τ-EXTRAPOLATION – THEORETICAL FOUNDATION, NUMERICAL EXPERIMENT AND APPLICATION TO NAVIER-STOKES EQUATIONS

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Abstract. This article deals with τ-extrapolation – a modification of the multigrid method, which leads to solutions with an improved convergence order. The number of numerical operations depends linearly on the problem size and is not much higher than for a multigrid method without this modification.
Section 1 and section 2 contain a short mathematical foundation of the τ-extrapolation. Section 3 deals with a careful tuning of some multigrid components necessary for a successful application of τ-extrapolation.
Section 4 presents numerical illustrations to the theoretical investigations for one-dimensional test problems.
Section 5 contains some experience with the use of τ-extrapolation for the Navier-Stokes equations.

Key words. Finite Differences, Extrapolation, Multigrid, Navier-Stokes equations.

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1. Introduction. We consider a boundary-value problem

$$Au = f$$

with the solution $u = u^*$ and a difference approximation with discretization parameter H

$$A_H u_H = f_H \,.$$

A and A_H are linear operators, f is related to f_H by

$$f_H = R_H f. \tag{1}$$

R_H is a restriction operator projecting the right hand side f into the image space of A_H.

The truncation error of the discrete problem is given by inserting a projection $\hat{R}_H u^*$ of the exact solution in the discrete equation:

$$\tau_H(u^*) = A_H \hat{R}_H u^* - f_H = A_H \hat{R}_H u^* - R_H A u^* \,. \tag{2}$$

Operators R_H and $\hat{R}_H$ can agree, if preimage and image of A coincide.

The approximation order of a discrete problem is defined by the relation

$$\tau_H(u) = A_H \hat{R}_H u - R_H A u = O(H^p) \quad \text{for} \quad u \in C^{o+p} \,,$$

where o is the order of the differential operator A and p is the approximation order of operator A_H.

An approximation of $\tau_H(u^*)$ can be used to improve the accuracy of the original discrete problem:

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Lemma 1.1. *For the right hand side of* $A_H u_H = \tilde{f}_H$
let $\tilde{f}_H = f_H + \tau_H(u^*) + O(H^\alpha)$ *with* $\alpha > p$
and suppose (1), (2) and $||A_H||^{-1} \leqslant M$, M *independent of* H.
Then if follows $||u_H - \hat{R}_H u^*|| = O(H^\alpha)$.

Proof: The difference of $A_H u_H = \tilde{f}_H$ and $A_H \hat{R}_H u^* = f_H + \tau_H(u^*)$ gives
$A_H(u_H - \hat{R}_H u^*) = O(H^\alpha)$, i.e. $||(u_H - \hat{R}_H u^*)|| \leqslant M \cdot O(H^\alpha)$ ■

Remark: A direct application of Lemma 1.1. presupposes $||\hat{R}_H u^* - I_H u^*|| \leqslant O(H^\alpha)$, this can be fulfilled most easily by $\hat{R}_H = I_H$. A correction of the right hand side f_H, which estimates $\tau_H(u^*)$ with an error of order $\alpha > p$, improves the accuracy of the solution of $A_H u_H = f_H$ to the same order.

2. Multigrid algorithm and τ-extrapolation. We assume, that the reader is familiar with the multigrid method including the usual notations. Detailed information can be found in [2], [14] or in [3]. A schematic representation of the multigrid algorithm (MG-algorithm) and the full-multigrid algorithm (FMG-algorithm) is given in the appendix of this article.
In the full approximation scheme (FAS) the problems to be solved on coarser grids can be written in the form

$$A_H u_H = f_H + \tau_h^H(u_h) \quad \text{with} \quad \tau_h^H(u_h) = A_H \hat{R}_h^H u_h - R_h^H A_h u_h \,. \tag{3}$$

The correction term on the right hand side can be considered as an estimation of the approximation error based on the solution on the finer grid. Supposing

$$\hat{R}_h^H \hat{R}_h = \hat{R}_H \text{ and } R_h^H R_h = R_H \tag{4}$$

one can show by a short calculation, that the accuracy of the solution u_H on the coarse grid is the same as that of u_h on the fine grid (see [1]). However, a higher order of accuracy can not be obtained in this way. Taking into consideration Lemma 1.1. we need a correction term which is a better approximation to $\tau_h^H(u^*)$. Such an approximation gives the following

Lemma 2.1. *Assume (3),* $H = 2h$, *and*
(A1) $\tilde{u}_h = \hat{R}_h(u^* + \xi)$, $\xi = O(H^q)$, $\xi \in C^{(o+p)}$
(A2) $\tau_h(u) = O(H^p)$ *for* $u \in C^{(o+p)}$
(A3) $R_h^H \tau_h(u^*) = \frac{1}{2^p}\tau_H(u^*) + O(H^\beta)$, $\beta > p$.
Then it follows
$\frac{2^p}{2^p-1}\tau_h^H(\tilde{u}_h) = \tau_H(u^*) + O(H^\alpha)$ *with* $\alpha = \min(p+q, \beta)$.

Proof: Based on

$$\begin{aligned}
\tau_h^H(\hat{R}_h u^*) &= A_H \hat{R}_h^H \hat{R}_h u^* - R_h^H A_h \hat{R}_h u^* \\
&= (A_H \hat{R}_H u^* - R_H A u^*) - (R_h^H A_h \hat{R}_h u^* - R_h^H R_h A u^*) \\
&= \tau_H(u^*) - R_h^H \tau_h(u^*) = \tau_H(u^*) - \frac{1}{2^p}\tau_H(u^*) + O(H^\beta) \\
&= \frac{2^p - 1}{2^p}\tau_H(u^*) + O(H^\beta)
\end{aligned}$$

$$\text{we obtain}\quad \tau_h^H(\tilde{u}_h) = \tau_h^H\left(\hat{R}_h(u^*+\xi)\right) = \tau_h^H(\hat{R}_h u^*) + \tau_h^H(\hat{R}_h\xi)$$
$$= \frac{2^p-1}{2^p}\tau_H(u^*) + O(H^\beta) + O(H^{p+q})\,. \qquad \blacksquare$$

Lemma 1.1. and Lemma 2.1. together lead to the following improved formulation of the problem for the coarse grid

$$A_H u_H = f_H + \frac{2^p}{2^p-1}\tau_h^H(\tilde{u}_h)\,. \tag{5}$$

If h is the discretization parameter on the finest grid, equation (5) gives a higher order approximation on the grid with the discretization parameter $H = 2h$.

$$u_H = \hat{R}_H u^* + O(H^\alpha) \;\text{ with }\; \alpha > p\,.$$

Using the usual formulation (3) of the problems for coarser grids the improved accuracy can be carried over up to the coarsest grid.

In the correction step the solution on grid H can be transfered to grid h in such a way, that the order α for the low-frequency part remains unchanged.

The following post-smoothing step with the right hand side f_h tends to reduce the improved approximation order from α back to p. This effect can be counteracted in several ways:

1. The post-smoothing step on the finest grid is supressed.
2. Only one iteration for post-smoothing is done – the effect on low-frequency parts of the solution is small.
3. The right hand side f_h is corrected by

$$\tilde{f}_h = f_h + \frac{1}{2^p-1}P_H^h\tau_h^H(\tilde{u}_h)\,. \tag{6}$$

   This can be done together with the τ-extrapolation (5). P_H^h is a prolongation operator. Because of $\tau_H(u^*) = O(H^p)$, $\tau_h(u^*) = O(h^p)$ and $H = 2h$ this **post-smoothing correction** is one fourth of the correction of f_H .

If a multigrid cycle on grid level h has been finished, one can go to a new still finer grid with discretization parameter $\frac{h}{2}$. This is done by prolongation of the solution u_h. On the new grid the problem

$$A_{\frac{h}{2}} u_{\frac{h}{2}} = f_{\frac{h}{2}}$$

must be solved. Without any correction it would again tend to a solution of order p. A new application of τ-extrapolation – now for the correction of f_h and eventually for a post-smoothing correction of $f_{\frac{h}{2}}$ – improves the order to α.
Before performing the pre-smoothing step on the new grid, it is possible to correct the right hand side $f_{\frac{h}{2}}$ in a **fine-grid correction** step, defined as

$$\tilde{f}_{\frac{h}{2}} = f_{\frac{h}{2}} + \frac{1}{2^p(2^p-1)}P_h^{\frac{h}{2}}P_H^h\tau_h^H(\tilde{u}_h)\,. \tag{7}$$

This step can be performed at the same time as the post-smoothing correction, when the finest grid has not been reached.

If all components of the FMG-algorithm are carefully chosen, the τ-extrapolation leads to a solution of the original problem with order $O(H^\alpha)$, $\alpha > p$. The computational amount for the τ-extrapolation itself is very small.

In the appendix a dashed box shows the place, where the FAS-MG-algorithm has to be changed. The τ-extrapolation replaces the step for calculation of F_{k-1} by the sequence of steps given below. If post-smoothing correction and fine grid correction are omitted, there remains only a small modification in the calculation of F_{k-1}.

3. Conditions for a multigrid method with τ-extrapolation.

3.1. Prolongation. In the FMG-algorithm prolongation occures in two different situations. One situation is the correction of a fine grid solution u_h with the solution u_H from the coarser grid, the other is the first interpolation of a solution on a new grid in the FMG-algorithm (FMG-prolongation). In booth situations it is necessary to save the accuracy reached on the coarser grid and to avoid too strong high-frequency errors on the finer grid.

3.1.1. Prolongation in the correction step of the MG-algorithm. The correction step can be written as

$$u_h := u_h + P_H^h(u_H - \hat{R}_h^H u_h) .$$

In the case $\hat{R}_h^H \neq I_h$ the term $\hat{R}_h^H u_h$ causes an error, which after the correction step can be found in the low-frequency part of the error of u_h.
Operator P_H^h primarily produces high-frequency errors. These errors depend on the order of magnitude of the function to be interpolated and on the interpolation formula. The interpolation error of a smooth function by an interpolation polynomial of order $(n-1)$ is of order $O(H^n)$ (the proof is given in [13]).
Table 1 shows the order of the errors of u_H and u_h, the order of magnitude of $u_H - \hat{R}_h^H u_h$ and finally the order of the low- and high-frequency parts of the error of u_h after the correction.
The conditions for the quality of the interpolation given in Table 1 are based on the following considerations:

1. The low-frequency errors caused by $\hat{R}_h^H u_h$ must be not larger than the (low-frequency with respect to the fine grid) errors in u_H.

2. The high-frequency errors from the prolongation have a strong effect on the magnitude of the defects. If o is the order of the differential equation to be solved, then errors in u_h appear o times larger in the defects. In order to guarantee that such high-frequency parts of the defects do not cause trouble, the magnitude of the oscillations must not exceed the magnitude of the defects themselves. Practical experiments show, that even interpolation errors of the same order of magnitude may decrease the convergence rate of the MG-algorithm.

In most cases there is no problem to fulfill the conditions in the table. However, the choices $n = o$ or $s = \alpha - p$ can cause a loss of accuracy, which has to be compensated by a larger number of smoothing steps or by replacing V-cycles by F-cycles.

TABLE 1

Order of errors caused by prolongation and conditions for s and n

Multigrid-method	Error of		Low-frequency error of $\hat{R}_h^H u_h$ Restriction by		Order of magnitude of	High-frequency error of
	u_H	u_h	Averaging ($s < \infty$)	Injection ($s = \infty$)	$(u_H - \hat{R}_h^H u_h)$	$P_H^h(u_H - \hat{R}_h^H u_h)$
without τ-extrapolation	p	p	$p+s$	∞	p	$p+n$
Conditions						$n \geqslant o$
with τ-extrapolation without (7)	α	p	$p+s$	∞	p	$p+n$
Conditions			$s \geqslant \alpha - p$			$n \geqslant o$
with τ-extrapolation with (6), (7)	α	α	$\alpha+s$	∞	α	$\alpha+n$
Conditions						$n \geqslant o$

Parameters:

p – approximation order of the discrete operator
α – order of accuracy of the MG-method with tauextrapolation
s – order of accuracy of $\hat{R}_h^H$
n – order of the error for polynomial prolongation with degree $(n-1)$
o – order of the differential equation to be solved

3.1.2. Prolongation in the FMG-algorithm. The FMG-prolongation $\tilde{P}_h^{\frac{h}{2}}$ produces the initial solution for a MG-cycle on a new grid. The quality of this interpolation has a great influence on the accuracy of the whole method. An essential difference to the last section is that we have to interpolate the solution, i.e. a function with an order of magnitude $O(1)$, and not a correction to the solution.
If o is the order of the differential operator the errors caused by the interpolation of the solution u_h should be at least o orders smaller than the defects. This can be obtained by

$$n - o \geqslant p\,, \text{ i.e. } n \geqslant p + o$$

in the case of the FMG-algorithm without τ-extrapolation and in the case of τ-extrapolation without fine grid correction

and by $n - o \geqslant \alpha\,,$ i.e. $n \geqslant \alpha + o$

in the case of τ-extrapolation with post-smoothing- and fine-grid correction. The conditions do not guarantee smooth defects after prolongation but oscillations decrease with the same order as the defects, if the grids become increasingly finer.

With the usual number of pre-smoothing steps oscillations in the defects can not be smoothed completely. If the restriction $A_h u_h$ in (3) is performed with an averaging operator, the remainig wiggles can be tolerated because $R_h^H A_h u_h$ operates like a filter which removes them. However, if R_h^H is an injection operator, the conditions $n \geqslant p+o$ and $n \geqslant \alpha + o$ can be insufficient. In this case the interpolation should be taken one or two orders higher.

3.2. Combination of the restriction operators $\hat{R}_h^H$ and R_h^H in the context of τ-extrapolation.

3.2.1. Linear problems. In the MG-algorithm without τ-extrapolation the two restriction operators $\hat{R}_h^H$ und R_h^H can be chosen independently. For the restriction of u_h, injection $\hat{R}_h^H = I_h$ is sufficient, because u_h is a smooth function. For the restriction of $A_h u_h$, however, an averaging operator is a better choice, because the defects are often less smooth.
In the case of τ-extrapolation, the two restriction operators must fit together, as the following considerations will show.
Special requirements must be fulfilled indeed only on the current finest level of the FMG-algorithm, which is used for the estimation of the discretization error. On the coarser grids it is possible to work with the usual combination, i.e. injection for the solution – averaging for the defect.

The essential assumption of Lemma 2.1. is

$$\text{(A3):} \quad R_h^H \tau_h(u^*) = \frac{1}{2^p} \tau_H(u^*) + O(h^\beta)\,, \quad \beta > p\,.$$

<u>**Case A:**</u> $\hat{R}_h^H = R_h^H = I_h^H,\ \hat{R}_H = R_H = I_H,\ \hat{R}_h = R_h = I_h$

(Injection – Injection)

If all restriction operatores are injection operators, assumption (A3) follows immediately from the representations of the approximation errors on two consequtive grids. Suppose:

$$\tau_H(u^*) = \tau_H^{I,I}(u^*) = A_H I_H u^* - I_H A u^* = I_H c(x) H^p + O(H^r)$$

and

$$\tau_h(u^*) = \tau_h^{I,I}(u^*) = A_h I_h u^* - I_h A u^* = I_h c(x) h^p + O(h^r)\,.$$

Because of $\quad I_h^H I_h c(x) h^p = \frac{1}{2^p} I_H c(x) H^p \quad$ we obtain

$$I_h^H \tau_h^{I,I}(u^*) = \frac{1}{2^p} \tau_H^{I,I}(u^*) + O(h^\beta) \quad \text{with} \quad \beta = r > p\,.$$

In the literature the great majority of articles uses this combination of restriction operators in the τ-extrapolation step. Only in [10], [2], [7] some hints at other possibilities are given. In the case of staggered grids, injection for $\hat{R}_h^H$, R_h^H and because of (4) for $\hat{R}_H$, R_H too is excluded, for points of the coarser grid are not collocated with points of the finer grid. In this situation it is necessary to work with averaging operators. Such operators can be favourable also for nonstaggered grids, because they have a stabilizing effect on the τ-extrapolation.
However, there are combinations of R_h^H and $\hat{R}_h^H$ that are inappropriate for the τ-extrapolation , though (4) is satisfied.

<u>**Case B:**</u> $\hat{R}_h^H = I_h^H,\ \hat{R}_H = I_H,\ \hat{R}_h = I_h, \quad R_h^H = M_h^H,\ R_H = M_H,\ R_h = I_h$

(Injection – Averaging)

<u>Remark</u>: This combination can be used for nonstaggered grids, M_h^H and M_H are averaging operators, wich work on $A_h u_h$ and Au^*. Conditions (4) are fulfilled if $M_H = M_h^H I_h$ is valid.

With $\tau_H^{I,M}(u^*) = A_H I_H u^* - M_H A u^*$ we get

$$\tau_H^{I,M}(u^*) - \tau_H^{I,I}(u^*) = (M_H - I_H) A u^* \,.$$

Supposing

(8) $\quad (M_H - I_H)\varphi = C H^s I_H \varphi^{(s)} + O(H^t), \quad s$–order of accuracy of M_H, $t > s$

it follows $\tau_H^{I,M}(u^*) - \tau_H^{I,I}(u^*) = C H^s I_H (A u^*)^{(s)} + o(H^s) = O(H^s)$ and therefore

$$\begin{aligned}\tau_H^{I,M}(u^*) &= \tau_H^{I,I}(u^*) + O(H^s) = I_H c(x) H^p + O(H^r) + O(H^s) \\ &= I_H c(x) H^p + O(H^{\min(r,s)}) \,.\end{aligned}$$

As for Case A it holds

$$\tau_h(u^*) = \tau_h^{I,I}(u^*) = I_h c(x) h^p + O(h^r) \,.$$

With (8) we get $\quad M_h^H \tau_h^{I,I}(u^*) = M_H c(x) h^p + O(h^r) = I_H c(x) h^p + O(H^{\min(r,p+s)})$

$$= \frac{1}{2^p} \tau_H^{I,M}(u^*) + O(h^\beta) \,, \quad \beta = \min(r, s, p+s) \,.$$

Assumption (A3) in Lemma 2.1. ($\beta > p$) is fulfiled only for $s > p$. In the case of second order central difference approximations ($p = 2$, $r = 4$) even $s > p + 1$ must be required. Otherwise the effect of τ-extrapolation is limited from the first to $\alpha = 3$.

If the right hand side of the original problem has the property $(Au^*)^{(s)} = f^{(s)} \equiv 0$, then it follows that $\tau_H^{I,M}(u^*) \equiv \tau_H^{I,I}(u^*)$, because the difference of the two terms depends on $(Au^*)^{(s)}$ and on the derivatives of this term, which are all zero. The value of β then is $\beta = \min(r, p+s)$.

<u>Remark:</u> The τ-extrapolation can work without premise (3) of Lemma 2.1., if parts of the error in the estimation of $\tau_H(u^*)$ are compensated by a suitable restriction for f_H in (5), which differs from M_H. Examples for this choice can be found in [10]. Postsmooting- and fine grid correction probably will not work in this case.

<u>Case C:</u> $\quad \hat{R}_h^H = R_h^H = M_h^H, \; \hat{R}_H = R_H = M_H, \; \hat{R}_h = R_h = I_h$

(Equal averaging)

Equations (4) are fulfilled as before, if $M_H = M_h^H I_h$. For linear operators A_H in the case of (8) we have

$$\begin{aligned}\tau_H^{M,M}(u^*) - \tau_H^{I,I}(u^*) &= A_H (M_H - I_H) u^* - (M_H - I_H) A u^* \\ &= A_H C H^s I_H (u^*)^{(s)} - C H^s I_H (A u^*)^{(s)} + O(H^t) \\ &= C H^s [A_H I_H (u^*)^{(s)} - I_H A (u^*)^{(s)}] + O(H^t) \\ &= C H^s \tau_H^{I,I}((u^*)^{(s)}) + O(H^t) = O(H^{\min(p+s,t)}) \,.\end{aligned}$$

As in case B, it follows that $M_h^H \tau_h^{I,I}(u^*) = \frac{1}{2^p}\tau_H^{M,M}(u^*)+O(h^\beta)$, $\beta = \min(r, p+s, t)$.

An essential supposition is, that the restriction of the solution and the right hand side is performed by the same averaging operator. A higher accuracy of this restriction is not necessary up to now, because p and s sum up.

However from Lemma 1.1. and Lemma 2.1. it follows that $\|u_H - \hat{M}_H u^*\| = O(H^\alpha)$, i.e. u_H approximates with high accuracy an interpolation of u^* with low accuracy and is therefore only of the low accuracy $O(H^s)$.
Nevertheless it is possible to show convergence of order α for the solution on the fine grid, if the correction step from the coarse to the fine grid is taken into consideration.

$$
\begin{aligned}
u_h &= P_H^h u_H + (I_h - P_H^h \hat{R}_h^H)\tilde{u}_h \\
&= P_H^h \hat{R}_h^H I_h u^* + (I_h - P_H^h \hat{R}_h^H)\tilde{u}_h + O(H^\alpha) \\
&= (I_h - P_H^h \hat{R}_h^H)(\tilde{u}_h - I_h u^*) + I_h u^* + O(H^\alpha) \\
&= I_h u^* + (I_h - P_H^h \hat{R}_h^H)\cdot O(H^q) + O(H^\alpha) \\
&= I_h u^* + O(H^{\tilde{\alpha}}), \quad \tilde{\alpha} = \min(n+q, s+q, \alpha) = \min(n+q, s+q, p+q, r, p+s, t).
\end{aligned}
$$

Even post-smootiong- and fine grid correction will work in this case, because $\tau_H^{(M,M)}(u^*) = \tau_H^{(I,I)}(u^*)+O(H^{min(p+s,t)})$ and Lemma 2.1. result in values for $\tau_h^H(\tilde{u}_h)$, which in case A and case C fall together up to terms of order α with
$\alpha = \min(p+q, r, p+s, t)$.

<u>**Case D:**</u> $\hat{R}_h^H = \hat{M}_h^H$, $\hat{R}_H = \hat{M}_H = \hat{M}_h^H I_h$, $R_h^H = M_h^H$, $R_H = M_H = M_h^H I_h$, $\hat{R}_h = R_h = I_h$
(Nonequal averaging)

Let $(\hat{M}_H - I_H)\varphi = C_1 H^{s_1} I_H \varphi^{(s_1)} + o(H^{s_1})$, $(M_H - I_H)\varphi = C_2 H^{s_2} I_H \varphi^{(s_2)} + o(H^{s_2})$.

If A_H is linear, we get

$$
\begin{aligned}
\tau_H^{\hat{M},M}(u^*) - \tau_H^{I,I}(u^*) &= A_H(\hat{M}_H - I_H)u^* - (M_H - I_H)Au^* \\
&= A_H C_1 H^{s_1} I_H (u^*)^{(s_1)} - C_2 H^{s_2} I_H (Au^*)^{(s_2)} + o(H^{\min(s_1,s_2)}) \\
&= O(H^{(s)}) \quad \text{with } s = \min(s_1, s_2),
\end{aligned}
$$

and it follows $M_h^H \tau_h^{I,I}(u^*) = \frac{1}{2^p}\tau_H^{\hat{M},M}(u^*) + O(h^\beta)$, $\beta = \min(r, s)$.

<u>Remark:</u> With $s = \min(s_1, s_2)$ we assume the worst case. In the case of $s_1 = s_2$ leading error terms may cancel.

3.2.2. τ-extrapolation for nonlinear problems. Booth the FMG-algorithm and the τ-extrapolation can be used for nonlinear problems. To carry over the theoretical results from the linear case to the nonlinear case the following assumptions are to be made:

1. $||\xi_H|| \leqslant M||A_H(R_H u^* + \xi_H) - A_H R_H u^*||$,

2. $\tau_h^H(\hat{R}_h(u^*) + \xi_h) = \tau_h^H(\hat{R}_h(u^*)) + O(H^{p+q})$ for $\xi_h = O(H^p)$,

3. $||A_H \hat{R}_H u^* - A_H I_H u^*|| \leqslant C||\hat{R}_H u^* - I_H u^*||$.

A systematic investigation of the nonlinear case is not intended. In the following only a remark on the restriction cases from above is given:

Cases A and B from 3.2.1. do not depend on the linearity of A_H or A_h.

Equation $\tau_H^{M,M}(u^*) - \tau_H^{I,I}(u^*) = O(H^{\min(p+s,t)})$,
which was shown for case C, however, is not valid for nonlinear operators. This can be shown explicitly by a simple example (see [1]). As in case D we can show only $\tau_H^{M,M}(u^*) - \tau_H^{I,I}(u^*) = O(H^s)$ for nonlinear problems, so that higher order restrictions are required.

3.2.3. Conditions for a multigrid method with τ-extrapolation. All conditions of 3.2. concerning restriction are concentrated in Table 2 below:

Remark: The table contains only those restriction operators, which are necessary for the implementation of τ-extrapolation. $\hat{R}_h = I_h$, $R_h = I_h$, $\hat{R}_H = \hat{R}_h^H I_h$, $R_H = R_h^H I_h$ is assumed for the other restriction operators in all cases.

Conclusions:

1. The first choice for the restriction operators (case A) can be used for nonstaggered grids without essential restraints both for linear and nonlinear problems. If the possible order of accuracy for the solution is not reached, the cause may be nonsmooth defects in the τ-extrapolation step. A higher order of the FMG-prolongation or a higher number of smoothing steps will give better results in this situation.

2. Because of the averaging in the restriction of $A_h u_h$ τ-extrapolation combined with the second choice of restriction operators (case B) is less sensible to the quality of smoothing and FMG-prolongation.
   The increased order of accuracy of the restriction R_h^H ($s \geqslant p+1$ or $s \geqslant p+2$) in the case $f^{(s)} \not\equiv 0$ leads to some additional work.

3. Restriction operators according to case C for linear problems give the advantage of case B without an increased accuracy of the restriction operators R_h^H and $\hat{R}_h^H$. For nonlinear problems higher order restriction operators are needed.

4. Case D is the most obvious generalization of case A. The restriction operators are chosen independently as averaging operators with high accuracy (close to injection operators). In the case of staggered grids case D is the only way to use τ-extrapolation.

TABLE 2
Conditions in coherence with restriction and τ-extrapolation

Case of restriction	α, β as function of n, p, q, r, s, t	Condition for $\alpha \geqslant p+1$	Condition for $\alpha \geqslant p+2$	Condition for $\alpha \geqslant 4$ if $p=2$
A: (Injection-Injection) $\hat{R}_h^H = I_h^H$, $R_h^H = I_h^H$	$\beta = r$ $\alpha = \min(p+q,\, r)$		$q \geqslant 2$ $r \geqslant p+2$	$q \geqslant 2$ $r \geqslant 4$
B:(Injektion-Averaging) $\hat{R}_h^H = I_h^H$, $R_h^H = M_h^H$	$\beta = \min(r,\, s)$ $\alpha = \min(p+q,\, r,\, s)$	$\mathbf{s \geqslant p+1}$	$q \geqslant 2$ $r \geqslant p+2$ $\mathbf{s \geqslant p+2}$	$q \geqslant 2$ $r \geqslant 4$ $\mathbf{s \geqslant 4}$
if $f^{(s)} \equiv 0$:	$\beta = \min(r,\, p+s)$ $\alpha = \min(p+q, r, p+s)$		$q \geqslant 2$ $r \geqslant p+2$ $s \geqslant 2$	$q \geqslant 2$ $r \geqslant 4$ $s \geqslant 2$
C: (Equal averaging) $\hat{R}_h^H = M_h^H$, $R_h^H = M_h^H$ linear Problems:	$\beta = \min(r,\, p+s,\, t)$ $\tilde{\alpha} = \min(n+q, s+q, r, p+q, p+s, t)$	$t \geqslant p$	$q \geqslant 2, n \geqslant 2$ $r \geqslant p+2$ $s \geqslant 2$ $t \geqslant p+2$	$q \geqslant 2, n \geqslant 2$ $r \geqslant 4$ $s \geqslant 2$ $t \geqslant 4$
nonlinear Problems:	see case **B**, upper part			
D:(Nonequal averaging) $\hat{R}_h^H = \hat{M}_h^H$, $R_h^H = M_h^H$	in general as case **B**, upper part			

Parameters:

α – approximation order of the multigrid method with τ-extrapolation
β – order of the error in $R_h^H \tau_h(u^*) = \frac{1}{2^p}\tau_H(u^*) + O(H^\beta)$
q – approximation order of u_h before application of τ-extrapolation
p – approximation order of the discrete operator A_h
o – order of the differential equation
r – order of the second term of the appoximation error $\tau_H(\xi) = c(x)H^p + O(h^r)$
n – order of accuracy of prolongation with a polynomial of degree $(n-1)$
s – order of accuracy of the restriction operator $R_H = M_H$
or minimal order if two different restriction operators are used
t – order of the second error term of the restriction, $(M_H - I_H)(\xi) = d(x)H^s + O(H^t)$

4. Experiments with one-dimensional problems. Many properties of the multigrid method are independent of the dimension of the problem to be solved. In this section we use the following one-dimensional test problems to verify the results from the last section.

$$-u'' = f(x) = \pi^2 \cos(\pi x)\,, \qquad u(-1) = u(1) = 0 \tag{T1}$$

with the solution $u(x) = \cos x$,

$$-u'' = f(x) = k(k-1)x^{k-2}, \qquad u(-1) = u(1) = 0 \tag{T2}$$

with the solution $u(x) = 1 - x^k$ for $k = 4,\ 10$

and as a nonlinear problem

$$u u_x - \nu u_{xx} = 0\,, \quad u(-1) = \tanh(\tfrac{1}{2\nu}) \approx 1,\; u(1) = -\tanh(\tfrac{1}{2\nu}) \approx -1 \tag{T3}$$
$$\text{with the solution} \quad u(x) = -\tanh(\tfrac{x}{2\nu}) \quad \text{for } \nu = 0.1,\; 0.005\,.$$

Problems (T1), (T2), (T3) are discretized by the standard central differences of second order. We use a sequence of nonstaggered equally spaced grids with $2^N + 1$ points ($N = N_{min}, N_{min} + 1, \ldots, N_{max}$; $N_{min} \geqslant 1$, $N_{max} \leqslant 11$). The use of such a large number of grids is not realistic for problems with two ore three dimensions. It was done only to assure, that the improved accuracy occures not only for two or three grids.

In all cases the ω-Jacobi method

$$\hat{u}_h = u_h + \omega D_A^{-1}(f_h - A_h u_h)$$

with $\omega = \frac{2}{3}$ was used for smoothing. D_A is a diagonal matrix with the elements of the main diagonal of A_h.

Test 1: Effect of τ-extrapolation

Figure 1 shows the errors for the results of the FMG-method for problem (T2) with the solution $u(x) = 1 - x^{10}$. The left-hand side contains the errors without τ-extrapolation, on the right-hand side the results with τ-extrapolation are plotted. The calculation was done with 7 grids starting with 33 up to 2049 gridpoints. For the calculation without τ-extrapolation one V-cycle with one pre- and one post-smoothing step was used at each grid level. The calculation with τ-extrapolation was performed with two such V-cycles at each grid.

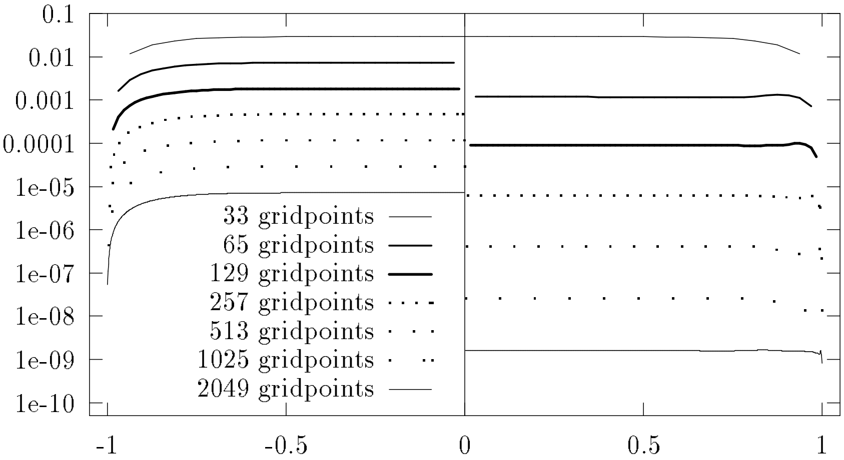

FIG. 1. *Error without and with τ-extrapolation*

Results: The second approximation order of the FMG-algorithm is raised to 4th order by τ-extrapolation. This is typical for central differences, because the second term of the error expansion is not of third but of fourth order ($p = 2, r = 4$). With τ-extrapolation the grid with 257 points gives a higher accuracy as the eight times finer grid without τ-extrapolation.

<u>**Test 2:**</u> Influence of restriction and prolongation

Table 3 shows convergence rates ρ for the normal MG-algorithm and the largest errors for the FMG-method with eight grids. The results refer to problem (T1) with the solution $u(x) = \cos x$ and 1025 points on the finest grid. In the notation $T(i,j)$ for the type of the cycle we have $T = \begin{cases} V, & \text{V-cycle} \\ F, & \text{F-cycle} \end{cases}$; i, j are the numbers of pre- and post-smoothing iterations. To avoid any negative influence on the FMG-results, quintic FMG-prolongation was used.

TABLE 3
Convergence rates and accuracy in dependence on restriction and prolongation

Multigrid algorithm	Type of Cycle	R_h^H is averaging operator		R_h^H is injection operator	
		P_h^H linear $(n = o)$	P_h^H cubic	P_h^H linear $(n = o)$	P_h^H cubic
MG-algorithm	$V(1,1)$	$\rho = 0.31$	$\rho = 0.10$	$\rho = 0.30$	$\rho = 0.10$
without τ-xtrapolation	$V(2,2)$	$\rho = 0.11$	$\rho = 0.04$	$\rho = 0.11$	$\rho = 0.04$
FMG-algorithm	$V(1,1)$	$0.56E{-}09$	$\mathbf{0.41E{-}11}$	$0.58E{-}09$	$0.28E{-}10$
with τ-extrapolation	$V(2,2)$	$0.73E{-}11$	$0.59E{-}11$	$\mathbf{0.16E{-}11}$	$\mathbf{0.37E{-}11}$
without post-smoothing	$F(1,1)$	$\mathbf{0.36E{-}11}$	$\mathbf{0.34E{-}11}$	$0.27E{-}10$	$0.27E{-}10$
and fine-grid correction	$F(2,2)$	$0.59E{-}11$	$0.59E{-}11$	$\mathbf{0.16E{-}11}$	$\mathbf{0.32E{-}11}$
FMG-Algorithm	$V(1,1)$	$\mathbf{0.46E{-}09}$	$\mathbf{0.15E{-}11}$	$0.60E{-}09$	$0.27E{-}10$
with τ-extrapolation	$V(2,2)$	$\mathbf{0.12E{-}11}$	$\mathbf{0.93E{-}12}$	$0.92E{-}11$	$0.74E{-}11$
with post-smooting-	$F(1,1)$	$\mathbf{0.14E{-}11}$	$\mathbf{0.98E{-}12}$	$0.26E{-}10$	$0.25E{-}10$
and fine-grid correction	$F(2,2)$	$\mathbf{0.98E{-}12}$	$\mathbf{0.98E{-}12}$	$0.64E{-}11$	$0.62E{-}11$

<u>Remark:</u> For the FMG-algorithm the headline "R_h^H ist injection operator" is meant for the current finest grid only (we call it τ-restriction). On coarser grids the defects were restricted as usual by an averaging operator. To perform the τ-extrapolation, the restriction operator $\hat{R}_h^H$ on the current finest level was chosen in agreement with R_h^H (see Table 2, cases A and C). On coarser grids and in the MG-algorithm without τ-extrapolation $\hat{R}_h^H = I_h^H$ was used.

<u>**Conclusions:**</u>

1. Table 3 shows that the condition $n \geqslant o$ (see Table 1) for the minimal accuracy of prolongation is correct. In fact, linear prolongation in the case $o = 2$ is possible, but in some situations the ability of the numerical algorithms is not trated fully in this way. The convergence rates of the MG-algorithm with cubic prolongation are better. In the case of τ-extrapolation for the $V(1,1)$-cycle the fourth order of accuracy occures only on the first grids, on finer grids the order reduces to a value between two and three. However, this effect can be easily removed by increasing the number of smoothing steps.
2. For the accuracy of the FMG-algorithm with τ-extrapolation without post-smoothing correction Table 3 shows the following behaviour: If the amount for smoothing is small restriction with averaging operators gives the better results, because averaging leads to a better estimation of the discretization error. In the

case of more smoothing iterations restriction by injection gives solutions with higher accuracy.
An explanation of the observed effect can be given by the accuracy of the error estimation and by the low-frequency error, which is proceeded from a restriction with $s = \alpha - p$ (see Table 1).

3. In the case of injective τ-restriction the algorithm of post-smoothing- and fine-grid correction is poor. Only a $V(3,3)$-cycle with three additional pre-smoothing iterations on the current finest level reduces the error to $0.35E{-}12$. This gain of accuracy cannot justify the increased amount of work.
In the case of τ-restriction by averaging operators we get reduced errors with a less increased amount of work. Comparable results can be obtained, if in a calculation without post-smoothing- and fine-grid correction the post-smoothing step at the finest grid is omitted (see Table 4 below). Only the defects are less smooth in this case. Obviously post-smoothing- and fine-grid correction can give a somewhat higher accuracy, if the correction term in the $\tau-$extrapolation step has been calculated accurately. The use of averaging operators for restriction in this context is more efficient than an increased number of smoothing steps.

Remark on the application of τ-extrapolation in combination with averaging operators for restriction: For test problem (T2) the first attempts with τ-extrapolation in combination with averaging operators failed – the method provided results with second order accuracy, as in the case without τ-extrapolation.
The reason for this problem was the influence of boundary points. Indeed a restriction $M_h^H A_h u_h$ cannot work for the neighbors of boundary points, because $A_h u_h$ is not defined at the boundary. The use of $I_h^H A_h u_h$ for these points seems to be a possible compensation. Unfortunately this affects the error estimates. Every deviation from M_h^H in the first inner gridpoint influences the error estimate at the next point. A simple way to overcome this problem is the omission of the τ-extrapolation at the first inner gridpoints. The second order accuracy for the related equations does not spoil the overall fourth order for the solution. The accuracy is only somewhat reduced by an additional error term of fourth order. This term is small, when compared with the difference between second and fourth order accuracy.
The described effect was observed only for problem (T2). This can be explained by the fact, that the second derivative of the solution of the other problems vanishes at the boundary.

Test 3: The efficiency of different variants of τ-extrapolation can be studied in Table 4

The variants are

Algorithm 1:	τ-extrapolation without post-smoothing- and fine-grid correction
Algorithm 2:	– same as Algorithm 1, but without the post-smoothing step on the finest grid
Algorithm 3:	τ-extrapolatioin with post-smoothing correction
Algorithm 4:	τ-extrapolation with post-smoothing- and fine-grid correction

Table 4 contains the largest absolute values for the error of problem (T1). The errors are given for all grid levels. The calculation was performed with a $V(1,1)$-cycle, cubic prolongation, quintic FMG-prolongation and restriction by an averaging operator (for u_h on the current finest grid only).

TABLE 4
Accuracy of different variants of τ-extrapolation

Grid		Algorithm 1	Algorithm 2	Algorithm 3	Algorithm 4	Factor
1	(9 points)	$0.13E-01$	$0.13E-01$	$0.13E-01$	$0.13E-01$	
2	(17 points)	$0.27E-04$	$0.27E-04$	$0.18E-04$	$0.18E-04$	707.78
3	(33 points)	$0.40E-05$	$0.40E-05$	$0.14E-05$	$0.14E-05$	12.83
4	(65 points)	$0.24E-06$	$0.24E-06$	$0.74E-07$	$0.73E-07$	19.42
5	(129 points)	$0.15E-07$	$0.15E-07$	$0.48E-08$	$0.48E-08$	15.39
6	(257 points)	$0.96E-09$	$0.96E-09$	$0.31E-09$	$0.31E-09$	15.18
7	(513 points)	$0.62E-10$	$0.62E-10$	$0.21E-10$	$0.21E-10$	14.57
8	(1025 points)	$0.41E-11$	$0.17E-11$	$0.15E-11$	$0.15E-11$	14.11

Remark: The last column of the table contains the reciprocal values of the quotients of consecutive errors for Algorithm 4. The fourth order accuracy can be seen for the third and all finer grids, on grid 2 we have a jump from second order accuracy (calculation without τ-extrapolation on the first grid) to fourth order.

Conclusions:

1. On the finest grid the result of Algorithm 2 is slightly better than that of Algorithm 1, because the post-smoothing step of Algorithm 1 recovers errors of second order. This effect increases with the number of post-smoothing iterations.
2. Algorithm 3 with post-smoothing correction on all grids beginning with the second is more accurate than Algorithm 1 by a factor two to three. The second order error parts which normally are introduced by post-smoothing do not appear. On the finest grid the same accuracy as with Algorithm 2 is reached.
3. Algorithm 4 gives the same results as Algorithm 3. At least for linear problems the fine-grid correction does not pay. The avoidance of a small low-frequency error in the pre-smoothing step is unnecessary because this error can hardly influence the error estimation but can easily be removed on coarser grids.

Test 4: Influence of the FMG-prolongation

Problem (A1) is solved with a $V(1,1)$-cycle and cubic prolongation P_H^h under the same conditions as in test 2 above. The purpose of the test is a comparison of cubic and quintic FMG-prolongation $\tilde{P}_H^h$.

Table 5 shows the decrease of the error on the consequtive grids, the factors indicate, how much smaller is an error in comparison with the previous grid.

Conclusions:

1. In agreement with 3.1.2 in the case of τ-restriction by averaging cubic FMG-prolongation ($n = p + o$) is sufficient to obtain fourth order convergence. With τ-restriction by injection cubic FMG-prolongation gives only second order convergence; the mechanism of τ-extrapolation fails, because the defects after pre-smoothing are not smooth. A $V(2,2)$- or a $F(3,3)$-cycle have second order too. Only an extremly high number of smoothing iterations leads to fourth order.

2. Quintic FMG-prolongation in combination with τ-restriction by averaging gives a higher accuracy than cubic FMG-prolongation. The combination of quintic FMG-prolongation and injective τ-restriction results in a fourth order method with somewhat reduced accuracy.

TABLE 5
Table 5: Accuracy of the solution in dependence on the FMG-prolongation

	τ-restriction by averaging				τ-restriction by injection			
Grid	cubic FMG-Prol.		quintic FMG-Prol.		cubic FMG-Prol.		quintic FMG-Prol.	
9	$0.13E-01$	–	$0.13E-01$	–	$0.13E-01$	–	$0.12E-01$	–
17	$0.17E-03$	76.97	$0.27E-04$	484.80	$0.11E-01$	1.16	$0.22E-03$	57.92
33	$0.16E-04$	10.35	$0.40E-05$	6.67	$0.31E-02$	3.54	$0.28E-04$	7.87
65	$0.11E-05$	15.14	$0.24E-06$	16.73	$0.80E-03$	3.93	$0.18E-05$	15.86
129	$0.70E-07$	15.29	$0.15E-07$	15.88	$0.20E-03$	3.98	$0.12E-06$	15.39
257	$0.46E-08$	15.33	$0.96E-09$	15.67	$0.50E-04$	4.00	$0.72E-08$	16.22
513	$0.31E-09$	14.91	$0.62E-10$	15.44	$0.13E-04$	4.00	$0.45E-09$	15.86
1025	$0.21E-10$	14.53	$0.41E-11$	15.14	$0.31E-05$	4.00	$0.28E-10$	16.29

Remark: Besides cubic and quintic FMG-prolongation an interpolation with the difference approximation of the original equation was tested. The results were close to those of quintic prolongation.

Test 5: τ-extrapolation for nonlinear problems

We consider problem (T3), the Burger's equation. Apart from the fact, that a combination of restriction operators according to case 3 is not favourable, the FMG-algorithm with τ-extrapolation works in the nonlinear case too. However, for our test problem we have to pay attention to some specialities.

1. With $\nu \gg 1$ the change from the left boundary value $u(-1) \approx 1$ to the right boundary value $u(1) \approx -1$ takes place in a very narrow region. This excludes grids, which have not enough gridpoints in this region. As an orientation we can take the stability constraint for the difference scheme $Re_h = \frac{uh}{\nu} < 2$. For the number of gridpoints this means $n > \nu^{-1}$, i.e. $n_{min} = 17$ for $\nu = 0.1$ and $n_{min} = 257$ for $\nu = 0.005$.
2. In the same context we must take into consideration, that the error of derivatives of functions, which are to be interpolated, becomes very large for small values of ν.

Results for $\nu = 0.1$:

A $V(1,1)$-cycle in the case of 6 grids (from 33 to 1025 points) has fourth order of convergence. If a 7th grid with 17 points is added, the order of convergence is not much larger than two. Even with a $V(3,3)$-cycle the order of convergence is below three. A $F(1,1)$-cycle is more robust. In the case of seven grids it gives fourth order of convergence beginning with the third grid (65 points). If eight grids are used (the coarsest has 9 points only) fourth order is reached on the last four grids. Table 6 shows factors for the convergence and the absolut error on the finest grid.

TABLE 6
Convergence of the FMG-method for diffrent cycles

Number of gridpoints	Type of MG-algorithm $V(1,1)$ 6 grids	$V(1,1)$ 7 grids	$V(3,3)$ 7 grids	$F(1,1)$ 7 grids	$F(1,1)$ 8 grids
9					–
17		–	–	–	1.5
33	–	5.2	7.6	5.1	6.7
65	19.8	4.8	6.0	26.4	7.3
129	16.1	5.1	7.0	333.5	14.8
257	20.5	5.0	6.8	5.1	22.6
513	15.1	5.1	7.0	16.2	37.0
1025	19.3	5.1	7.0	49.6	52.0
Error	$0.74E-08$	$0.40E-05$	$0.62E-06$	$0.37E-09$	$0.79E-08$

Results for $\nu = 0.005$:

Figure 2 shows the solutions of a difference method for *one* grid with 129 ((a), diamonds) and 257 (b) points in the interval $[-0.05, 0.05]$. The change from oscillating solutions to physically correct approximations is evident. A two-grid method (c) with τ-extrapolation and 257 points on the finest grid yields some gain of accuracy, although the coarse grid with 129 points for the one-grid method is too coarse. (The problem on this grid is stabilized by the correction term on the right hand side in the multigrid method.) The most accurate solution with five grids from 129 to 2049 gridpoints (line (d)) has a maximal deviation from the correct solution of $0.16E-5$. Without τ-extrapolation this deviation is $0.14E-2$ (see Table 7).

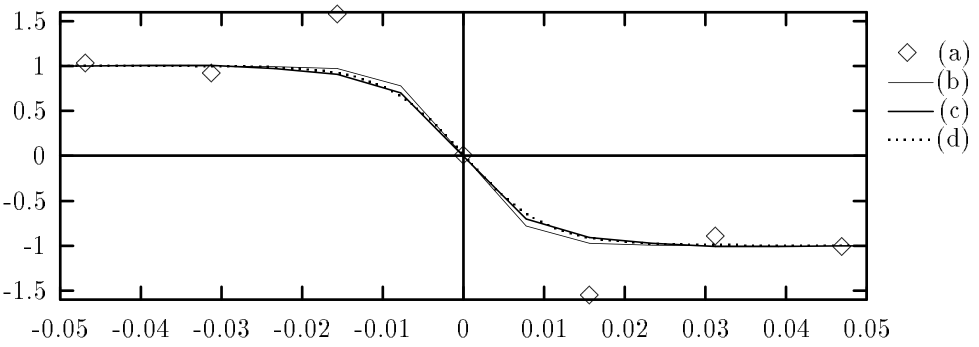

FIG. 2. *Zoom of solutions for the Burgers equation with* $\nu = 0.005$

Remarks on Table 7:

1. Taking into consideration the structure of the solution, three pre- and post-smoothing iterations were performed only in the small range, where the solution actually changes. Outside of this region one iteration was sufficient. A better investigated variant of such a strategy can be found in [6]. All calculations, excluding the last, were done with Algorithm 2, i.e. without post-smoothing- and fine-grid correction and without the post-smoothing step on the finest grid.
2. If nonlinearity and/or the use of relative coarse grids cause a noticeable change

TABLE 7
Convergence of the FMG-method for Burgers equation with $\nu = 0.005$

FMG-method	Maximal error on grid 1	grid 2		grid 3		grid 4		grid 5 (2049 Points)	
$F(3,3)$-cycle without τ-extrapolation	0.62E+0	0.67E–1	(9, 2)	0.37E–1	(1.8)	0.49E–2	(7.6)	0.14E–2	(3.5)
$V(3,3)$-cycle	0.62E+0	0.24E+0	(2.5)	0.18E–1	(13.6)	0.17E–2	(10.7)	0.28E–3	(5.9)
$F(3,3)$-cycle	0.62E+0	0.24E+0	(2.5)	0.43E–1	(5.7)	0.17E–2	(25.2)	0.33E–4	(51.6)
$F(3,3)$-cycle, $\gamma > 1$ MG-cycles on grids $2-4$									
$\gamma = 2$ in FMG:	0.62E+0	0.22E+0	(2.7)	0.12E–1	(19.2)	0.16E–3	(74.4)	0.63E–5	(24.6)
$\gamma = 3$ in FMG:	0.62E+0	0.53E–1	(11.5)	0.54E–2	(9.8)	0.13E–3	(42.0)	0.22E–5	(58.2)
$\gamma = 2$ in FMG, see remark 2:	0.62E+0	0.11E+0	(5.5)	0.35E–2	(31.7)	0.22E–3	(16.5)	0.16E–5	(132.1)
as above, Algorithm 4	0.62E+0	0.91E–1	(6.7)	0.10E–2	(91.1)	0.30E–4	(33.9)	0.35E–5	(8.3)

of the solution from one grid to the next, then it is advantageous to perform $\gamma > 1$ MG-cycles on each grid level of the FMG-method. On the last two grids, however, it was possible to work with $\gamma = 1$ without loss of accuracy.
Moreover it is possible to perform the first MG-cycle on a new grid without τ-extrapolation (see [12]). Obviously in the second MG-cycle the error can be estimated more precisely than immediately after FMG-prolongation and pre-smoothing. During the two last calculations on grid 2 and grid 3 the τ-extrapolation was done only in the second F-cycle.

Conclusions:

1. Even in the case of the $V(3,3)$-cycle some additional MG-cycles on the finest grid reduced the error to a value of about 0.5E-5. This shows, that τ-extrapolation works also for strongly nonlinear problems. However, the nonlinearity of a problem should be treated already on coarser grids. This can be tried by using the F-cycle and by the modification described in remark 2 above, as the last two calculatons show.
2. In comparison with algorithm 2 the algorithm of post-smoothing- and fine-grid correction gives a little gain of accuracy on the coarser grids. On the finest grid the accuracy is not improved.

5. Solution of Navier-Stokes equations with τ-extrapolation . There are different ways to produce difference methods with higher order accuracy, for instance the use of difference formulas over an enlarged stencil [8], [9] or compact difference schemes [16], [5]. The τ-extrapolation seems to be attractive because it is easy to implement and has low expense. Moreover, there arises a possibility to estimate the error of the solution.
The change from one-dimensional test problems to the case of the two-dimensional Navier-Stokes equations includes the increase of space dimensions and the change

from one equation to a system of equations.
Examples for the use of τ-extrapolation for scalar equations on multidimensional regions can be found in [11] for the Poisson equation over the unit square or in [15] for the same equation on a three-dimensional region, which is defined by three overlapping cylindrical grids.
Beside the use for scalar equations (Poisson equation, nonlinear potential equation – with special respect to Neumann boundary conditions) in [12], τ-extrapolation is applied to the solution of the shell problem for the calculation of stresses and deformations in weakly curved thin elastic shells. This problem leads to a system of four poisson-like equations with nonlinear coupling.
In most cases it was possible to improve the convergence order from two to four or to a value close to four by implementing the τ-extrapolation. For smooth solutions this should be attainable for the Navier-Stokes equations too.

5.1. Discretization of the Navier-Stokes equations. The Navier-Stokes equations are considered in the form

$$\begin{aligned} \nabla \cdot \mathbf{u}\,\mathbf{u} - \nu\,\Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in} \quad \Omega \\ \nabla \cdot \mathbf{u} &= 0 && \text{in} \quad \Omega \\ \mathbf{u} &= \mathbf{u}_\Gamma && \text{on} \quad \partial\Omega \end{aligned} \tag{9}$$

over a rectangular region Ω. In these equations $\mathbf{u}$ stands for the velocity with components u and v, p denotes the pressure, ν is the kinematic viscosity and $\mathbf{f}$ is an external force with components f_x and f_y. Equation (9) is discretized by a second order difference approximation on staggered grids:

$$\begin{aligned} \frac{u_{he}u_e - u_{hw}u_w}{2h_x} + \frac{v_{hn}u_n - v_{hs}u_s}{2h_y} - \nu\left(\frac{u_e - 2u + u_w}{h_x^2} + \frac{u_n - 2u + u_s}{h_y^2}\right) + \frac{p_{he} - p_{hw}}{h_x} &= f_x \\ \frac{u_{he}v_e - u_{hw}v_w}{2h_x} + \frac{v_{hn}v_n - v_{hs}v_s}{2h_y} - \nu\left(\frac{v_e - 2v + v_w}{h_x^2} + \frac{v_n - 2v + v_s}{h_y^2}\right) + \frac{p_{hn} - p_{hs}}{h_y} &= f_y \\ \frac{u_{he} - u_{hw}}{h_x} + \frac{v_{hn} - v_{hs}}{h_y} &= 0\,. \end{aligned}$$

The first two equations refer to the inner points of the grid for the u- and v-components of the velocity. The discrete continuity equation is written for the centers of the meshes, where the pressure is defined. Positions, which are shifted by h_x in x-direction and by h_y in y-direction, are called w, e, s, n; shift operations by half steps are indicated by hw, he, hs, hn. In the convection term this leads to the calculation of averages. Fictive grid points outside the region Ω are introduced at the upper and the lower boundary for u and on the left and the right boundary for v. Values for the tangential velocity components in these points must be extrapolated from the interior of Ω.

5.2. Implementation of the τ-extrapolation . A detailed presentation of the MG-method, which was used for the Navier-Stokes equations is not intended at this place. Only components, which are related to the τ-extrapolation are discussed in the following subsection. At first we consider the calculation of the τ-extrapolation terms. The system of discretized equations can be written in the form

$$\begin{aligned} A_h(\mathbf{u}_h)\mathbf{u}_h + GRAD_h\,p_h &= \mathbf{f}_h \\ DIV_h\,\mathbf{u}_H &= g_h\,, \end{aligned}$$

where the first equation is a vector equation with two components. Right hand sides $g_h \neq 0$ are introduced by the MG-method. On the finest grid we have $g_h = 0$. Using $\tau-$extrapolation the problem on a coarser grid is

$$\begin{aligned} A_H(\mathbf{u}_H)\mathbf{u}_H + GRAD_H\, p_H &= R_h^H \mathbf{f}_h + \tfrac{4}{3}\tau_h^H(\mathbf{u}_h,\, p_h) \\ DIV_H\, \mathbf{u}_H &= \mathsf{R}_h^H g_h + \tfrac{4}{3}\sigma_h^H(\mathbf{u}_h) \end{aligned}$$

with

$$\begin{aligned} \tau_h^H(\mathbf{u}_h,\, p_h) &= A_H(\hat{R}_h^H \mathbf{u}_h)\hat{R}_h^H \mathbf{u}_h + GRAD_H \hat{\mathcal{R}}_h^H p_h - R_h^H A_h(\mathbf{u}_h)\mathbf{u}_h - R_h^H GRAD_h\, p_h \\ \sigma_h^H(\mathbf{u}_h) &= DIV_H\, \hat{R}_h^H \mathbf{u}_h - \mathsf{R}_h^H DIV_h\, \mathbf{u}_h\ . \end{aligned}$$

The choice of the restriction operators is explained in 5.3.2.

5.3. Components of the multigrid method.

5.3.1. Smoothing. The Smoothing step is based on the ω-Jacobi method. At first this method is applied on the two components of the momentum equation. Then a correction of the calculated velocity field $\tilde{\mathbf{u}}$ is performd. On the current finest level this correction makes the velocity divergence-free; on coarser grids it takes into consideration the right hand sides g which are defined by the MG-algorithm (indices H or h are omitted in the following). From equations

$$\hat{\mathbf{u}} = \tilde{\mathbf{u}} + GRAD\,\delta p, \quad \hat{p} = p + \delta p \tag{10}$$

and $DIV\,\hat{\mathbf{u}} = g$, where $GRAD$ and DIV are discrete counterparts of the corresponding differential operators, we obtain

$$DIV\ GRAD\,\delta p = g - DIV\,\tilde{\mathbf{u}}. \tag{11}$$

With the solution δp of this Poisson equation the fields $\tilde{\mathbf{u}}, p$ are updated to $\hat{\mathbf{u}}, \hat{p}$ according to (10). Then the whole smoothing cycle can be repeated.
The solution of the Poisson equation (11) is done with an inner multigrid method.

Bevor starting a smoothing step, velocity components for fictive points must be calculated. For the MG-method without τ-extrapolation a first order extrapolation is sufficient. In the case of τ-extrapolation third order extrapolation is necessary.

5.3.2. Restriction. Outside the τ-extrapolation step linear restriction operators can be used. In connection with the τ-extrapolation the situation is more complicated:

Because of the staggered grids for the velocity components only cases C and D from Table 2 are possible.
Hence the restriction of $\mathbf{u}_h$ and $A_h\mathbf{u}_h$ for the momentum equation can be performed by the same averaging operator $\hat{R}_h^H = R_h^H$ with fourth order accuracy.

For the pressure again a fourth order restriction operator (that means a bicubic interpolation) is needed, because the two restriction operators $\hat{\mathcal{R}}_h^H$ and R_h^H are not defined on the same grid.

In the case of the continuity equation, linear restriction for $DIV\mathbf{u}$ is possible (operator R_h^H) independently from the cubic restriction of $\mathbf{u}$. As in case B this linear restriction causes no errors, because the right hand side of the original problem vanishes. The cubic restriction for $\mathbf{u}$ leads to a fourth order error, if we compare it with injection. This has no influence on the τ-extrapolation.

Taking into account the restriction of the right hand side of (11) it must be warned of any "better" interpolation for $DIV\mathbf{u}$. Operator $DIV\,GRAD$ has an one-dimensional null space. Consequently the right hand side of (11) has to fulfill a solvability condition (the sum of all components must be zero) and this relation must be conseved by the restriction. This is done by linear restriction, because the values from the finer grid are summed up in groups only.

5.3.3. Prolongation. The errors of prolongation in the MG-algorithm should be no larger than of second order for the velocity components ($n \geqslant o$, $o = 2$) and of first order for the pressure ($o = 1$). With linear prolongation for $\mathbf{u}$ and p these conditions are fulfilled.

In the case of FMG-prolongation according to 3.1.2 (condition $n \geqslant p+o$) fourth order for the velocity ($p = 2, o = 2$) and third order for pressure ($p = 2, o = 1$) is needed. This means cubic FMG-prolongation for $\mathbf{u}$ and quadratic or cubic prolongation for the pressure too.

Unsymmetrical interpolation formulae at the boundaries cause larger interpolation errors than symmetrical formulae of the same order in the interior. For this reason near the boundary interpolation of an order higher than three was used.

5.4. Test calculations for two problems.

5.4.1. Problem 1. At first we consider the straight flow through a channel. The calculation is done in the unit square. At the upper and lower boundary the flow profile is given, at the left and right boundary we apply no slip boundary conditions. Using a parabolic profile would not show any effect for τ-extrapolation, because the difference method would give the exact solution without τ-extrapolation. Analogously with a polynomial of fourth order the method with τ-extrapolation should give the exact solution.

Therefore the calculations were performed with the profile

$$u(x) = 0,\ v(x) = 0.2\,x\,(x-1)(x+1)(x-2)(x-10)\,,$$

which over $x \in [0,1]$ looks similar to a parabola. If this profile is taken to be the solution for $v(x,y)$ and if $u(x,y)$ vanishes, the following right hand side must be taken

$$f_x(x,y) \equiv 0,\ f_y(x,y) = -\nu\frac{\partial^2 v}{\partial x^2} = -\nu \cdot 0.4(10x^3 - 72x^2 + 12).$$

The related pressure field is $p \equiv 0$. (A pressure field, which falls linearly with y can be obtained, if the number 12 on the right hand side is omitted. The velocity remains unchanged in this case.) The calculation was performed with $\nu = 0.1$, because the smoothing method works for small Reynolds numbers only.

Results: Problem 1 was solved on a sequence of grids with $4*4$, $8*8$, …, to $256*256$ meshes. One F-cycle was performed on each grid. On the current finest level two pre-smoothing iterations were performed and excluding the finest grid two post-smoothing iterations too. In all other situations one smoothing iteration was sufficient. Table 8 shows the maximal error of the v-component of the velocity for the FMG-method without and with use of τ-extrapolation. Postsmoothing- and fine-grid

TABLE 8
Maximal errors and factors for the decrease of errors for problem

Grid	FMG – Method			
	without τ-extrapolation		with τ-extrapolation	
$4*4$	$0.21E-02$	–	$0.21E-10$	–
$8*8$	$0.37E-03$	5.60	$0.37E-03$	5.60
$16*16$	$0.34E-04$	0.39	$0.18E-04$	20.80
$32*32$	$0.36E-05$	3.89	$0.93E-06$	19.33
$64*64$	$0.17E-05$	5.00	$0.73E-07$	12.69
$128*128$	$0.40E-06$	4.24	$0.51E-08$	14.33
$256*256$	$0.10E-06$	4.03	$0.31E-09$	16.52

correction were not applied. The first use of τ-extrapolation was done on the grid with $16*16$ meshes.

Table 8 shows, that the algorithm of τ-extrapolation works for the Navier-Stokes equations too, which are more complicated than the one-dimensional problems in the last section. However, our problem treates a very simple flow, which may be not representative.

5.4.2. Problem 2. As a second example problem we use a rotating flow

$$u(x,y) = \sin \pi x \cos \pi y \,,\; v(x,y) = -\cos \pi x \sin \pi y$$

in the square $[0,1]\times[0,1]$ with a viscosity parameter $\nu = 0.01$.
Setting this solution in (9) we get the right hand side

$$f_x(x,y) = \pi \sin \pi x\,(\cos \pi x + 2\pi\nu \cos \pi y)\,,\; f_y(x,y) = \pi \sin \pi y\,(\cos \pi y - 2\pi\nu \cos \pi x)\,.$$

At the boundary now the normal components of the velocity are zero, while the tangential components are functions of x or y.

Results: On a sequence of grids with $8*8$ to $256*256$ meshes on each grid one to three F-cycles with one pre- and one post-smoothing iteration were performed. On the current finest level one additional pre-smoothing iteration was done, on the finest grid the post-smoothing step was supressed. Again post-smoothing- and fine-grid correction were not applied. Table 9 shows the maximal error for the v-component of the solution The first use of τ-extrapolation was done on the third grid, which caused a remarkable decrease of the error. A single F-cycle, however, can not exploite the possible increase of accuracy. To do this by additional cycles on the finest grid is inefficient; the better way is to use a larger number of F-cycles on the coarser grids. The solution on the $64*64$-grid in this case is more accurate than a solution without τ-extrapolation on a $256*256$-grid.
Again the solution of our problem was obtained with fourth order convergence and an computational cost, which depends linearly on the number of grid points.

6. Conclusions. The authors experience with τ-extrapolation can be summarized as follows:

1. The τ-extrapolation is a very efficient method to improve the accuracy of multigrid methods. It can be applied for linear as well as for nonlinear problems, provided the solution is sufficiently smooth.

TABLE 9
Convergence for different FMG-cycles

	FMG – method with tauextrapolation						without tau-	
Grid	$F(1,1)$-cycle		– two times		– three times		extrapolation	
$8*8$	0.28E-02		0.28E-02		0.28E-02		0.28E-02	
$16*16$	0.81E-02	0.34	0.83E-03	3.35	0.97E-03	2.68	0.97E-03	2.68
$32*32$	0.17E-03	48.06	0.19E-04	44.36	0.74E-05	131.29	0.27E-03	3.62
$64*64$	0.15E-04	11.35	0.11E-05	16.69	0.84E-06	8.79	0.68E-04	3.96
$128*128$	0.40E-05	3.70	0.33E-06	3.43	0.51E-07	16.42	0.17E-04	3.99
$256*256$	0.41E-06	9.68	0.13E-07	24.68	0.19E-08	26.48	0.42E-05	4.00
additional	0.12E-06	33.71	0.46E-08	70.72	0.18E-08	28.31		
single cycles	0.28E-07	143.33	0.15E-08	216.93				

2. The most reliable way to implement the τ-extrapolation algorithm is the use of injective restriction operators in the τ-extrapolation step combined with an improved number of smoothing steps and a high order FMG-prolongation.

3. The application of τ-extrapolation combined with restriction operators, which are averaging operators, is more complicated. This is caused by the fact, that not all combinations of restriction operators are applicable (see 3.2.3). However, with averaging operators the improved accuracy can be obtained with lower computational work (because no special requirements for the MG-algorithm must be fulfilled).

4. Postsmoothing- and especially fine-grid correction are appendixes to the τ-extrapolation, which give only a small gain of accuracy of the solution in comparison with Algorithm 1. These improvements of the basic algorithm should not be applied. On the other hand it can be recommended to omit the last post-smoothing step (Algorithm 2).

5. In the case of nonlinear problems it is important to solve the problem on the coarsest grid with sufficient accuracy. The coarsest grid must not be too coarse.

6. To exauste the full potential of the τ-extrapolation algorithm, it can be necessary to perform more than one MG-cycle on each grid level. Before setting γ to a value of two or three all other possibilities for a failure of the τ-extrapolation should be excluded. Even in cases, where $\gamma > 1$ is necessary for the coarser grids, on the finer grids one MG-cycle can be sufficient.

7. Besides a study of the behavior of the solution an experimental analysis of the τ-extrapolation algorithm should include a study of the defects. Only a look on the behaviour of the defects permits a deeper understanding of some properties of the method.

8. Staggered grids do not exclude the application of τ-extrapolation. However, they make it's application more complicated and require the use of the most expensive variant for the restriction operators in the τ-extrapolation step.

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Appendix: MG- and FMG-algorithm and τ-extrapolation

$\mathrm{MG}(U_k, F_k)$:	$\mathrm{FMG}(kk)$:
IF $k = 1$ THEN	$U_1 \leftarrow A_1 U_1 = F_1$
$\quad U_1 \leftarrow A_1 U_1 = F_1$	DO $k = 2, kk$
ELSE	$\quad U_k = \tilde{P}_{k-1}^k (U_{k-1})$
$\quad U_k \leftarrow S_k^{\nu_1}(U_k, F_k)$	$\quad U_k \leftarrow \mathrm{MG}^\gamma(U_k, F_k)$
$\quad Y_{k-1} \leftarrow \hat{R}_k^{k-1} U_k$	END DO
$\quad \boxed{F_{k-1} \leftarrow R_k^{k-1}(F_k - A_k U_k) + A_{k-1} Y_{k-1}}$	
$\quad U_{k-1} \leftarrow \mathrm{MG}^\mu(Y_{k-1}, F_{k-1})$	
$\quad U_k \leftarrow U_k + \sigma P_{k-1}^k (U_{k-1} - Y_{k-1})$	
$\quad U_k \leftarrow S_k^{\nu_2}(U_k, F_k)$	
END IF	

τ-extrapolation :

$$\tau_{k-1}^k = A_{k-1} Y_{k-1} - R_k^{k-1} A_k U_k$$

$$F_{k-1} = R_k^{k-1} F_k + \frac{2^p}{2^p - 1} \tau_{k-1}^k$$

$$F_k = I_k f + \frac{1}{2^p - 1} P_{k-1}^k \tau_{k-1}^k \qquad \text{(post-smoothing correction)}$$

$$F_{k+1} = I_{k+1} f + \frac{1}{2^p(2^p - 1)} P_k^{k+1} P_{k-1}^k \tau_{k-1}^k \qquad \text{(fine grid correction)}$$

Symbols:

A_k	discrete operator	p	approximation order of A_k
U_k	discrete solution	I_k	injection operator
F_k	right hand side	f	right hand side of the continous problem
S_k	smoothing operator	ν_1	number of pre-smoothing steps
$\hat{R}_k^{k-1}$	restriction operator for U_k	ν_2	number of post-smothing steps
R_k^{k-1}	restriction operator for $A_k U_k$	μ	parameter for the type of MG-cycle
P_{k-1}^k	prolongation operator	σ	damping parameter for nonlinear problems
$\tilde{P}_{k-1}^k$	FMG-prolongation operator	γ	parameter for the type of FMG-cycle
τ_{k-1}^k	discretization error	k	number of grids
		kk	index for the current level