Preconditioning the Pseudo–Laplacian for Finite Element Simulation of Incompressible Flow

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Abstract

In this paper, we investigate the question of the spectrally equivalence of the so-called Pseudo-Laplacian to the usual discrete Laplacian in order to use hierarchical preconditioners for this more complicate matrix. The spectral equivalence is shown to be equivalent to a Brezzi-type inequality, which is fulfilled for the finite element spaces considered here.

0. Introduction

In general, numerical simulation of unsteady incompressible flows involves complex geometries. The finite element method is a natural tool of discretization in such cases. The questions of error estimates, existence and uniqueness are strong connected with the so called LBB-condition (due to LADYSHENSKAJA/ BABUŠKA/BREZZI) on the pair of F.E. - spaces $\mathbb{V}_h \subset H_0^1(\Omega)^d$ approximating the velocity $\mathbf{u}$ and $\mathbb{X}_h \subset L_2(\Omega)$ approximating the pressure p.

A very popular element among the conforming ones is the quadratic (biquadratic) approximation of the velocity in a 6-node triangle (9-node quadrilateral) and the linear (bilinear) approximation of the pressure with unknowns belonging to the vertice nodes. Another variant with the same approximation of the pressure on triangles (quadrilaterals) T_h and linear (bilinear) approximation of $\mathbf{u}$ in the mesh $T_{h/2}$ has near the same properties. Here the 6-node triangle (9-node quadrilateral) is used as a macro element of four smaller triangles (resp. quadrilaterals), see [1].

For the time dependent problems, additionally to the discretization in space we need a suitable treatment of time stepping that guarantees a ratherly correct behaviour in time. For the Navier-Stokes equations a total implicit time discretization would lead to a complicate nonlinear problem in each time step. So, we prefer a semi-implicit projection method in a variant proposed by GRESHO/CHAN [5] with two linear equations on each time step. The iterative solution of these equations is considered here. We prove the possibility of using hierarchical preconditioners of the YSERENTANT [9] type in $2D$ (or of the BRAMBLE/PASCIAC/XU type in $2D$ and $3D$ [2]) for the Pseudo-Laplacian matrix occuring in the pressure correction. The reason for this, the spectral equivalence to the Laplacian – matrix depends on a LBB-like condition, which is fulfilled for the finite elements considered here.

1. The Continuous Problem and its F.E. Discretization

Let Ω be a bounded domain in $\mathbb{R}^d (d = 2$ or $3)$ with regular boundary $\Gamma = \partial\Omega$. We consider the unsteady Navier-Stokes equation in Ω

$$\begin{aligned} \dot{\mathbf{u}} - \nu \triangle \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= f(x,t) \\ \nabla \cdot \mathbf{u} &= 0 \qquad \text{in } \Omega \end{aligned} \tag{1.1}$$

with boundary conditions $\quad \mathbf{u}(x,t) = 0 \quad x \in \Gamma$
and initial conditions $\quad \mathbf{u}(x,0) = u_\circ(x) \ x \in \Omega$

(for in- and outflow situations the b.c. could be generalized).
The usual finite element discretization is based on the weak formulation of (1.1).

$$\begin{aligned} \langle \dot{\mathbf{u}}, \mathbf{v}\rangle_d + \nu\, a(\mathbf{u},\mathbf{v}) + c(\mathbf{u};\mathbf{u},\mathbf{v}) - b(\mathbf{v},p) &= \langle f, \mathbf{v}\rangle_d \\ b(\mathbf{u},q) &= 0 \end{aligned} \tag{1.2}$$

$$\forall \mathbf{v} \in H^1_\circ(\Omega)^d, \quad \forall q \in L_2(\Omega),$$
$$\mathbf{u} \in H^1_\circ(\Omega)^d\ , \quad p \in L_2(\Omega),$$

here

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v}\rangle_d &= \int_\Omega \mathbf{u} \cdot \mathbf{v}\ dx, \\ a(\mathbf{u},\mathbf{v}) &= \int_\Omega (\nabla \mathbf{u}) : (\nabla \mathbf{v})\ dx, \\ b(\mathbf{u},q) &= \int_\Omega (div\ \mathbf{u})\ q\ dx \end{aligned}$$

(for $q \in H^1(\Omega)$, equivalently $b(\mathbf{u},q) = -\int_\Omega \mathbf{u} \cdot \nabla q\ dx$).

The non-linear term in (1.1) leads to $c(\mathbf{u};\mathbf{u},\mathbf{v})$ with some variants, equal in $H^1_o(\Omega)^2$ but non-identical in the discrete case, see [3].

Let

$$\Phi = (\varphi_1(x)\mathbf{e}_1, \cdots, \varphi_N(x)\mathbf{e}_1, \varphi_1(x)\mathbf{e}_2 \cdots \varphi_N(x)\mathbf{e}_2) \tag{1.3}$$

the row vector of the finite element base functions in $\mathbb{V}_h \subset H^1_o(\Omega)^2$ $(\mathbf{e}_1 = (1,0)^T, \mathbf{e}_2 = (0,1)^T$, analogously in $3D)$ and

$$\Psi = (\psi_1(x), \cdots, \psi_m(x)) \tag{1.4}$$

the row vector of the nodal finite element base functions in $\mathbb{X}_h \subset H^1(\Omega) \subset L_2(\Omega)$, then the F.E. function $\mathbf{u} \in \mathbb{V}_h$ is uniquely mapped to the $2N$-vector $\underline{u}$

$$\text{by} \qquad \mathbf{u} = \Phi \underline{u} \tag{1.5}$$

respectively for the pressure we have

$$p = \Psi \underline{p} \tag{1.6}$$

with $\underline{p} \in \mathbb{R}^m$. The finite element discretization of (1.2) yields the matrices

$$A = \begin{pmatrix} A_\circ \mathbb{O} \\ \mathbb{O} A_\circ \end{pmatrix} \quad \text{with} \quad A_\circ = (a_\circ(\varphi_j, \varphi_i))\ _{i,j=1}^{N} \tag{1.7}$$

with the bilinear functional $a_\circ(\cdot, \cdot)$ belonging to the usual Laplacian operator $-\triangle$:

$$a_\circ(\varphi, \mu) = \int_\Omega \nabla\varphi \cdot \nabla\mu \ \ dx,$$

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad \text{with} \quad B_k = (b(\varphi_i \mathbf{e}_k, \psi_j))\ _{i=1}^{N}\ _{j=1}^{m} \tag{1.8}$$

and

$$M = \begin{pmatrix} M_\circ \mathbb{O} \\ \mathbb{O} M_\circ \end{pmatrix} \quad \text{with} \quad M_\circ = (\langle \varphi_j, \varphi_i \rangle)\ _{i,j=1}^{N}. \tag{1.9}$$

For later use we additionally define

$$A_p = (a_\circ(\psi_j, \psi_i))\ _{i,j=1}^{m} \tag{1.10}$$

2. Semiimplicit Projection Method

The semi-implicit projection method of GRESHO/CHAN performs one time step from

$$(\mathbf{u}^n, p^n) \sim (\mathbf{u}(x, t_n), p(x, t_n)) \ \text{ to } \ (\mathbf{u}^{n+1}, p^{n+1}) \sim (\mathbf{u}(x, t_{n+1}), p(x, t_{n+1}))$$

in the following way written in the vector space ($\mathbf{u}^n = \Phi \underline{u}^n, p^n = \Psi \underline{p}^n$ and so on):

$$(M + \triangle t \nu A)\underline{\tilde{u}}^{n+1} = M\underline{u}^n + \triangle t(F_{n+1} - \underline{c}(\underline{u}^n) + MM_L^{-1} B\underline{p}^n) \tag{2.1}$$

$$B^T M_L^{-1} B \underline{\tilde{p}}^{n+1} = -B^T \underline{\tilde{u}}^{n+1} \tag{2.2}$$

$$\begin{array}{rcl} \underline{u}^{n+1} & := & \underline{\tilde{u}}^{n+1} + M_L^{-1} B \underline{\tilde{p}}^{n+1} \\ \underline{p}^{n+1} & := & \underline{p}^n + (\gamma / \triangle t)\underline{\tilde{p}}^{n+1} \quad (1 \le \gamma \le 2) \end{array}$$

Here, $\underline{c}(\underline{u}^n) \in \mathbb{R}^{2N}$ results from the nonlinear term $c(\mathbf{u}; \mathbf{u}, \mathbf{v})$, containing the values $c(\mathbf{u}^n; \mathbf{u}^n, \varphi_j \mathbf{e}_k)$, $(k = 1, 2$ and $j = 1, \cdots, N)$, F_n is the right hand side $(\langle f(t_n), \varphi_j \mathbf{e}_k \rangle_2)$ and M_L denotes the lumped mass matrix M.

So, we have to solve two linear equations per time step. The first one is ratherly easy due to the small condition number of the matrix $(M + \triangle t \ \nu A)$ at least for small time steps and

small ν (large Re). (Note that in practice A contains the balancing tensor diffusivity term for stabilizing). A simple Jacobi preconditioner $diag\ (M + \triangle t\nu A)$ is recommended for the case of large changes of the size of the elements over the domain Ω.
The other matrix $B^T M_L^{-1} B$, the so called Pseudo-Laplacian causes some difficulties due to a very large condition number $\mathcal{O}(h^{-2})$. The construction of preconditioners (such as incomplete factorizations) which depend on the matrix elements is for large m (fine discretization) nearly impossible, because we never will form this matrix explicitly, we have only a matrix-vector multiply routine using B and M_L^{-1}. This is especially important for parallel calculations which are based on the domain decomposition as basic idea for data distribution. Here, the matrices $A, B, M \cdots$ are splitted over the processors (compare [6, 7]).

3. Preconditioning the Pseudo Laplacian

From the similarity

$$\begin{array}{ccl} B & \sim & \text{grad and} \\ B^T & \sim & \text{div} \end{array}$$

we should think about $B^T M_L^{-1} B$ as a discretization of a second order partial differential operator such as $(-\triangle)$ and should use YSERENTANT's hierachical preconditioner for a quick solution of the linear systems with that matrix.
To be more precise, let C be the $(m \times m)$–matrix belonging to a hierarchical preconditioning. We have $C^{-1} = QQ^T$ in the simplest YSERENTANT case, here Q is the matrix of basis transformation of the usual nodal basis Ψ in $\mathbb{X}_h$ into a hierarchical basis of the same space (see [6, 9]). There are some important advantages in using such C:

1. The preconditioner depends mainly on the mesh but <u>not</u> on the elements of the matrix.
2. The action of the preconditioner is very cheep (2 m operations).
3. The preconditioner is very easily used in parallel in connection with the domain decomposition [6,7].
4. The resulting condition number (for the discrete Laplacian A_p in the space $\mathbb{X}_h$) grows very slowly with $h \to 0$. We have (in the sence of positive definiteness)

$$\frac{c}{(J+1)^2}\, C \le A_p < c\, C \tag{3.1}$$

with a constant c and J the number of levels of subdividing a given course mesh into finer elements of half mesh size.

If a spectral equivalence estimation

$$\underline{\beta} A_p \le B^T M_L^{-1} B \le \bar{\beta} A_p \tag{3.2}$$

with constants $\underline{\beta}$, $\bar{\beta}$ independent of h is valid, the hierarchical preconditioner C used in the preconditioned conjugate gradient iteration for the linear system (2.2) leads to a nearly optimal solver for this step. The estimation (3.2) is proven in the next chapter for our F.E. spaces.

4. The spectral equivalence

The inequality (3.2) is intuitively valid from the fact that both matrices approximate 2nd order partial differential operators. But the explicit proof of the following theorem indicates the dependence on the finite element pair of the spaces $\mathbb{V}_h$ and $\mathbb{X}_h$.

Theorem: For the matrices A_p, B and M as defined in (1.7) to (1.10) we have

$$\beta^2 A_p \leq B^T M^{-1} B \leq A_p \tag{4.1}$$

with the constant β from the LBB-like condition

$$\sup_{\substack{\mathbf{u}\in\mathbb{V}_h \\ \mathbf{u}\neq 0}} \frac{|\, b(\mathbf{u},p)\,|}{\|\mathbf{u}\|_{L_2(\Omega)}} \geq \beta\, a_\circ(p,p)^{1/2} \quad \forall p \in \mathbb{X}_h \tag{4.2}$$

Proof: We start with the inner product in $\mathbb{R}^m$:

$$\begin{aligned} (B^T M^{-1} B\underline{p}, \underline{p}) &= \|\, M^{-1/2} B \underline{p}\,\|^2 \\ &= \max_{\underline{v}\in\mathbb{R}^{2N}} \frac{(\underline{v}^T M^{-1/2} B\underline{p})^2}{\|\,\underline{v}\,\|^2} \\ &= \max_{\underline{u}\in\mathbb{R}^{2N}} \frac{(\underline{u}^T B\underline{p})^2}{\underline{u}^T M \underline{u}} \text{ (with } \quad \underline{u} = M^{-1/2}\underline{v}). \end{aligned}$$

From the definition (1.5), (1.6) and (1.7) to (1.10) we have

$$\begin{aligned} \underline{u}^T M \underline{u} &= \langle \mathbf{u}, \mathbf{u}\rangle_2 \\ \underline{u}^T B \underline{p} &= b(\mathbf{u}, p) \\ \underline{p}^T A_p \underline{p} &= a_\circ(p,p) \end{aligned}$$

In our case ($\mathbb{X}_h \subset H^1(\Omega) \subset L_2(\Omega)$ and $\mathbf{u} = 0 \,|_\Gamma\ \forall \mathbf{u} \in \mathbb{V}_h$) we have

$$b(\mathbf{u},p) = \int_\Omega div\ \mathbf{u}\cdot p\ dx = -\int_\Omega \mathbf{u}\cdot \nabla p\ dx, \tag{4.3}$$

so,

$$\begin{aligned} \mid b(\mathbf{u},p) \mid^2 &\leq \langle \mathbf{u},\mathbf{u}\rangle_2 \cdot \langle \nabla p, \nabla p\rangle_2 \\ &= \langle \mathbf{u},\mathbf{u}\rangle_2 \cdot a_\circ(p,p) \end{aligned}$$

which proves the upper inequality in (4.1). The lower inequality is exactly the LBB-like condition, (4.2).

Remark: For our finite elements the condition (4.2) is proven in [1,4], where the constant β is near $0.4 \sin\Theta$ with the smallest angle Θ of all triangles of the mesh. In [8] the "right" LBB-condition for these elements

$$\sup_{\mathbf{u} \in \mathbb{V}_h} \frac{\mid b(\mathbf{u},p) \mid}{\parallel \mathbf{u} \parallel_{H^1_\circ(\Omega)}} \geq \gamma \parallel p \parallel_{L_2(\Omega)} \quad \forall p \in \mathbb{X}_h$$

was deduced from the condition (4.2).

The dependence on the smallest angle coincides with numerical tests on hierarchical preconditioning the Pseudo-Laplacian.

The inequality (3.2) with constant $\underline{\beta}, \bar{\beta}$ follows from the well-known spectral equivalence of M and M_L.

5. Problems in Introducing the Coarse Mesh Solver

The behaviour of the preconditioner is much improved, if some additional coarse mesh solver is introduced on the coarsest level. Let the first n_0 nodes belong to the coarse mesh of Level 0, then the preconditioner used in practice is described by

$$C^{-1} = Q \begin{pmatrix} C_0^{-1} & \mathbb{O} \\ \mathbb{O} & I \end{pmatrix} Q^T, \tag{5.1}$$

where C_0 is an $(n \times n)$–symmetric matrix approximating the linear system on the coarsest mesh. For defining C_0 we have two possibillities:

> Either C_0 is the true F.E. matrix of the problem under consideration belonging to the starting Level–0–triangulation or C_0 is a spectrally equivalent approximation of the true coarse grid matrix.

From the complicate structure of $B^T M_L^{-1} B$ even on coarsest level we consider an approximation due to Bramble/Pasciak/Schatz [10]:
Due to the spectral equivalence of $B^T M_L^{-1} B$ and A_p we will use the F.E. assembly of the "element" matrix

$$G = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

over all pairs of nodes of the coarse grid having a common edge. This matrix results from the bilinear form

$$a_G(p,p) = \sum_{i,j} \left(p(x_i) - p(x_j)\right)^2$$

where the sum is taken over all pairs (i,j) defining an edge of the coarse mesh (cf. [10]). After removing rows and columns of nodes with Dirichlet type boundary conditions, we usually obtain a nonsingular matrix C_0. These are at least 2 nodes: the both ends of the

boundary part Γ_D with 1st type boundary conditions.
In application to our pseudo–Laplacian problem the situation is more complicate due to the fact that the pressure may not be prescribed all along the boundary.
A more thorough study of the boundary conditions leads to 3 special situations:

<u>Situation 1:</u>

We consider the boundary conditions as (1.1), i.e. we have a closed domain Ω without instream and outstream or both instream and outstream are prescribed on parts of $\partial\Omega$. Here, the pressure is free overall on $\partial\Omega$ and $B^T M_L^{-1} B$ is a singular matrix due to the following calculation:

$$\begin{aligned}(B_k \underline{e})_i &= \sum_{j=1}^{m} b(\varphi_i \mathbf{e}_k, \psi_j), \quad k = 1,2 \\ &= b(\varphi_i \mathbf{e}_k, \sum \psi_j) \\ &= b(\varphi_i \mathbf{e}_k, 1)\end{aligned} \tag{5.2}$$

With (4.3) we have

$$b(\varphi_i \mathbf{e}_k, 1) = -\int \varphi_i \mathbf{e}_k \cdot \bigtriangledown 1 dx = 0, \qquad \text{so } B\underline{e} = 0.$$

Often we write

$$\int_\Omega p dx = 0 \tag{5.3}$$

for uniqueness of the pressure in this case. This has to be used in C_0.

<u>Situation 2:</u>

More general situations with one outstream boundary part $\Gamma_{D,p}$ can be simulated by

$$\begin{array}{llll}\mathbf{u}(x,t) = 0 & \text{(or prescribed inflow)} & \text{on} & x \in \Gamma_{D,u} \\ p(x,t) = 0 & & \text{on} & x \in \Gamma_{D,p}\end{array}$$

$$\text{and} \qquad \partial\Omega = \Gamma_{D,u} \cup \Gamma_{D,p}.$$

Then $B^T M_L^{-1} B$ is a regular matrix and we will arrive at a regular preconditioner C_0 by "removing" rows and columns belonging to the nodes of $\Gamma_{D,p}$.
Note that formula (4.3) is true because the boundary term vanishes

$$\begin{array}{rlcll}(\text{ either} & x \in \Gamma_{D,u} & \Longrightarrow & u = 0 & \forall u \in \mathbb{V}_h \\ \text{or} & x \in \Gamma_{D,p} & \Longrightarrow & p = 0 & \forall p \in \mathbb{X}_h).\end{array}$$

<u>Situation 3:</u>

The most complicate situation considers one or more outstream boundaries without prescribing the pressure.
Here we have $\mathbf{u}(x,t) = 0$ (or prescribed) for $x \in \Gamma_{D,u}$ and $\Gamma_{out} = \partial\Omega \setminus \Gamma_{D,u}$ has to be considered especially. The transformation of (1.1) into (1.2) produces a "natural" boundary condition on Γ_{out} : $\quad p\mathfrak{n} = \nu \frac{\partial \mathbf{u}}{\partial \mathfrak{n}}$
with $\mathfrak{n}$ the unit outer normal at $x \in \Gamma_{out}$. So the formula (4.3) is no longer true and the matrix $B^T M_L^{-1} B$ is regular.

For the construction of the resulting preconditioner C_0 these 3 situations have to be distinguished:

Situation 1:

The starting matrix C_0 is singular, but we have to work in the subspace

$$\underline{p} \perp \underline{e} = (1, \cdots, 1)^T \in \mathbb{R}^m \qquad \text{(the same in } \mathbb{R}^{n_0}\text{)}$$

due to (5.3). The conjugate gradient solver for the matrix $B^T M_L^{-1} B$ works immediately in this subspace, because the right hand side $B^T \underline{u}$ belongs to it and the start vector is zero, so only the problem of constructing a regular preconditioner via regular C_0 remains:

The singular matrix C_0 (assembly of G's over all pairs of nodes having a common edge in the coarse mesh) is build up and we calculate the Cholesky decomposition $C_0 = R_0 R_0^T$. Here R_0 is upper triangular and the elements of R_0 have to be calculated from last to first row backwards. So, this decomposition detects the singularity at the last calculated element $(R_0)_{11} \approx 0$. If we set $(R_0)_{11}^{-1} = 0$ (the inverse values of $(R_0)_{kk}$ are usually stored on the main diagonal), we are able to presolve a linear coarse mesh system

$$R_0 R_0^T \underline{w}_0 = \underline{r}_0,$$

with the first entry $(\underline{w}_0)_1 = 0$. Then the vector $\underline{w}_0$ obtained is orthogonalized with respect to $\underline{e} \in \mathbb{R}^{n_0}$:

$$\underline{w}_0 := \underline{w}_0 - \left(\frac{\underline{e}^T \underline{w}_0}{n_0} \right) \underline{e}.$$

This trick defines $\underline{w}_0$ fulfilling both equations:

$$\begin{aligned} C_0 \underline{w}_0 &= \underline{r}_0 \\ \underline{e}^T \underline{w}_0 &= 0 \end{aligned}$$

and the resulting operator $\underline{r}_0 \longrightarrow \underline{w}_0$ is regular within the subspace orthogonal to $\underline{e}$, so the whole preconditioner C is.

Situation 2:

C_0 is nonsingular when rows and columns belonging to nodes on $\Gamma_{D,p}$ are "removed" and will be Cholesky decomposed as above without problems.

Situation 3:

Due to the nonsingularity of the matrix $B^T M_L^{-1} B$ a singular C_0 as obtained in the first step makes no sence, so the simple assembly of matrices G as indicated above is not enough in this case.
If we consider the proof of the Theorem in Chapter 4, we obtain a more complicate term for

$$(B^T M^{-1} B \underline{p}, \underline{p}) = \max_{\mathbf{u} \in \mathbb{V}_h} \frac{|b(\mathbf{u}, p)|^2}{\|\mathbf{u}\|_{L_2}^2},$$

now we have

$$b(\mathbf{u}, p) = -\int_\Omega \nabla p \cdot \mathbf{u}\; dx + \int_{\Gamma_{out}} \mathfrak{n} \cdot \mathbf{u}\; p\; ds.$$

The function $\mathfrak{n} \cdot \mathbf{u}$ is an arbitrary piecewise linear function (along a smooth Γ_{out}), so the extra term looks like a 3rd type boundary condition, which usually adds a positive entry to the main diagonal of the stiffness matrix. From this analongy we enlarge the main diagonal entries in C_0 belonging to nodes of Γ_{out} by $\alpha > 0$ leading to a nonsingular preconditioner again. In our tests some positive numbers from $\alpha = 0.1$ to $\alpha = 100$ had no large influence to the resulting number of CG iterations, so we use $\alpha = 1$.

6. Numerical Example

We present some numerical tests on a simple back ward facing step. The domain Ω consists of 32 rectangular triangles with edges of length 1 (y-direction) and x_{scale} (x-direction). In enlarging $x_{scale} = 1, 2, 4, 8$ we obtain more and more worse examples (smaller angles Θ).

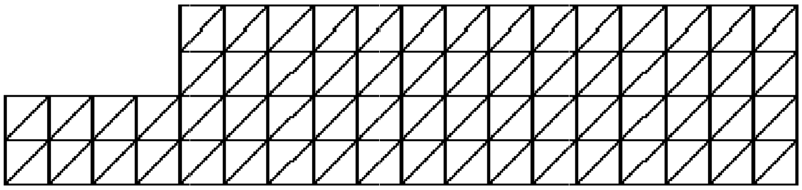

Figure 1: Level–1–Mesh, $x_{scale} = 1$

We have subdivided a triangle of Level L into 4 equal subtriangles of the Level $L+1$. On the fine level we used the linear T_h/ linear $T_{h/2}$ combination of the elements as proposed in Chapter 0. We used Yserentant's hierarchical preconditioner for solving

$$B^T M_L^{-1} B \underline{\tilde{p}} = \underline{b},$$

with a coarse grid solver depending on the boundary conditions as proposed in Chapter 5. The test run on a 32 processor *MIMD* parallel computer (each processor worked on a subdomain, which coincides with one coars mesh triangle).

As boundary conditions we have used:

- non slip condition on the walls and
- prescribed instream at $x = 0$:

$$u(0, y) = \begin{pmatrix} 4y(1-y) \\ 0 \end{pmatrix}.$$

Additionally we consider at the outstream boundary ($x_{out} = 9 \cdot x_{scale}$):

<u>Situation 1:</u>

- prescribed outstream $\quad u(x_{out}, y) = \begin{pmatrix} \frac{1}{2}y(2-y) \\ 0 \end{pmatrix}$, so $\Gamma_{D,u} = \partial\Omega$.

<u>Situation 2:</u>

- prescribed pressure $p(x_{out}, y) = 0$ so $\Gamma_{D,p} = x_{out} \times \ [0,2]$.

<u>Situation 3:</u>

- "no" condition on $\Gamma_{out} = x_{out} \times \ [0,2]$
  (from the weak formulation we have $p\mathfrak{n} = \nu\frac{\partial u}{\partial \mathfrak{n}}$)

The tests run with $\nu = 0.01$, and solution to relative accuracy of 10^{-4}.

The following table illustrates the theoretic results. We have typically a growth of the numbers of CG-iterations as $|\ln h|$:

Situation 1:

Level	m / N	with coarse grid solver # CG-iterations for x_{scale}= 1.	2.	4.	8.	without coarse grid solver # CG-iterations for x_{scale}= 1.	2.	4.	8.
2	301 / 1113	30	36	48	92	34	46	74	145
3	1113 / 4372	39	44	59	103	42	57	89	173
4	4372 / 16737	43	51	71	115	50	71	114	222
5	16737 / 66241	48	58	80	128	61	92	156	299

Situation 2:

Level	m / N	with coarse grid solver # CG-iterations for x_{scale}= 1.	2.	4.	8.	without coarse grid solver # CG-iterations for x_{scale}= 1.	2.	4.	8.
2	301 / 1113	29	35	47	91	39	52	84	163
3	1113 / 4372	39	45	60	102	48	65	103	199
4	4372 / 16737	43	48	69	114	60	80	131	248
5	16737 / 66241	47	54	79	123	73	104	173	326

Situation 3:

Level	m / N	with coarse grid solver # CG-iterations for x_{scale}= 1.	2.	4.	8.	without coarse grid solver # CG-iterations for x_{scale}= 1.	2.	4.	8.
2	301 / 1113	40	40	49	92	46	54	84	163
3	1113 / 4372	58	55	65	103	64	72	103	200
4	4372 / 16737	80	69	81	115	93	99	134	249
5	16737 / 66241	107	90	99	136	139	140	189	326

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