loop system uniquely solvable. Even more, in Section 5 we will show, that it is sufficient to study a closely related condensed form to answer the question whether there exist a state and/or derivative feedback which makes the closed loop uniquely solvable or not.

From now on we will focus our analysis on the generalization of the remaining results from Section 2 for the condensed form (30) of Corollary 11.

Writing down the descriptor system equations that belongs to the matrix triple from Corollary 11, we get

$$\begin{array}{ll} (a) & \dot{x}_1(t) = A_{12}(t)x_2(t) + A_{14}(t)x_4(t) + A_{15}(t)x_5(t) + B_{12}(t)u_3(t) \\ (b) & \dot{x}_2(t) = A_{24}(t)x_4(t) + A_{25}(t)x_5(t) + B_{22}(t)u_3(t) \\ (c) & 0 = x_3(t) \\ (d) & 0 = x_1(t) \\ (e) & 0 = A_{52}(t)x_2(t) + A_{54}(t)x_4(t) + A_{55}(t)x_5(t) + u_1(t) \\ (f) & 0 = 0 \end{array} \tag{32}$$

From equation (32d) we see that $x_1(t) \equiv 0$. This implies $\dot{x}_1(t) \equiv 0$ and from inserting $\dot{x}_1(t) \equiv 0$ in (32a) we get an algebraic equation. This corresponds to passing from the form (30) to

$$\left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12}(t) & 0 & A_{14}(t) & A_{15}(t) \\ 0 & 0 & 0 & A_{24}(t) & A_{25}(t) \\ 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & A_{52}(t) & 0 & A_{54}(t) & A_{55}(t) \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12}(t) \\ 0 & B_{22}(t) \\ 0 & 0 \\ 0 & 0 \\ I_f & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{array}{c} s \\ d \\ a \\ s \\ f \\ u^l \end{array}, \tag{33}$$

for which we again compute characteristic values r, a, s, d, u^l, u^r and v.

This leads to an inductive definition of a sequence $(E_i(t), A_i(t), B_i(t))$, $i \in \mathcal{N}_0$ of matrix function triples, where $(E_0(t), A_0(t), B_0(t)) = (E(t), A(t), B(t))$ and $(E_{i+1}(t), A_{i+1}(t), B_{i+1}(t))$ is derived from $(E_i(t), A_i(t), B_i(t))$ by bringing it into the form (30) and passing them to the form above. Here we must assume that none of the values $r(t) \equiv r, f(t) \equiv f, a(t) \equiv a, s(t) \equiv s$ for every occurring pair of matrices. Connected with this sequence, we then have sequences $r_i, f_i, a_i, s_i, u_i^l, u_i^r, v_i,\ i \in \mathcal{N}_0$ of nonnegative integers.

The next Theorem shows that these sequences are indeed characteristic for a given triple $(E(t), A(t), B)t))$, i.e. they do not depend on the specific way they are obtained.

Theorem 13 *Let $(E(t), A(t), B(t))$, $(\bar{E}(t), \bar{A}(t), \bar{B}(t))$ be equivalent and of the form (30). Then the modified triples $(E_{\text{mod}}(t), A_{\text{mod}}(t), B_{\text{mod}}(t))$, $(\bar{E}_{\text{mod}}(t), \bar{A}_{\text{mod}}(t), \bar{B}_{\text{mod}}(t))$ obtained by passing to (33) are also equivalent.*

Proof. Assume that $(E(t), A(t), B(t))$, $(\bar{E}(t), \bar{A}(t), \bar{B}(t))$ are equivalent and of the form (30). Omitting arguments we get

$$P\bar{E} = EQ,\ P\bar{A} = AQ - E\dot{Q},\ P\bar{B} = BS$$

where P, Q and S are smooth, pointwise nonsingular matrix functions. From the first relation we get

$$\left[\begin{array}{ccccc} P_{11} & P_{12} & 0 & 0 & 0 \\ P_{21} & P_{22} & 0 & 0 & 0 \\ P_{31} & P_{32} & 0 & 0 & 0 \\ P_{41} & P_{42} & 0 & 0 & 0 \\ P_{51} & P_{52} & 0 & 0 & 0 \\ P_{61} & P_{62} & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccccc} Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} & Q_{25} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

if we partition P and Q according to Corollary 11.

With this we obtain for the third, fourth and sixth block rows of the second relation

$$\left[\begin{array}{ccccc} P_{34} & P_{35}\bar{A}_{52} & P_{33} & P_{35}\bar{A}_{54} & P_{35}\bar{A}_{55} \\ P_{44} & P_{45}\bar{A}_{52} & P_{43} & P_{45}\bar{A}_{54} & P_{45}\bar{A}_{55} \\ P_{64} & P_{65}\bar{A}_{52} & P_{63} & P_{65}\bar{A}_{54} & P_{65}\bar{A}_{55} \end{array}\right] = \left[\begin{array}{ccccc} Q_{31} & Q_{32} & Q_{33} & Q_{34} & Q_{35} \\ Q_{11} & Q_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

For the third to sixth block rows of the third relation we then deduce

$$\left[\begin{array}{cc} P_{35} & 0 \\ P_{45} & 0 \\ P_{55} & 0 \\ P_{65} & 0 \end{array}\right] = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ S_{11} & S_{12} \\ 0 & 0 \end{array}\right]$$

where we partition $S = \left[\begin{array}{cc} S_{11} & S_{12} \\ S_{21} & S_{22} \end{array}\right]$ according to Corollary 11.

In terms of the matricies Q and S we therefore have

$$P = \left[\begin{array}{cccccc} Q_{11} & 0 & P_{13} & P_{14} & P_{15} & P_{16} \\ Q_{21} & Q_{22} & P_{23} & P_{24} & P_{25} & P_{26} \\ 0 & 0 & Q_{33} & Q_{31} & 0 & P_{36} \\ 0 & 0 & 0 & Q_{11} & 0 & P_{46} \\ 0 & 0 & P_{53} & P_{54} & S_{11} & P_{56} \\ 0 & 0 & 0 & 0 & 0 & P_{66} \end{array}\right],$$

$$Q = \left[\begin{array}{ccccc} Q_{11} & 0 & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ Q_{31} & 0 & Q_{33} & 0 & 0 \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} & Q_{45} \\ Q_{51} & Q_{52} & Q_{53} & Q_{54} & Q_{55} \end{array}\right], \quad S = \left[\begin{array}{cc} S_{11} & 0 \\ S_{21} & S_{22} \end{array}\right]$$

and $Q_{11}, Q_{22}, Q_{33}, S_{11}, S_{22}, P_{66}, \left[\begin{array}{cc} Q_{44} & Q_{45} \\ Q_{54} & Q_{55} \end{array}\right]$ must be nonsingular. From the first two and the fifth block row of the second relation, we then get

$$\begin{bmatrix} Q_{11} & 0 & P_{15} \\ Q_{21} & Q_{22} & P_{25} \\ 0 & 0 & S_{11} \end{bmatrix} \begin{bmatrix} \bar{A}_{12} & \bar{A}_{14} & \bar{A}_{15} \\ 0 & \bar{A}_{24} & \bar{A}_{25} \\ \bar{A}_{52} & \bar{A}_{54} & \bar{A}_{55} \end{bmatrix} =$$
$$\begin{bmatrix} A_{12} & A_{14} & A_{15} \\ 0 & A_{24} & A_{25} \\ A_{52} & A_{54} & A_{55} \end{bmatrix} \begin{bmatrix} Q_{22} & 0 & 0 \\ Q_{42} & Q_{44} & Q_{45} \\ Q_{52} & Q_{54} & Q_{55} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ \dot{Q}_{22} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Similar, from the same block rows of the third equation, we deduce

$$\begin{bmatrix} Q_{11} & 0 & P_{15} \\ Q_{21} & Q_{22} & P_{25} \\ 0 & 0 & S_{11} \end{bmatrix} \begin{bmatrix} 0 & \bar{B}_{12} \\ 0 & \bar{B}_{22} \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ I & 0 \end{bmatrix} S$$

Let $(\bar{E}_{\text{mod}}, \bar{A}_{\text{mod}}, \bar{B}_{\text{mod}})$ be the modified triple which we obtained form the triple $(\bar{E}, \bar{A}, \bar{B})$. Then

$$\begin{aligned} &(\bar{E}_{\text{mod}}, \bar{A}_{\text{mod}}, \bar{B}_{\text{mod}}) \\ &\sim \left(\begin{bmatrix} Q_{11} & & & & & \\ Q_{21} & Q_{22} & & & & \\ & & I & & & \\ & & & I & & \\ & & & & S_{11} & \\ & & & & & I \end{bmatrix} \begin{bmatrix} 0 & & & & & \\ & I & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}, \right. \\ &\qquad \begin{bmatrix} Q_{11} & & & & & \\ Q_{21} & Q_{22} & & & & \\ & & I & & & \\ & & & I & & \\ & & & & S_{11} & \\ & & & & & I \end{bmatrix} \begin{bmatrix} 0 & \bar{A}_{12} & 0 & \bar{A}_{14} & \bar{A}_{15} \\ 0 & 0 & 0 & \bar{A}_{24} & \bar{A}_{25} \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & \bar{A}_{52} & 0 & \bar{A}_{54} & \bar{A}_{55} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ &\qquad\qquad \left. \begin{bmatrix} Q_{11} & & & & & \\ Q_{21} & Q_{22} & & & & \\ & & I & & & \\ & & & I & & \\ & & & & S_{11} & \\ & & & & & I \end{bmatrix} \begin{bmatrix} 0 & \bar{B}_{12} \\ 0 & \bar{B}_{22} \\ 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & 0 \end{bmatrix} \right) \end{aligned}$$

$$
\sim \left(\begin{bmatrix} 0 & & & & \\ & Q_{22} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \\ & & & & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & 0 & A_{14} & A_{15} \\ 0 & 0 & 0 & A_{24} & A_{25} \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & A_{52} & 0 & A_{54} & A_{55} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I & & & & \\ & Q_{22} & & & \\ & & I & & \\ & Q_{42} & & Q_{44} & Q_{45} \\ & Q_{52} & & Q_{54} & Q_{55} \end{bmatrix} \right.
$$

$$
\left. - \begin{bmatrix} 0 & & & & \\ & \dot{Q}_{22} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \\ & & & & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & 0 \end{bmatrix} S \right)
$$

$$
\sim \left(\begin{bmatrix} 0 & & & & \\ & Q_{22} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} I & & & & \\ & Q_{22}^{-1} & & & \\ & & I & & \\ & * & & * & * \\ & * & & * & * \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & 0 & A_{14} & A_{15} \\ 0 & 0 & 0 & A_{24} & A_{25} \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & A_{52} & 0 & A_{54} & A_{55} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right.
$$

$$
- \begin{bmatrix} 0 & & & & \\ & \dot{Q}_{22} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} I & & & & \\ & Q_{22}^{-1} & & & \\ & & I & & \\ & * & & * & * \\ & * & & * & * \end{bmatrix}
$$

$$
\left. - \begin{bmatrix} 0 & & & & \\ & Q_{22} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} 0 & & & & \\ & \dot{Q}_{22}^{-1} & & & \\ & & 0 & & \\ & * & & * & * \\ & * & & * & * \end{bmatrix}, \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & 0 \end{bmatrix} \right)
$$

$$
\sim \left(\begin{bmatrix} 0 & & & & \\ & I & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \\ & & & & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & 0 & A_{14} & A_{15} \\ 0 & X & 0 & A_{24} & A_{25} \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & A_{52} & 0 & A_{54} & A_{55} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & 0 \end{bmatrix} \right)
$$

where $X = -\left(\dot{Q}_{22}Q_{22}^{-1} + Q_{22}\dot{Q}_{22}^{-1}\right) = -\overbrace{\left(Q_{22}Q_{22}^{-1}\right)}^{\cdot} = -\dot{I} = 0.$ ■

Now we can state some basic properties of these quantities:

Lemma 14 *Let $E(t), A(t)$ and $B(t)$ in (1) be sufficiently smooth and such that the sequences $(E_i(t), A_i(t), B_i(t))$, $i \in \mathcal{N}_0$ and $r_i, f_i, a_i, s_i, d_i, u_i^l, u_i^r, v_i$, $i \in \mathcal{N}_0$ are well-defined by the above process. Let furthermore*

$$
\begin{aligned}
&(E_i(t), A_i(t), B_i(t)) \\
&\sim \left(\begin{bmatrix} I_{s_i} & 0 & 0 & 0 & 0 \\ 0 & I_{d_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right. \\
&\qquad \left. \begin{bmatrix} 0 & A_{12}^{(i)}(t) & 0 & A_{14}^{(i)}(t) & A_{15}^{(i)}(t) \\ 0 & 0 & 0 & A_{24}^{(i)}(t) & A_{25}^{(i)}(t) \\ 0 & 0 & I_{a_i} & 0 & 0 \\ I_{s_i} & 0 & 0 & 0 & 0 \\ 0 & A_{52}^{(i)}(t) & 0 & A_{54}^{(i)}(t) & A_{55}^{(i)}(t) \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12}^{(i)}(t) \\ 0 & B_{22}^{(i)}(t) \\ 0 & 0 \\ 0 & 0 \\ I_{f_i} & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{array}{l} s_i \\ d_i \\ a_i \\ s_i \\ f_i \\ u_i^l \end{array}
\end{aligned}
\tag{34}
$$

Then, we have (for all $t \in [t_1, t_2]$, $i \in \mathcal{N}$)

$$
\begin{array}{ll}
(a) & r_{i+1} = r_i - s_i \\
(b) & f_{i+1} = f_i + \operatorname{rank}(B_{12}^{(i)}(t)) \\
(c) & a_{i+1} = a_i + s_i + \operatorname{rank}(R_i(t)^*[A_{14}^{(i)}(t)A_{15}^{(i)}(t)]) \\
(d) & s_{i+1} = \operatorname{rank}(W_i(t)^* R_i(t)^* A_{12}^{(i)}(t)) \\
(e) & d_{i+1} = d_i - \operatorname{rank}(W_i(t)^* R_i(t)^* A_{12}^{(i)}(t)) \\
(f) & u_{i+1}^l = u_i^l + (s_i - \operatorname{rank}(R_i(t)^*[A_{12}^{(i)}(t))A_{14}^{(i)}(t)A_{15}^{(i)}(t)]) - \operatorname{rank}(B_{12}^{(i)}(t))) \\
(g) & u_{i+1}^r = u_i^r + (s_i - \operatorname{rank}(R_i(t)^*[A_{12}^{(i)}(t))A_{14}^{(i)}(t)A_{15}^{(i)}(t)])) \\
(h) & v_{i+1} = v_i - \operatorname{rank}(B_{12}^{(i)}(t))
\end{array}
\tag{35}
$$

with $R_i(t) = \operatorname{corange}(B_{12}^{(i)}(t))$ and $W_i(t) = \operatorname{corange}(R_i(t)^*[A_{14}^{(i)}(t)A_{15}^{(i)}(t)])$.

There exists a number $\nu \in \mathcal{N}_0$ defined by

$$
\nu = \min\{i \in \mathcal{N}_0 | s_i = 0\}
\tag{36}
$$

and the above sequences have the properties

$$
\begin{array}{llllll}
(a) & r_i > r_{i+1} & \text{for } i < \nu, & r_i = r_\nu & \text{for } i \geq \nu \\
(b) & f_i \leq f_{i+1} & \text{for } i < \nu, & f_i = f_\nu & \text{for } i \geq \nu \\
(c) & a_i < a_{i+1} & \text{for } i < \nu, & a_i = a_\nu & \text{for } i \geq \nu \\
(d) & s_i \geq s_{i+1} & \text{for } i < \nu, & s_i = 0 & \text{for } i \geq \nu \\
(e) & d_i \geq d_{i+1} & \text{for } i < \nu, & d_i = d_\nu & \text{for } i \geq \nu \\
(f) & u_i^l \leq u_{i+1}^l & \text{for } i < \nu, & u_i^l = u_\nu^l & \text{for } i \geq \nu \\
(g) & u_i^r \leq u_{i+1}^r & \text{for } i < \nu, & u_i^r = u_\nu^r & \text{for } i \geq \nu \\
(h) & v_i \geq v_{i+1} & \text{for } i < \nu, & v_i = v_\nu & \text{for } i \geq \nu
\end{array}
\tag{37}
$$

Proof. Replacing I_{s_i} by 0 in $E_i(t)$ we get (35a) from $r_{i+1} = \text{rank}(E_{i+1}(t))$. (35b) is then a consequence of $f_{i+1} = \text{rank}(Z_{i+1}(t)^* B_{i+1}(t))$, where $Z_{i+1}(t)$ is a basis of corange$(E_{i+1}(t))$. Since $a_{i+1} = \text{rank}(K_{i+1}(t)^* Z_{i+1}(t)^* A_{i+1}(t) T_{i+1}(t))$, where $K_{i+1}(t)$ is a basis of corange$(Z_{i+1}(t)^* B_{i+1}(t))$ and $T_{i+1}(t)$ is a basis of kernel$(E_{i+1}(t))$, we get (35c). (35d) follows now immediately from the definition (24) of s_{i+1}. By direct application of (24) we now get (35e-h).

$A_{12}^{(i)}(t)$ is an (s_i, d_i)–matrix, so that $s_i \geq s_{i+1}$ and s_i must become zero after a finite number of steps. Thus, (37) is a direct consequence of (35). ■

The quantities ν and $r_i, f_i, a_i, s_i,\ i \in \{0, \ldots, \nu\}$ are characteristic for a given descriptor system and the hope is that they are sufficient to describe the possible phenomena for (1). We now get a condensed form which reflects the above quantities similar to the condensed form of [6].

Theorem 15 *Let ν from Lemma 14 be well-defined for a triple $(E(t), A(t), B(t))$ of smooth matrix functions. Let $r_i, f_i, a_i, s_i, d_i, u_i^l, u_i^r, v_i,\ i \in 0, \ldots, \nu$ be the related characteristic values as above. Furthermore define (in the notation of Lemma 14)*

$$\begin{array}{lll} (a) & b_0 = a_0, & b_i = \text{rank}(R_{i-1}(t)^* [A_{14}^{(i-1)}(t) A_{15}^{(i-1)}(t)]), \\ (b) & g_0 = 0, & g_i = \text{rank}(B_{12}^{(i-1)}(t)), \\ (c) & c_0 = a_0 + s_0, & c_i = \text{rank}(R_{i-1}(t)^* [A_{12}^{(i-1)}(t) A_{14}^{(i-1)}(t) A_{15}^{(i-1)}(t)]) + g_i, \\ (d) & w_0^l = u_0^l, & w_i^l = u_i^l - u_{i-1}^l, \\ (e) & w_0^r = u_0^r, & w_i^r = u_i^r - u_{i-1}^r, \quad i = 1, \ldots, \nu. \end{array} \tag{38}$$

We then have

$$\begin{array}{ll} (a) & c_i = b_i + s_i, \quad i = 0, \ldots, \nu, \\ (b) & w_i^l = s_{i-1} - c_i - g_i \quad i = 1, \ldots, \nu, \\ (c) & w_i^r = s_{i-1} - c_i, \quad i = 1, \ldots, \nu \end{array} \tag{39}$$

and the triple $(E(t), A(t), B(t))$ is equivalent to a triple of matrix functions of the form

(without arguments)

$$\left(\begin{bmatrix} I & 0 & \cdots & 0 & 0 & * & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & F_\nu & & * \\ \vdots & \vdots & & \vdots & & \ddots & \ddots & \\ \vdots & \vdots & & \vdots & & & \ddots & F_1 \\ 0 & 0 & \cdots & 0 & & & & 0 \\ 0 & 0 & \cdots & 0 & 0 & E_\nu & & * \\ \vdots & \vdots & & \vdots & & \ddots & \ddots & \\ \vdots & \vdots & & \vdots & & & \ddots & E_1 \\ 0 & 0 & \cdots & 0 & & & & 0 \\ 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \end{bmatrix}, \begin{bmatrix} * & * & \cdots & * & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & I & & & \\ \vdots & \vdots & & & & \ddots & & \\ \vdots & \vdots & & & & & \ddots & \\ 0 & 0 & \cdots & 0 & & & & I \\ * & * & \cdots & * & 0 & \cdots & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0 & * \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \\ I & 0 \end{bmatrix} \right) \begin{array}{c} d_\nu \\ w^l_\nu \\ \vdots \\ \vdots \\ w^l_0 \\ c_\nu \\ \vdots \\ \vdots \\ c_0 \\ f_\nu \end{array} \tag{40}$$

where

$$\operatorname{rank}\left(\begin{bmatrix} F_i \\ E_i \end{bmatrix} \right) = c_i + w^l_i = s_{i-1} \leq c_{i-1}. \tag{41}$$

The second to the $\nu+2$-th block column have size w^r_ν to w^r_0.

The proof of Theorem 15 is given in Appendix B.

To complete the picture, we will conclude this section with some remarks about the generalization of the above process for the canonical form (29) of Theorem 10.

Generalizing the above process we again get an inductive definition of a sequence of matrix function triples $(E_i(t), A_i(t), B_i(t))$, $i \in \mathcal{N}_0$ and sequences of corresponding characteristic values. In this case we must assume additionally that $d^c(t) \equiv d^c$ and $s^c(t) \equiv s^c$ for every occuring pair of matricies.

Then we can generalize Theorem 13 and Lemma 14. But note, that neither for d^c_i, d^u_i, s^c_i or s^u_i we do get any recurrence formulas nor properties as in (37) are valid.

Finally, we can generalize Theorem 15 and get the following canonical form.

Theorem 16 *Let ν from Lemma 14 be well-defined for a triple $(E(t), A(t), B(t))$ of smooth matrix functions and let the values $c_i,\ w_i^l,\ w_i^r,\ i = 0, \ldots, \nu$ be defined as in Theorem 15. The triple $(E(t), A(t), B(t))$ is then equivalent to a triple of matrix functions of the form (without arguments)*

$$\left(\begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \\ 0 & I & 0 & \cdots & 0 & 0 & * & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & F_\nu & & * & \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \vdots & & & & \ddots & F_1 \\ 0 & 0 & 0 & \cdots & 0 & & & & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & E_\nu & & * \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \vdots & & & & \ddots & E_1 \\ 0 & 0 & 0 & \cdots & 0 & & & & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \end{bmatrix}, \begin{bmatrix} * & * & * & \cdots & * & 0 & \cdots & \cdots & 0 \\ * & * & * & \cdots & * & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & I & & & \\ \vdots & \vdots & \vdots & & & & \ddots & & \\ \vdots & \vdots & \vdots & & & & & \ddots & \\ 0 & 0 & 0 & \cdots & 0 & & & & I \\ * & * & * & \cdots & * & 0 & \cdots & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0 & I \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \\ I & 0 \end{bmatrix} \right) \begin{array}{c} d_\nu^c \\ d_\nu^u \\ w_\nu^l \\ \vdots \\ \vdots \\ w_0^l \\ c_\nu \\ \vdots \\ \vdots \\ c_0 \\ f_\nu \end{array} \tag{42}$$

and (39) and (41) are still valid.

Note that we have assumed additionally that $d^c(t) \equiv d^c$ and $s^c(t) \equiv s^c$ for every occuring pair of matrices, but in the normal form (42) only the d_i^c's occur. Therefore, one can use weaker assumptions to prove Theorem 16.

Until now we have studied only equivalence transformations of the form (22) and (23). For constant coefficient system feedback canonical forms have been studied in [25]. The canonical form (42) gives us the possibility to generalize these results for the variable coefficient case.

5 Regularization by feedback

In this final section we answer the question whether the system is regularizable by proportional and/or derivative feedbacks, i.e. that the closed loop system is uniquely solvable for all consistent initial vectors.

For constant coefficient systems regularizability has been studied by several authors, for example [3, 4]. An approach similar to the one we present in this paper for linear descriptor systems with variable coefficients has been studied in [6].

Using the results of Section 4 we can transform (1) to an equivalent descriptor system of a very special structure. Note that equivalence here means that there is a one–to–one correspondence of the solutions, that is we get a descriptor system which has the same solutions as the original system (1) for every consistent initial value and any given control.

Theorem 17 *Let ν from (36) be well–defined for the triple $(E(t), A(t), B(t))$ in (1). Then (1) is equivalent to a descriptor system of the form*

$$\begin{array}{llcl} (a) & \dot{x}_1(t) & = & A_{13}(t)x_3(t) + B_{12}(t)u_2(t) \\ (b) & 0 & = & x_2(t) \\ (c) & 0 & = & A_{31}(t)x_1(t) + A_{33}(t)x_3(t) + u_1(t) \\ (d) & 0 & = & 0 \end{array} \tag{43}$$

d_ν, a_ν and u_ν^r are the number of the differential, algebraic and undetermined components of the unknown x in (43) and f_ν^l and u_ν^l are the number of equations in (43c) and (43d).

Proof. We transform the triple $(E(t), A(t), B(t))$ to the form (30) and pass to (33). From Lemma 14 we know that we can repeat this process ν–times until $s_\nu = 0$. This yields a triple of matrices of the form

$$\begin{array}{c} \left(\begin{bmatrix} I_{d_\nu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & A_{13}(t) \\ 0 & I_{a_\nu} & 0 \\ A_{31}(t) & 0 & A_{33}(t) \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12}(t) \\ 0 & 0 \\ I_{f_\nu} & 0 \\ 0 & 0 \end{bmatrix} \right) \end{array} \begin{array}{l} d_\nu \\ a_\nu \\ f_\nu \\ u_\nu^l \end{array} \tag{44}$$

where the last block columns of the first and second matrix have width u_ν^r and all these steps are reversible ∎

Note that some solution components of (43b) which are constrained to zero come from uncontrollable higher index components of (1). The other uncontrollable higher index components are fulfilled trivially and can be found in (43d).

Before we can answer the question posed in the beginning of this section, we have to define what we understand under regularizability.

Definition 18 (see [6], Definition 7)

(a) *The descriptor system (1) is called regularizable by proportional feedback if there exists a (proportional state) feedback* $u(t) = F(t)x(t) + w(t)$ *such that the closed loop system*

$$E(t)\dot{x}(t) = (A(t) + B(t)F(t))x(t) + B(t)w(t), \quad x(t_0) = x_0$$

is uniquely solvable for every consistent initial value x_0 *and any given control* $w(t)$.

(b) *The descriptor system (1) is called regularizable by derivative feedback if there exists a (derivative) feedback* $u(t) = G(t)\dot{x}(t) + w(t)$ *such that the closed loop system*

$$(E(t) + B(t)G(t))\dot{x}(t) = A(t)x(t) + B(t)w(t), \quad x(t_0) = x_0$$

is uniquely solvable for every consistent initial value x_0 *and any given control* $w(t)$.

(c) *The descriptor system (1) is called regularizable by combined derivative and proportional state feedback if there exists a feedback* $u(t) = G(t)\dot{x}(t) + F(t)x(t) + w(t)$ *such that the closed loop system*

$$(E(t) + B(t)G(t))\dot{x}(t) = (A(t) + B(t)F(t))x(t) + B(t)w(t), \quad x(t_0) = x_0$$

is uniquely solvable for every consistent initial value x_0 *and any given control* $w(t)$.

Now we can formulate the main theorem of this section. It gives necessary and sufficient conditions whether the descriptor system (1) is regularizable by proportional or derivative feedback.

Theorem 19 *Let the* ν *from (36) be well–defined for the triple* $(E(t), A(t), B(t))$ *in (1).*

(a) *The descriptor system (1) can be regularized by proportional state feedback if and only if* $u_\nu^r = f_\nu$.

(b) *The descriptor system (1) can be regularized by derivative feedback if and only if* $u_\nu^r = f_\nu$.

(c) *The descriptor system (1) can be regularized by combined derivative and proportional state feedback if and only if* $u_\nu^r = f_\nu$.

Proof. From Theorem 17 we know that it is sufficient to analyse the descriptor system (43). Therefore, we assume that (1) is of the form (43).

In order to show that the condition $u_\nu^r = f_\nu$ is necessary observe that the last block rows (43(d)) are fullfilled trivially and we can leave these equations off altogether. If $u_\nu^r > f_\nu$ the remaining system (43(a)–(c)) has more columns then rows, i.e. we can choose components of x arbitrarily and the solution will not be unique. If $u_\nu^r < f_\nu$ the system (43(a)–(c)) has more rows then columns and we cannot apply arbitrary controls.

Assume now that $u^r_\nu = f_\nu$. We can choose the proportional feedback

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} -A_{31}(t) & 0 & I_{f_\nu} - A_{33}(t) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

and the closed loop system is then of the form

$$\begin{bmatrix} I_{d_\nu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & A_{13}(t) \\ 0 & I_{a_\nu} & 0 \\ 0 & 0 & I_{f_\nu} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 & B_{12}(t) \\ 0 & 0 \\ I_{f_\nu} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}.$$

The corresponding DAE has the characteristic values $d_{DAE} = d_\nu, a_{DAE} = a_\nu + f_\nu, s_{DAE} = 0, u^r_{DAE} = 0$. Since $u^l_{DAE} = u^l_\nu$ and since the last block row of $B(t)$ of the closed loop system is zero, we get from [24, Corollary 20] that the closed loop system is uniquely solvable for every consistent initial value x_0 and any given control $w(t)$.

In the case of derivative feedback we choose

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & I_{f_\nu} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

and get the closed loop system

$$\begin{bmatrix} I_{d_\nu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{f_\nu} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & A_{13}(t) \\ 0 & I_{a_\nu} & 0 \\ A_{31}(t) & 0 & A_{33}(t) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 & B_{12}(t) \\ 0 & 0 \\ I_{f_\nu} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}.$$

which is as required. (c) follows now immediately. ∎

Corollary 20 *Let ν from (36) be well-defined for the triple $(E(t), A(t), B(t))$ of a square system (1), i.e. $n = l$. The system (1) can be regularized by a proportional state, a derivative or a combined derivative proportional feedback if and only if $u^l_\nu = 0$.*

Theorem 19 and Corollary 20 show that it is sufficient to study the reduction process based on the condensed form (30). Note that there is still a lot of freedom in the choice of the feedback and the canonical form (29) can maybe used to improve robustness of the system or guarantee controllability of the regularized system. For constant coefficient systems this is done in [3, 4, 13] but so far it is not really clear what robustness or controllability means for linear coefficients systems with variable coefficients.

6 Conclusion

We have presented local and global equivalences and corresponding canonical forms for linear descriptor systems with variable coefficients. The global canonical forms and the global condensed forms, which are not as far reduced as the canonical forms, are powerful tools in the analysis of this type of descriptor systems. Based on a condensed form we found under what conditions a linear descriptor system is regularizable, i.e., there exists a derivative and/or proportional state feedback such that the closed loop system is uniquely solvable for all consistent initial vectors. These conditions are necessary and sufficient.

While the global forms are not suitable for numerical computations the numerical accessibility of local quantities which give essential information on the global solution behaviour are of great importance in the development of numerical methods.

We assumed that sequences of characteristic values are constant. As for differential algebraic equations weaker assumptions such as jumps at isolated points connected with a weak solvability concept can be considered.

A Proof of Theorem 10

To proof Theorem 10 we make use of the following property [21, 30]

Lemma 21 *Let* $E \in C^{\ell}([t_1, t_2],\ \mathcal{C}^{n,n})$*,* $\ell \in \mathcal{N}_0$ *and* $\operatorname{rank} E(t) = r$ *for all* $t \in [t_1, t_2]$*. Then there exist* $U, V \in C^{\ell}([t_1, t_2], \mathcal{C}^{n,n})$ *with* $U(t), V(t)$ *nonsingular (unitary) for every* $t \in [t_1, t_2]$ *such that*

$$U(t)^* E(t) V(t) = \begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix},\ t \in [t_1, t_2], \tag{45}$$

where $\Sigma \in C^{\ell}([t_1, t_2], \mathcal{C}^{r,r})$*.*

Proof of Theorem 10. From now on, we will omit the argument t in the proofs and use the word "new" on top of the equivalence operator if we have changed the notation according to the new block structure of the matrices. Using Lemma 21, we have

$$\begin{aligned}
(E, A, B) &\sim (U_1^* \Sigma_1 V_1, U_1^* A V_1 - U_1^* E \dot{V}_1, U_1^* B) \\
&\overset{\text{new}}{\sim} \left(\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} \right) \\
&\overset{\text{new}}{\sim} \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} \right) \\
&\overset{\text{new}}{\sim} \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} \right) \\
&\overset{\text{new}}{\sim} \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ I_f & 0 \\ 0 & 0 \end{bmatrix} \right)
\end{aligned}$$

$$\sim \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12}V_3 \\ A_{21} & A_{22}V_3 \\ U_3^* A_{31} & U_3^* A_{32} V_3 \end{bmatrix} - \begin{bmatrix} I_r & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & \dot{V}_3 \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ I_f & 0 \\ 0 & 0 \end{bmatrix} \right)$$

$$\overset{\text{new}}{\sim} \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & I_a & 0 \\ A_{41} & 0 & 0 \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ I_f & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

$$\overset{\text{new}}{\sim} \left(\begin{bmatrix} E_{11} & E_{12} & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & I_a & 0 & 0 \\ \Sigma_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ I_f & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

$$\overset{\text{new}}{\sim} \left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ I_f & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

$$\overset{\text{new}}{\sim} \left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ 0 & 0 \\ 0 & 0 \\ I_f & 0 \\ 0 & 0 \end{bmatrix} \right)$$

$$\overset{\text{new}}{\sim} \left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} & A_{15} \\ 0 & A_{22} & A_{23} & A_{24} & A_{25} \\ 0 & A_{32} & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & A_{52} & A_{53} & A_{54} & A_{55} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ 0 & 0 \\ 0 & 0 \\ I_f & 0 \\ 0 & 0 \end{bmatrix} \right)$$

$$\sim \left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} & A_{15} \\ 0 & A_{22} & A_{23} & A_{24} & A_{25} \\ 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & A_{52} & A_{53} & A_{54} & A_{55} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ 0 & 0 \\ 0 & 0 \\ I_f & 0 \\ 0 & 0 \end{bmatrix} \right)$$

$$
\stackrel{\text{new}}{\sim} \left(\begin{bmatrix} I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.
$$

$$
\left. \begin{bmatrix} 0 & 0 & A_{13} & 0 & A_{15} & A_{16} & A_{17} \\ 0 & 0 & A_{23} & 0 & A_{25} & A_{26} & A_{27} \\ 0 & 0 & A_{33} & 0 & A_{35} & A_{36} & A_{37} \\ 0 & 0 & 0 & I_a & 0 & 0 & 0 \\ I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{73} & 0 & A_{75} & A_{76} & A_{77} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_{s^c} & 0 \\ 0 & 0 & 0 \\ 0 & B_{32} & B_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)
$$

$$
\stackrel{\text{new}}{\sim} \left(\begin{bmatrix} I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 \\ -B_{32} & 0 & I_d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.
$$

$$
\left. \begin{bmatrix} 0 & 0 & A_{13} & 0 & A_{15} & A_{16} & A_{17} \\ 0 & 0 & A_{23} & 0 & A_{25} & A_{26} & A_{27} \\ 0 & 0 & A_{33} & 0 & A_{35} & A_{36} & A_{37} \\ 0 & 0 & 0 & I_a & 0 & 0 & 0 \\ I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{73} & 0 & A_{75} & A_{76} & A_{77} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_{s^c} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)
$$

$$\overset{\text{new}}{\sim}\left(\begin{bmatrix} I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\right.$$

$$\left.\begin{bmatrix} A_{11} & 0 & A_{13} & 0 & A_{15} & A_{16} & A_{17} \\ A_{21} & 0 & A_{23} & 0 & A_{25} & A_{26} & A_{27} \\ A_{31} & 0 & A_{33} & 0 & A_{35} & A_{36} & A_{37} \\ 0 & 0 & 0 & I_a & 0 & 0 & 0 \\ I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 \\ A_{71} & 0 & A_{73} & 0 & A_{75} & A_{76} & A_{77} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_{s^c} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\right.$$

$$\left.\begin{bmatrix} 0 & 0 & A_{13} & 0 & A_{15} & A_{16} & A_{17} \\ 0 & 0 & A_{23} & 0 & A_{25} & A_{26} & A_{27} \\ 0 & 0 & A_{33} & 0 & A_{35} & A_{36} & A_{37} \\ 0 & 0 & I_a & 0 & 0 & 0 & 0 \\ I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{73} & 0 & A_{75} & A_{76} & A_{77} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_{s^c} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$
\overset{\text{new}}{\sim}
\left(
\begin{bmatrix}
I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{d^c} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{d^u} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\right.
$$

$$
\left.
\begin{bmatrix}
0 & 0 & A_{13} & A_{14} & 0 & A_{16} & A_{17} & A_{18} \\
0 & 0 & A_{23} & A_{24} & 0 & A_{26} & A_{27} & A_{28} \\
0 & 0 & A_{33} & A_{34} & 0 & A_{36} & A_{37} & A_{38} \\
0 & 0 & A_{43} & A_{44} & 0 & A_{46} & A_{47} & A_{48} \\
0 & 0 & 0 & 0 & I_a & 0 & 0 & 0 \\
I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_{83} & A_{83} & 0 & A_{86} & A_{87} & A_{88} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & I_{s^c} & 0 \\
0 & 0 & 0 \\
0 & 0 & I_{d^c} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
I_f & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\right)
$$

$$
\sim
\left(
\begin{bmatrix}
I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Q_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{d^u} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\right.
$$

$$
\left.
\begin{bmatrix}
0 & 0 & A_{13} & A_{14} & 0 & A_{16} & A_{17} & A_{18} \\
0 & 0 & A_{23} & A_{24} & 0 & A_{26} & A_{27} & A_{28} \\
0 & 0 & A_{33}Q_5 - \dot{Q}_5 & A_{34} & 0 & A_{36} & A_{37} & A_{38} \\
0 & 0 & A_{43} & A_{44} & 0 & A_{46} & A_{47} & A_{48} \\
0 & 0 & 0 & 0 & I_a & 0 & 0 & 0 \\
I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_{83} & A_{83} & 0 & A_{86} & A_{87} & A_{88} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & I_{s^c} & 0 \\
0 & 0 & 0 \\
0 & 0 & I_{d^c} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
I_f & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\right)
$$

Q_5 was chosen to be the solution of the initial value problem

$$\dot{Q}_5 - A_{33}Q_5, \;\; Q_5(t_0) = I,$$

which is nonsingular at every point $t \in [t_1, t_2]$, i.e.

$$
\begin{aligned}
&(E, A, B) \\
&\overset{\text{new}}{\sim} \left(\begin{bmatrix} I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{d^c} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{d^u} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right. \\
&\qquad \left. \begin{bmatrix} 0 & 0 & A_{13} & A_{14} & 0 & A_{16} & A_{17} & A_{18} \\ 0 & 0 & A_{23} & A_{24} & 0 & A_{26} & A_{27} & A_{28} \\ 0 & 0 & 0 & A_{34} & 0 & A_{36} & A_{37} & A_{38} \\ 0 & 0 & A_{43} & A_{44} & 0 & A_{46} & A_{47} & A_{48} \\ 0 & 0 & 0 & 0 & I_a & 0 & 0 & 0 \\ I_{s^c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{83} & A_{83} & 0 & A_{86} & A_{87} & A_{88} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_{s^c} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{d^c} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)
\end{aligned}
$$

which at last leads to (29) by a similar final transformation step. ■

B Proof of Theorem 15

From

$$
\begin{aligned}
&\operatorname{rank}(R_{i-1}(t)^*[A_{12}^{(i-1)}(t) A_{14}^{(i-1)}(t) A_{15}^{(i-1)}(t)]) \\
&\qquad = \operatorname{rank}(R_{i-1}(t)^*[A_{14}^{(i-1)}(t) A_{14}^{(i-1)}(t)]) + \operatorname{rank}(W_{i-1}(t)^* R_{i-1}(t)^* A_{12}^{(i-1)}(t))
\end{aligned}
$$

we obtain (39a), while (39b,c) are consequences of Lemma 14 (35f,g).

Starting from Corollary 11 in the permuted form (if we do not perform the last elimination step, i.e. do not zero out A_{22})

$$
\left(\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ A_{61} & A_{62} & A_{63} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I & 0 \end{bmatrix} \right) \begin{array}{c} d_0 \\ s_0 \\ w_0^l \\ s_0 \\ b_0 \\ f_0 \end{array}
$$

we obtain the following equivalent triples of matrix functions in the i–th step (omitting subscripts$^{(i)}$ and denoting by $[A_{ij} \dots A_{ij}]$ a block entry A_{ij} which extends over several block columns)

$$
\left(
\left[\begin{array}{cc|ccccc|cc|cccc}
I & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 & * & \dots & \dots & * \\
0 & 0 & 0 & \dots & \dots & \dots & 0 & I & 0 & * & \dots & \dots & * \\
\hline
 & & 0 & & & & & & & F_i & & & * \\
 & & & \ddots & & & & & & 0 & \ddots & & \\
 & & & & \ddots & & & & & & \ddots & \ddots & \\
 & & & & & \ddots & & & & & & \ddots & F_1 \\
 & & & & & & 0 & & & & & & 0 \\
\hline
 & & & & & & & 0 & 0 & W_{11} & * & \dots & * \\
 & & & & & & & 0 & 0 & W_{21} & * & \dots & * \\
\hline
 & & & & & & & & & 0 & E_{i-1} & & * \\
 & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & \ddots & E_1 \\
 & & & & & & & & & & & & 0 \\
\hline
 & & & & & & & 0 & 0 & * & \dots & \dots & *
\end{array}\right]
\right.
$$

$$
\left.
\left[\begin{array}{cc|ccccc|cc|cccc}
A_{11} & A_{12} & A_{13} & \dots & \dots & \dots & A_{13} & & & & & & \\
A_{21} & A_{22} & A_{23} & \dots & \dots & \dots & A_{23} & & & & & & \\
\hline
 & & 0 & & & & & & & & & & \\
 & & & \ddots & & & & & & & & & \\
 & & & & \ddots & & & & & & & & \\
 & & & & & \ddots & & & & & & & \\
 & & & & & & 0 & & & & & & \\
\hline
 & & & & & & & I & 0 & & & & \\
 & & & & & & & 0 & I & & & & \\
\hline
 & & & & & & & & & I & & & \\
 & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & I \\
\hline
* & * & * & \dots & \dots & \dots & * & 0 & 0 & 0 & \dots & \dots & 0
\end{array}\right],
\left[\begin{array}{cc}
0 & B_{12} \\
0 & B_{22} \\
\hline
0 & 0 \\
\vdots & \vdots \\
\vdots & \vdots \\
\vdots & \vdots \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\hline
0 & 0 \\
\vdots & \vdots \\
\vdots & \vdots \\
0 & 0 \\
\hline
I & 0
\end{array}\right]
\right)
\begin{array}{c}
d_i \\ s_i \\ w_i^l \\ \vdots \\ \vdots \\ \vdots \\ w_0^l \\ s_i \\ b_i \\ c_{i-1} \\ \vdots \\ \vdots \\ c_0 \\ f_i
\end{array}
$$

$$
\overset{\text{new}}{\sim}
\left(
\left[\begin{array}{ccc|ccccc|ccc|cccc}
I & 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 & 0 & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & I & 0 & 0 & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & I & 0 & * & \dots & \dots & * \\
\hline
 & & & 0 & & & & & & & & F_i & & & * \\
 & & & & \ddots & & & & & & & 0 & \ddots & & \\
 & & & & & \ddots & & & & & & & \ddots & \ddots & \\
 & & & & & & \ddots & & & & & & & \ddots & F_1 \\
 & & & & & & & 0 & & & & & & & 0 \\
\hline
 & & & & & & & & 0 & 0 & 0 & W_{11} & * & \dots & * \\
 & & & & & & & & 0 & 0 & 0 & W_{21} & * & \dots & * \\
 & & & & & & & & 0 & 0 & 0 & W_{31} & * & \dots & * \\
\hline
 & & & & & & & & & & & 0 & E_{i-1} & & * \\
 & & & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & & & \ddots & E_1 \\
 & & & & & & & & & & & & & & 0 \\
\hline
 & & & & & & & & 0 & 0 & 0 & * & \dots & \dots & *
\end{array}\right]
\left[\begin{array}{ccc|ccccc|ccc|cccc}
A_{11} & A_{12} & A_{13} & A_{14} & \dots & \dots & \dots & A_{14} & & & & & & & \\
A_{21} & A_{22} & A_{23} & A_{24} & \dots & \dots & \dots & A_{24} & & & & & & & \\
A_{31} & A_{32} & A_{33} & A_{34} & \dots & \dots & \dots & A_{34} & & & & & & & \\
\hline
 & & & 0 & & & & & & & & & & & \\
 & & & & \ddots & & & & & & & & & & \\
 & & & & & \ddots & & & & & & & & & \\
 & & & & & & \ddots & & & & & & & & \\
 & & & & & & & 0 & & & & & & & \\
\hline
 & & & & & & & & I & 0 & 0 & & & & \\
 & & & & & & & & 0 & I & 0 & & & & \\
 & & & & & & & & 0 & 0 & I & & & & \\
\hline
 & & & & & & & & & & & I & & & \\
 & & & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & & & I \\
\hline
* & * & * & * & \dots & \dots & \dots & * & 0 & 0 & 0 & 0 & \dots & \dots & 0
\end{array}\right],
\left[\begin{array}{ccc}
0 & B_{12} & B_{13} \\
0 & I & 0 \\
0 & 0 & 0 \\
\hline
0 & 0 & 0 \\
\vdots & & \vdots \\
\vdots & & \vdots \\
\vdots & & \vdots \\
0 & 0 & 0 \\
\hline
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\hline
0 & 0 & 0 \\
\vdots & & \vdots \\
\vdots & & \vdots \\
0 & 0 & 0 \\
\hline
I & 0 & 0
\end{array}\right]
\right)
\begin{array}{c}
d_i \\
g_{i+1} \\
s_i - g_{i+1} \\
w_i^l \\
\vdots \\
\vdots \\
\vdots \\
w_0^l \\
g_{i+1} \\
s_i - g_{i+1} \\
b_i \\
c_{i-1} \\
\vdots \\
\vdots \\
c_0 \\
f_i
\end{array}
$$

$$
\overset{\text{new}}{\sim}\left(\left[\begin{array}{cc|ccccc|ccc|cccc}
I & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & * & \cdots & \cdots & * \\
0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & I & 0 & * & \cdots & \cdots & * \\
\hline
 & & 0 & & & & & & & & F_i & & & * \\
 & & & \ddots & & & & & & & 0 & \ddots & & \\
 & & & & \ddots & & & & & & & \ddots & \ddots & \\
 & & & & & \ddots & & & & & & & \ddots & F_1 \\
 & & & & & & 0 & & & & & & & 0 \\
\hline
 & & & & & & & 0 & 0 & 0 & W_{11} & * & \cdots & * \\
 & & & & & & & 0 & 0 & 0 & W_{21} & * & \cdots & * \\
 & & & & & & & 0 & 0 & 0 & W_{31} & * & \cdots & * \\
\hline
 & & & & & & & & & & 0 & E_{i-1} & & * \\
 & & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & & \ddots & E_1 \\
 & & & & & & & & & & & & & 0 \\
\hline
 & & & & & & & * & 0 & 0 & * & \cdots & \cdots & *
\end{array}\right],\right.
$$

$$
\left[\begin{array}{cc|ccccc|ccc|cccc}
A_{11} & A_{12} & A_{13} & \cdots & \cdots & \cdots & A_{13} & & & & & & & \\
A_{21} & A_{22} & A_{23} & \cdots & \cdots & \cdots & A_{23} & & & & & & & \\
\hline
 & & 0 & & & & & & & & & & & \\
 & & & \ddots & & & & & & & & & & \\
 & & & & \ddots & & & & & & & & & \\
 & & & & & \ddots & & & & & & & & \\
 & & & & & & 0 & & & & & & & \\
\hline
 & & & & & & & I & 0 & 0 & & & & \\
 & & & & & & & 0 & I & 0 & & & & \\
 & & & & & & & 0 & 0 & I & & & & \\
\hline
 & & & & & & & & & & I & & & \\
 & & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & & I \\
\hline
* & * & * & \cdots & \cdots & \cdots & * & 0 & 0 & 0 & 0 & \cdots & \cdots & 0
\end{array}\right],
$$

$$
\left.\left[\begin{array}{cc}
0 & B_{12} \\
0 & 0 \\
\hline
0 & 0 \\
\vdots & \vdots \\
\vdots & \vdots \\
\vdots & \vdots \\
0 & 0 \\
\hline
0 & 0 \\
0 & 0 \\
0 & 0 \\
\hline
0 & 0 \\
\vdots & \vdots \\
\vdots & \vdots \\
0 & 0 \\
\hline
I & 0
\end{array}\right]\right)
\quad
\begin{array}{c}
d_i \\
s_i - g_{i+1} \\
w_i^l \\
\vdots \\
\vdots \\
\vdots \\
w_0^l \\
g_{i+1} \\
s_i - g_{i+1} \\
b_i \\
c_{i-1} \\
\vdots \\
\vdots \\
c_0 \\
f_{i+1}
\end{array}
$$

$$
\overset{\text{new}}{\sim}\left(
\left[\begin{array}{ccc|ccccc|cccc|cccc}
I & 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & U_{11} & U_{12} & 0 & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & U_{21} & U_{22} & 0 & * & \dots & \dots & * \\
\hline
 & & & 0 & & & & & & & & & F_i & & & * \\
 & & & & \ddots & & & & & & & & 0 & \ddots & & \\
 & & & & & \ddots & & & & & & & & \ddots & \ddots & \\
 & & & & & & \ddots & & & & & & & & \ddots & F_1 \\
 & & & & & & & 0 & & & & & & & & 0 \\
\hline
 & & & & & & & & 0 & & & & W_{11} & * & \dots & * \\
 & & & & & & & & & \ddots & & & W_{21} & * & \dots & * \\
 & & & & & & & & & & \ddots & & W_{31} & * & \dots & * \\
 & & & & & & & & & & & 0 & W_{41} & * & \dots & * \\
\hline
 & & & & & & & & & & & & 0 & E_{i-1} & & * \\
 & & & & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & & & & \ddots & E_1 \\
 & & & & & & & & & & & & & & & 0 \\
\hline
 & & & & & & & & * & 0 & 0 & 0 & * & \dots & \dots & *
\end{array}\right],
\right.
$$

$$
\left[\begin{array}{ccc|ccccc|cccc|cccc}
A_{11} & A_{12} & A_{13} & A_{14} & \dots & \dots & \dots & A_{14} & & & & & & & & \\
A_{21} & I & 0 & 0 & \dots & \dots & \dots & 0 & & & & & & & & \\
A_{31} & 0 & 0 & 0 & \dots & \dots & \dots & 0 & & & & & & & & \\
\hline
 & & & 0 & & & & & & & & & & & & \\
 & & & & \ddots & & & & & & & & & & & \\
 & & & & & \ddots & & & & & & & & & & \\
 & & & & & & \ddots & & & & & & & & & \\
 & & & & & & & 0 & & & & & & & & \\
\hline
 & & & & & & & & I & & & & & & & \\
 & & & & & & & & & I & & & & & & \\
 & & & & & & & & & & I & & & & & \\
 & & & & & & & & & & & I & & & & \\
\hline
 & & & & & & & & & & & & I & & & \\
 & & & & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & & & & I \\
\hline
* & * & * & * & \dots & \dots & \dots & * & 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0
\end{array}\right],
$$

$$
\left.
\left[\begin{array}{cc}
0 & B_{12} \\
0 & 0 \\
0 & 0 \\
\hline
0 & 0 \\
\vdots & \vdots \\
\vdots & \vdots \\
\vdots & \vdots \\
0 & 0 \\
\hline
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\hline
0 & 0 \\
\vdots & \vdots \\
\vdots & \vdots \\
0 & 0 \\
\hline
I & 0
\end{array}\right]\right)
\begin{array}{c}
d_i \\
b_{i+1} \\
s_i - g_{i+1} - b_{i+1} \\
w_i^l \\
\vdots \\
\vdots \\
\vdots \\
w_0^l \\
g_{i+1} \\
b_{i+1} \\
s_i - g_{i+1} - b_{i+1} \\
b_i \\
c_{i-1} \\
\vdots \\
\vdots \\
c_0 \\
f_{i+1}
\end{array}
$$

$$
\underset{\sim}{\text{new}} \left(\left[\begin{array}{cccc|ccccc|c|cccc}
I & 0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & * & \dots & \dots & * \\
0 & I & 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & U_{11} & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & U_{21} & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & U_{31} & * & \dots & \dots & * \\
\hline
 & & & & 0 & & & & & & F_i & & & * \\
 & & & & & \ddots & & & & & 0 & \ddots & & \\
 & & & & & & \ddots & & & & & \ddots & \ddots & \\
 & & & & & & & \ddots & & & & & \ddots & F_1 \\
 & & & & & & & & 0 & & & & & 0 \\
\hline
 & & & & & & & & & 0 & E_i & * & \dots & * \\
\hline
 & & & & & & & & & & 0 & E_{i-1} & & * \\
 & & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & & \ddots & E_1 \\
 & & & & & & & & & & & & & 0 \\
\hline
 & & & & & & & & & * & * & \dots & \dots & *
\end{array} \right], \right.
$$

$$
\left[\begin{array}{cccc|ccccc|c|cccc}
A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & \dots & \dots & \dots & A_{15} & & & & & \\
A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & \dots & \dots & \dots & A_{25} & & & & & \\
A_{31} & A_{41} & I & 0 & 0 & \dots & \dots & \dots & 0 & & & & & \\
0 & I & 0 & 0 & 0 & \dots & \dots & \dots & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & & & & & \\
\hline
 & & & & 0 & & & & & & & & & \\
 & & & & & \ddots & & & & & & & & \\
 & & & & & & \ddots & & & & & & & \\
 & & & & & & & \ddots & & & & & & \\
 & & & & & & & & 0 & & & & & \\
\hline
 & & & & & & & & & I & & & & \\
\hline
 & & & & & & & & & & I & & & \\
 & & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & & I \\
\hline
* & * & * & * & * & \dots & \dots & \dots & * & 0 & 0 & \dots & \dots & 0
\end{array} \right],
$$

$$
\left. \left[\begin{array}{cc}
0 & B_{12} \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\hline
0 & 0 \\
\vdots & \vdots \\
\vdots & \vdots \\
\vdots & \vdots \\
0 & 0 \\
\hline
0 & 0 \\
\hline
0 & 0 \\
\vdots & \vdots \\
\vdots & \vdots \\
0 & 0 \\
\hline
I & 0
\end{array} \right] \right) \quad
\begin{array}{c}
d_{i+1} \\
s_{i+1} \\
b_{i+1} \\
s_{i+1} \\
w^l_{i+1} \\
w^l_i \\
\vdots \\
\vdots \\
\vdots \\
w^l_0 \\
c_i \\
c_{i-1} \\
\vdots \\
\vdots \\
c_0 \\
f_{i+1}
\end{array}
$$

$$
\overset{\text{new}}{\sim}\left(
\left[\begin{array}{cccc|ccccc|c|cccc}
I & 0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & * & * & \dots & \dots & * \\
0 & I & 0 & 0 & 0 & \dots & \dots & \dots & 0 & * & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & W_{21} & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & W_{11} & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & F_{i+1} & * & \dots & \dots & * \\
\hline
 & & & & 0 & & & & & & F_i & & & * \\
 & & & & & \ddots & & & & & 0 & \ddots & & \\
 & & & & & & \ddots & & & & & \ddots & \ddots & \\
 & & & & & & & \ddots & & & & & \ddots & F_1 \\
 & & & & & & & & 0 & & & & & 0 \\
\hline
 & & & & & & & & & 0 & E_i & * & \dots & * \\
\hline
 & & & & & & & & & & 0 & E_{i-1} & & * \\
 & & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & & \ddots & E_1 \\
 & & & & & & & & & & & & & 0 \\
\hline
 & & & & & & & & & * & * & \dots & \dots & *
\end{array}\right],\right.
$$

$$
\left[\begin{array}{cccc|ccccc|c|cccc}
A_{11} & 0 & 0 & A_{14} & A_{15} & \dots & \dots & \dots & A_{15} & & & & & \\
A_{21} & 0 & 0 & A_{24} & A_{25} & \dots & \dots & \dots & A_{25} & & & & & \\
0 & 0 & I & 0 & 0 & \dots & \dots & \dots & 0 & & & & & \\
0 & I & 0 & 0 & 0 & \dots & \dots & \dots & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & & & & & \\
\hline
 & & & & 0 & & & & & & & & & \\
 & & & & & \ddots & & & & & & & & \\
 & & & & & & \ddots & & & & & & & \\
 & & & & & & & \ddots & & & & & & \\
 & & & & & & & & 0 & & & & & \\
\hline
 & & & & & & & & & I & & & & \\
\hline
 & & & & & & & & & & I & & & \\
 & & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & & I \\
\hline
* & 0 & 0 & * & * & \dots & \dots & \dots & * & 0 & 0 & \dots & \dots & 0
\end{array}\right],
$$

$$
\left.\left[\begin{array}{cc}
0 & B_{12} \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\hline
0 & 0 \\
\vdots & \vdots \\
\vdots & \vdots \\
\vdots & \vdots \\
0 & 0 \\
\hline
0 & 0 \\
\hline
0 & 0 \\
\vdots & \vdots \\
\vdots & \vdots \\
0 & 0 \\
\hline
I & 0
\end{array}\right]\right)
\begin{array}{c}
d_{i+1} \\
s_{i+1} \\
b_{i+1} \\
s_{i+1} \\
w^l_{i+1} \\
w^l_i \\
\vdots \\
\vdots \\
\vdots \\
w^l_0 \\
c_i \\
c_{i-1} \\
\vdots \\
\vdots \\
c_0 \\
f_{i+1}
\end{array}
$$

$$
\overset{\text{new}}{\sim}
\left(
\left[\begin{array}{cc|ccccc|cc|cccc}
I & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & * & * & \dots & * \\
0 & 0 & 0 & 0 & \dots & \dots & 0 & I & 0 & * & * & \dots & * \\
\hline
 & & 0 & & & & & 0 & 0 & F_{i+1} & * & \dots & * \\
 & & & 0 & & & & & & 0 & F_i & & * \\
 & & & & \ddots & & & & & & \ddots & \ddots & \\
 & & & & & \ddots & & & & & & \ddots & F_1 \\
 & & & & & & 0 & & & & & & 0 \\
\hline
0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & W_{11} & * & \dots & * \\
0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & W_{21} & * & \dots & * \\
\hline
 & & & & & & & & & 0 & E_i & & * \\
 & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & \ddots & E_1 \\
 & & & & & & & & & & & & 0 \\
\hline
 & & & & & & & & & * & * & \dots & *
\end{array}\right]
\right.
$$

$$
\left.
\left[\begin{array}{cc|ccccc|cc|cccc}
A_{11} & A_{12} & A_{13} & A_{14} & \dots & \dots & A_{14} & & & & & & \\
A_{21} & A_{22} & A_{23} & A_{24} & \dots & \dots & A_{24} & & & & & & \\
\hline
 & & 0 & & & & & & & & & & \\
 & & & \ddots & & & & & & & & & \\
 & & & & \ddots & & & & & & & & \\
 & & & & & \ddots & & & & & & & \\
 & & & & & & 0 & & & & & & \\
\hline
 & & & & & & & I & & & & & \\
 & & & & & & & & I & & & & \\
\hline
 & & & & & & & & & I & & & \\
 & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & I \\
\hline
* & * & * & * & \dots & \dots & * & 0 & 0 & 0 & \dots & \dots & 0
\end{array}\right],
\left[\begin{array}{cc}
0 & B_{12} \\
0 & 0 \\
\hline
0 & 0 \\
\vdots & \vdots \\
\vdots & \vdots \\
\vdots & \vdots \\
0 & 0 \\
\hline
0 & 0 \\
0 & 0 \\
\hline
0 & 0 \\
\vdots & \vdots \\
\vdots & \vdots \\
0 & 0 \\
\hline
I & 0
\end{array}\right]
\right)
\begin{array}{c}
d_{i+1} \\
s_{i+1} \\
w^l_{i+1} \\
\vdots \\
\vdots \\
\vdots \\
w^l_0 \\
s_{i+1} \\
b_{i+1} \\
c_i \\
c_{i-1} \\
\vdots \\
c_0 \\
f_{i+1}
\end{array}
$$

Thus (40) follows by induction and (41) holds, since $\begin{bmatrix} F_{i+1} \\ E_{i+1} \end{bmatrix}$ is obtained by nonsingular transformations applied to $[0\ U\ 0]$, with nonsingular U, where U is the transformation used above in the third step. ■

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