Technische Universität Chemnitz

Sonderforschungsbereich 393

Numerische Simulation auf massiv parallelen Rechnern

Sergey I. Solov'ev

Convergence of the modified subspace iteration method for nonlinear eigenvalue problems

Preprint SFB393/99-35

Abstract

The existence of eigenvalues of a finite-dimensional eigenvalue problem with nonlinear entrance of a spectral parameter is studied. The modified subspace iteration method is suggested for solving the problem. The convergence and the error of this method for computing eigenvalues are investigated.

Preprint-Reihe des Chemnitzer SFB 393

SFB393/99-35 **November 1999**

Author's address:

Sergey I. Solov'ev
Kazan State University
Faculty of computer science and cybernetics
Kremlevskaya 18, 420008 Kazan, Russia

e–mail: sergei.solovyev@ksu.ru

Introduction

The classical subspace iteration method is one of the most effective methods for computing a group of the smallest eigenvalues of a finite–dimensional symmetric eigenvalue problem (see, for example, [1]). A possibility for constructing the modified subspace iteration method is indicated in the paper [2], the convergence and the error of this method are investigated in the papers [3, 4] (see also [5]). In the present paper, the modified subspace iteration method is proposed for solving symmetric nonlinear eigenvalue problems. Nonlinear finite–dimensional eigenvalue problems arise after the discretization of infinite–dimensional nonlinear eigenvalue problems (see, for example, [6–17]).

In section 1 of the present paper, the statement of a symmetric eigenvalue problem in a finite–dimensional space with a nonlinear entrance of a spectral parameter is given. In section 2, the Rayleigh–Ritz approximation of the initial problem in a subspace is formulated. In section 3, results about existence and properties of the eigenvalues of the nonlinear eigenvalue problem in a subspace are proved. Similar results were obtained earlier in the papers [6–8, 10–15]. In section 4, we describe auxiliary results obtained in the papers [3, 4]. These results are used further for constructing and investigating the iterative method. In sections 5 and 6, the modified subspace iteration method for the nonlinear eigenvalue problem is formulated, the convergence and the error of this method for computing eigenvalues are investigated.

1. Formulation of the problem

Let H be an N–dimensional real Euclidean space with the scalar product $(.,.)$ and the norm $\|.\|$, and let Λ be an interval on the real axis $\mathbb{R}$, $\Lambda = (\alpha, \beta)$, $0 \le \alpha < \beta \le \infty$. Introduce the operators $A(\mu)$ and $B(\mu)$ that, for fixed $\mu \in \Lambda$, are symmetric linear operators from H to H satisfying the following conditions:

a) positive definiteness, i.e. there exist positive continuous functions $\alpha_1(\mu)$ and $\beta_1(\mu)$, $\mu \in \Lambda$, such that

$$(A(\mu)v, v) \ge \alpha_1(\mu)\|v\|^2, \quad (B(\mu)v, v) \ge \beta_1(\mu)\|v\|^2 \quad \forall v \in H, \mu \in \Lambda;$$

b) continuity with respect to the numerical argument, i.e.

$$\|A(\mu) - A(\eta)\| \to 0, \quad \|B(\mu) - B(\eta)\| \to 0,$$

as $\mu \to \eta$, $\mu, \eta \in \Lambda$. By $\|.\|$ also denote the norm of an operator from H to H.

Define the Rayleigh quotient by the formula:

$$R(\mu, v) = \frac{(A(\mu)v, v)}{(B(\mu)v, v)}, \quad v \in H \setminus \{0\}, \mu \in \Lambda.$$

Assume that the following additional conditions are fulfilled:

c) the Rayleigh quotient $R(\mu, v)$, $\mu \in \Lambda$, is, for fixed $v \in H$, a nonincreasing function of the numerical argument, i.e.

$$R(\mu, v) \geq R(\eta, v), \quad \mu < \eta, \mu, \eta \in \Lambda, v \in H \setminus \{0\};$$

d) there exists $\eta \in \Lambda$ such that

$$\eta - \min_{v \in H \setminus \{0\}} R(\eta, v) \leq 0;$$

e) there exists $\eta \in \Lambda$ such that

$$\eta - \max_{v \in H \setminus \{0\}} R(\eta, v) \geq 0.$$

Consider the following variational eigenvalue problem: find $\lambda \in \Lambda$, $u \in H \setminus \{0\}$, such that

$$(A(\lambda)u, v) = \lambda(B(\lambda)u, v) \quad \forall v \in H. \tag{1.1}$$

This problem is equivalent to the operator eigenvalue problem: find $\lambda \in \Lambda$, $u \in H \setminus \{0\}$, such that

$$A(\lambda)u = \lambda B(\lambda)u.$$

The number λ that satisfies (1.1) is called an eigenvalue, and the element u is called an eigenelement of problem (1.1) corresponding to λ. The set $U(\lambda)$ that consists of the eigenelements corresponding to the eigenvalue λ and the zero element is a closed subspace in H, which is called the eigensubspace corresponding to the eigenvalue λ. The dimension of this subspace is called a multiplicity of the eigenvalue λ.

2. Approximation of the problem in the subspace

Let V be a k–dimensional subspace of the space H, $1 \leq k \leq N$.

It is not difficult to see that conditions d) and e) imply the following properties:

f) there exists $\eta \in \Lambda$ such that

$$\eta - \min_{v \in V \setminus \{0\}} R(\eta, v) \leq 0;$$

g) there exists $\eta \in \Lambda$ such that

$$\eta - \max_{v \in V \setminus \{0\}} R(\eta, v) \geq 0.$$

Problem (1.1) is approximated by the following variational eigenvalue problem in the subspace: find $\lambda = \lambda(V) \in \Lambda$, $u = u(V) \in V \setminus \{0\}$, such that

$$(A(\lambda)u, v) = \lambda(B(\lambda)u, v) \quad \forall v \in V. \tag{2.1}$$

Problem (2.1) is called the Rayleigh–Ritz approximation of problem (1.1) in the subspace V.

Variational problem (2.1) is equivalent to the operator eigenvalue problem: find $\lambda = \lambda(V) \in \Lambda$, $u = u(V) \in V \setminus \{0\}$, such that

$$A(\lambda, V)u = \lambda B(\lambda, V)u,$$

where $A(\mu, V) = P_V A(\mu) P_V$ and $B(\mu, V) = P_V B(\mu) P_V$ are operators from V to V for fixed $\mu \in \Lambda$, and P_V is the orthogonal projector from H onto V.

Remark 1. Assume that v_i, $i = 1, 2, \ldots, k$, is a basis of the subspace V. Then problem (2.1) is equivalent to the matrix problem: find $\lambda = \lambda(V) \in \Lambda$, $y = y(V) \in \mathbb{R}^k \setminus \{0\}$, such that

$$\mathbf{A}(\lambda, V)y = \lambda \mathbf{B}(\lambda, V)y, \tag{2.2}$$

where $\mathbb{R}^k$ is the space of vectors $x = (x_1, x_2, \ldots, x_k)^{\mathrm{T}}$, $x_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$, $y = (y_1, y_2, \ldots, y_k)^{\mathrm{T}}$, y_i, $i = 1, 2, \ldots, k$ are the coefficients in the expansion of the eigenelement $u \in V$, $u = \sum\limits_{i=1}^{k} y_i v_i$, the matrices $\mathbf{A}(\mu, V)$ and $\mathbf{B}(\mu, V)$ of order k for fixed $\mu \in \Lambda$ are defined by the formulas:

$$\mathbf{A}(\mu, V) = \{a_{ij}(\mu, V)\}_{ij=1}^{k}, \quad \mathbf{B}(\mu, V) = \{b_{ij}(\mu, V)\}_{ij=1}^{k},$$

$$a_{ij}(\mu, V) = (A(\mu)v_i, v_j), \quad b_{ij}(\mu, V) = (B(\mu)v_i, v_j), \quad i, j = 1, 2, \ldots, k,$$

for $\mu \in \Lambda$.

Remark 2. Suppose $A(\mu)$ and $B(\mu)$ are matrices of order N for fixed $\mu \in \Lambda$, $H = \mathbb{R}^N$, V is a k–dimensional subspace of the space H, $1 \leq k \leq N$. Then the matrices of matrix problem (2.2) have the representation

$$\mathbf{A}(\mu, V) = Q^{\mathrm{T}} A(\mu) Q, \quad \mathbf{B}(\mu, V) = Q^{\mathrm{T}} B(\mu) Q, \quad \mu \in \Lambda,$$

where $Q = (v_1, v_2, \ldots, v_k)$ is the matrix with the columns v_i, $i = 1, 2, \ldots, k$.

3. Existence of the eigenvalues

For fixed $\mu \in \Lambda$ we introduce the auxiliary linear eigenvalue problem: find $\gamma(\mu, V) \in \mathbb{R}$, $u = u(\mu, V) \in V \setminus \{0\}$, such that

$$(A(\mu)u, v) = \gamma(\mu, V)(B(\mu)u, v) \quad \forall v \in V. \tag{3.1}$$

Variational problem (3.1) with $\mu \in \Lambda$ is equivalent to the operator eigenvalue problem: find $\gamma(\mu, V) \in \mathbb{R}$, $u = u(\mu, V) \in V \setminus \{0\}$, such that

$$A(\mu, V)u = \gamma(\mu, V) B(\mu, V)u.$$

Remark 3. For fixed $\mu \in \Lambda$ problem (3.1) is equivalent to the matrix problem: find $\gamma(\mu, V) \in \mathbb{R}$, $y = y(\mu, V) \in \mathbb{R}^k \setminus \{0\}$, such that

$$\mathbf{A}(\mu, V)y = \gamma(\mu, V)\mathbf{B}(\mu, V)y, \tag{3.2}$$

the matrices $\mathbf{A}(\mu, V)$ and $\mathbf{B}(\mu, V)$ of order k are defined in Remarks 1 and 2 for fixed $\mu \in \Lambda$.

For a symmetric positive definite linear operator A from V to V, denote by V_A the Euclidean space of elements from V with the scalar product $(u, v)_A = (Au, v)$ and the norm $\|v\|_A = (v, v)_A^{1/2}$, $u, v \in V_A$.

Lemma 1. *For fixed $\mu \in \Lambda$ problem (3.1) has k real positive eigenvalues $0 < \gamma_1(\mu, V) \leq \gamma_2(\mu, V) \leq \ldots \leq \gamma_k(\mu, V)$. The eigenelements $u_i = u_i(\mu, V)$, $i = 1, 2, \ldots, k$, correspond to these eigenvalues:*

$$(A(\mu)u_i, u_j) = \gamma_i(\mu, V)\delta_{ij}, \quad (B(\mu)u_i, u_j) = \delta_{ij}, \quad i, j = 1, 2, \ldots, k.$$

The elements $u_i = u_i(\mu, V)$, $i = 1, 2, \ldots, k$, form an orthonormal basis of the space $V_{B(\mu)}$.

The proof is given, for example, in [18].

Lemma 2. *The formula of the minimax principle is valid:*

$$\gamma_i(\mu, V) = \min_{W_i \subset V} \max_{v \in W_i \setminus \{0\}} R(\mu, v), \quad i = 1, 2, \ldots, k,$$

where W_i is an i–dimensional subspace of the space V. In particular, the following relations hold:

$$\gamma_1(\mu, V) = \min_{v \in V \setminus \{0\}} R(\mu, v), \quad \gamma_k(\mu, V) = \max_{v \in V \setminus \{0\}} R(\mu, v).$$

The proof is given, for example, in [18].

Set

$$\alpha_{1,min}(a, b) = \min_{\mu \in [a,b]} \alpha_1(\mu), \quad \beta_{1,min}(a, b) = \min_{\mu \in [a,b]} \beta_1(\mu),$$

for a fixed segment $[a, b]$ on Λ.

Lemma 3. *Suppose that*

$$\frac{\|A(\mu) - A(\eta)\|}{\alpha_{1,min}(a, b)} \leq \frac{1}{2}$$

for $\mu, \eta \in [a, b]$. Then the following inequality is valid:

$$|\gamma_i(\mu, V) - \gamma_i(\eta, V)| \leq 2\left(\frac{\|A(\mu) - A(\eta)\|}{\alpha_{1,min}(a, b)} + \frac{\|B(\mu) - B(\eta)\|}{\beta_{1,min}(a, b)}\right)\gamma_i(a, V)$$

for $i = 1, 2, \ldots, k,\ \mu, \eta \in [a, b]$.

Proof. Denote by $E_i(\mu, V)$ the subspace spanned on the eigenelements $u_j = u_j(\mu, V)$, $j = 1, 2, \ldots, i$, which correspond to the eigenvalues $\gamma_j(\mu, V)$, $j = 1, 2, \ldots, i$, of problem (3.1) for fixed $\mu \in \Lambda$, $1 \le i \le k$. Using the minimax principle of Lemma 2, we obtain

$$\begin{aligned} \gamma_i(\mu, V) &= \min_{W_i \subset V} \max_{v \in W_i \setminus \{0\}} R(\mu, v) \le \\ &\le \max_{v \in E_i(\eta, V) \setminus \{0\}} R(\mu, v) \le \\ &\le \max_{v \in E_i(\eta, V) \setminus \{0\}} R(\eta, v) + \max_{v \in E_i(\eta, V) \setminus \{0\}} |R(\mu, v) - R(\eta, v)| = \\ &= \gamma_i(\eta, V) + \sigma_i(\mu, \eta), \end{aligned}$$

where

$$\sigma_i(\mu, \eta) = \max_{v \in E_i(\eta, V) \setminus \{0\}} |R(\mu, v) - R(\eta, v)|, \quad \mu, \eta \in \Lambda.$$

Hence

$$|\gamma_i(\mu, V) - \gamma_i(\eta, V)| \le \max\{\sigma_i(\mu, \eta), \sigma_i(\eta, \mu)\}, \quad \mu, \eta \in \Lambda.$$

Let us estimate $\sigma_i(\mu, \eta)$, $\mu, \eta \in [a, b]$. It is easy to verify that

$$\begin{aligned} R(\mu, v) - R(\eta, v) &= R(\eta, v) \frac{(A(\mu)v, v) - (A(\eta)v, v)}{(A(\mu)v, v)} + \\ &+ R(\eta, v) \frac{(B(\eta)v, v) - (B(\mu)v, v)}{(B(\mu)v, v)} + \\ &+ (R(\mu, v) - R(\eta, v)) \frac{(A(\mu)v, v) - (A(\eta)v, v)}{(A(\mu)v, v)}, \quad \mu, \eta \in \Lambda. \end{aligned}$$

This relation implies the inequality

$$\sigma_i(\mu, \eta) \le \left(\frac{\|A(\mu) - A(\eta)\|}{\alpha_{1,min}(a, b)} + \frac{\|B(\mu) - B(\eta)\|}{\beta_{1,min}(a, b)} \right) \gamma_i(a, V) + \sigma_i(\mu, \eta) \frac{\|A(\mu) - A(\eta)\|}{\alpha_{1,min}(a, b)}$$

for $\mu, \eta \in [a, b]$. Consequently, the following estimate holds

$$\sigma_i(\mu, \eta) \le \frac{1}{1 - \frac{\|A(\mu) - A(\eta)\|}{\alpha_{1,min}(a,b)}} \left(\frac{\|A(\mu) - A(\eta)\|}{\alpha_{1,min}(a, b)} + \frac{\|B(\mu) - B(\eta)\|}{\beta_{1,min}(a, b)} \right) \gamma_i(a, V)$$

for $\mu, \eta \in [a, b]$. This proves the lemma.

Lemma 4. *The functions* $\gamma_i(\mu, V)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, k$, *are continuous nonincreasing functions with positive values. The following inequalities hold:* $\gamma_i(\mu, H) \le \gamma_i(\mu, V)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, k$.

Proof. The continuity of the functions $\gamma_i(\mu, V)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, k$, follows from Lemma 3 and condition b). Using the minimax principle of Lemma 2 and condition c), we obtain that the functions $\gamma_i(\mu, V)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, k$, are nonincreasing functions and $\gamma_i(\mu, H) \le \gamma_i(\mu, V)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, k$. Thus, the lemma is proved.

Lemma 5. *The functions $\mu - \gamma_i(\mu, V)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, k$, are continuous and strictly increasing functions with negative and positive values in the neighbourhoods of the points α and β, respectively.*

Proof. The increase of the functions $\mu - \gamma_i(\mu, V)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, k$, follows from Lemma 4.

Taking into account condition f), we obtain that there exists a number $\eta \in \Lambda$, for which the following relations are valid:

$$\mu - \gamma_i(\mu, V) < \eta - \gamma_i(\eta, V) \le \eta - \gamma_1(\eta, V) = \eta - \min_{v \in V \setminus \{0\}} R(\eta, v) \le 0$$

for $\mu \in (\alpha, \eta)$, $i = 1, 2, \ldots, k$.

According to condition g), there exists $\eta \in \Lambda$ such that the following inequalities hold:

$$\mu - \gamma_i(\mu, V) > \eta - \gamma_i(\eta, V) \ge \eta - \gamma_N(\eta, V) = \eta - \max_{v \in V \setminus \{0\}} R(\eta, v) \ge 0$$

for $\mu \in (\eta, \beta)$, $i = 1, 2, \ldots, k$. Thus, the lemma is proved.

Lemma 6. *A number $\lambda = \lambda(V) \in \Lambda$ is an eigenvalue of problem* (2.1) *if and only if the number λ is a solution of an equation from the set $\mu - \gamma_i(\mu, V) = 0$, $\mu \in \Lambda$, $i = 1, 2, \ldots, k$.*

Proof. If λ is a solution of the equation $\mu - \gamma_i(\mu, V) = 0$, $\mu \in \Lambda$, for some i, $1 \le i \le k$, then it follows from (2.1) and (3.1) that λ is an eigenvalue of problem (2.1). If λ is an eigenvalue of problem (2.1), then (2.1) and (3.1) imply $\lambda - \gamma_i(\lambda, V) = 0$ for some i, $1 \le i \le k$. This proves the lemma.

Theorem 1. *Problem* (2.1) *has k eigenvalues $\lambda_i = \lambda_i(V)$, $i = 1, 2, \ldots, k$, which are repeated according to their multiplicity: $\alpha < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_k < \beta$. Each eigenvalue λ_i is a unique root of the equation $\mu - \gamma_i(\mu, V) = 0$, $\mu \in \Lambda$, $i = 1, 2, \ldots, k$.*

Proof. By Lemma 5, each equation of the set $\mu - \gamma_i(\mu, V) = 0$, $\mu \in \Lambda$, $i = 1, 2, \ldots, k$, has a unique solution. Denote these solutions by λ_i, $i = 1, 2, \ldots, k$, i. e. $\lambda_i - \gamma_i(\lambda_i, V) = 0$, $i = 1, 2, \ldots, k$. To check that the numbers λ_i, $i = 1, 2, \ldots, k$, are put in an increasing order, let us assume the opposite, i. e. $\lambda_i > \lambda_{i+1}$. Then, according to Lemma 4, we obtain a contradiction, namely

$$\lambda_i = \gamma_i(\lambda_i, V) \le \gamma_i(\lambda_{i+1}, V) \le \gamma_{i+1}(\lambda_{i+1}, V) = \lambda_{i+1}.$$

By Lemma 6, the numbers λ_i, $i = 1, 2, \ldots, k$, are eigenvalues of problem (2.1). Thus, the theorem is proved.

For brevity we will put

$$\lambda_i = \lambda_i(H), \quad \gamma_i(\mu) = \gamma_i(\mu, H), \quad \mu \in \Lambda, \quad i = 1, 2, \ldots, N.$$

Theorem 2. *Problem* (1.1) *has N eigenvalues λ_i, $i = 1, 2, \ldots, N$, which are repeated according to their multiplicity: $\alpha < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N < \beta$. Each eigenvalue λ_i is a unique root of the equation $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$. The following inequalities are valid: $\lambda_i = \lambda_i(H) \leq \lambda_i(V)$, $i = 1, 2, \ldots, k$.*

The proof follows from Theorem 1 and Lemma 4.

Remark 4. If $\alpha = 0$, then conditions d) and f) follow from condition c).

Proof. Let us fix $\nu \in \Lambda$ and put $\eta = \min\{\gamma_1(\nu), \nu\}/2$. Taking into account condition c), Lemma 2, and the relation $\eta \leq \gamma_1(\nu)/2$, $\eta \leq \nu/2 < \nu$, we have

$$\eta - \min_{v \in V \setminus \{0\}} R(\eta, v) \leq \eta - \min_{v \in H \setminus \{0\}} R(\eta, v) = \eta - \gamma_1(\eta) \leq \gamma_1(\nu)/2 - \gamma_1(\nu) = -\gamma_1(\nu)/2 < 0.$$

Thus, conditions d) and f) are satisfied for chosen $\eta \in \Lambda$.

Remark 5. If $\beta = \infty$, then conditions e) and g) follow from condition c).

Proof. For fixed $\nu \in \Lambda$ put $\eta = 2\max\{\gamma_N(\nu), \nu\}$. Since $\eta \geq 2\gamma_N(\nu)$ and $\eta \geq 2\nu > \nu$, according to condition c) and Lemma 2, we obtain the relation:

$$\eta - \max_{v \in V \setminus \{0\}} R(\eta, v) \geq \eta - \max_{v \in H \setminus \{0\}} R(\eta, v) = \eta - \gamma_N(\eta) \geq 2\gamma_N(\nu) - \gamma_N(\nu) = \gamma_N(\nu) > 0,$$

which implies that conditions e) and g) are satisfied.

4. Auxiliary results

Assume that the symmetric positive definite linear operator $C(\mu)$ from H to H is given for fixed $\mu \in \Lambda$, and that there exist continuous functions $\delta_0(\mu)$, $\delta_1(\mu)$, $\mu \in \Lambda$, $0 < \delta_0(\mu) \leq \delta_1(\mu)$, $\mu \in \Lambda$, such that

$$\delta_0(\mu)(C(\mu)v, v) \leq (A(\mu)v, v) \leq \delta_1(\mu)(C(\mu)v, v), \quad v \in H, \quad \mu \in \Lambda.$$

Put

$$S(\mu, \eta) = I - \tau(\mu)C^{-1}(\mu)(A(\mu) - \eta B(\mu)),$$

where $\mu \in \Lambda$, $\eta \in \mathbb{R}$; I is the identity operator from H to H; $\tau(\mu)$, $\mu \in \Lambda$, is a given function.

For a given k–dimensional subspace V of the space H we define a subspace W of the space H and numbers ν^0 and ν^1 by the formulas:

$$W = S(\mu, \nu^0)V, \quad \nu^0 = \gamma_k(\mu, V), \quad \nu^1 = \gamma_k(\mu, W),$$

for fixed $\mu \in \Lambda$.

Lemma 7. *Let* $\tau(\mu) = \delta_1^{-1}(\mu)$*,* $\mu \in \Lambda$*. Then* W *is a* k*–dimensional subspace of the space* H*.*

The proof is given in [3, 4].

Lemma 8. *Let* $\gamma_k(\mu) = \gamma_k(\mu, H)$ *be an eigenvalue of problem* (3.1) *with* $\mu \in \Lambda$ *such that*

$$\gamma_k(\mu) = \ldots = \gamma_{k+s}(\mu) < \gamma_{k+s+1}(\mu),$$

$k \geq 1$*,* $s \geq 0$*,* $k + s + 1 \leq N$*. Assume that* $\nu^0 < \gamma_{k+s+1}(\mu)$*,* $\tau(\mu) = \delta_1^{-1}(\mu)$*,* $\mu \in \Lambda$*. Then* $\gamma_k(\mu) \leq \nu^1 \leq \nu^0$*, and the following estimate is valid:*

$$\nu^1 - \gamma_k(\mu) \leq \rho(\mu, \nu^0)(\nu^0 - \gamma_k(\mu)),$$

where $0 < \rho(\mu, \nu) < 1$*,*

$$\rho(\mu, \nu) = \frac{1 - \delta(\mu)(1 - \nu/\gamma_{k+s+1}(\mu))}{1 + \delta(\mu)(1 - \nu/\gamma_{k+s+1}(\mu))(\nu/\gamma_k(\mu) - 1)},$$

$$\delta(\mu) = \delta_0(\mu)/\delta_1(\mu), \quad \nu \in [\gamma_k(\mu), \gamma_{k+s+1}(\mu)), \quad \mu \in \Lambda.$$

The proof is given in [3, 4].

5. Convergence of the modified subspace iteration method

Introduce the functions $\varphi_n(\mu)$, $\mu \in \Lambda$, $n = 0, 1, \ldots$, by the formulas:

$$\varphi_n(\mu) = \gamma_k(\mu, H_k^n), \quad \mu \in \Lambda,$$

where H_k^n is a subspace of the space H, $n = 0, 1, \ldots$, $\gamma_k(\mu, V)$ is the k–th eigenvalue of problem (3.1) for fixed $\mu \in \Lambda$.

Consider the following iterative method:

$$H_k^{n+1} = S(\mu^n) H_k^n, \quad n = 0, 1, \ldots, \tag{5.1}$$

where the number μ^n is defined as a solution of the equation:

$$\mu - \varphi_n(\mu) = 0, \quad \mu \in \Lambda, \tag{5.2}$$

for $n = 0, 1, \ldots$ Here $S(\mu) = S(\mu, \mu)$, $\mu \in \Lambda$; $S(\mu, \eta)$, $\mu \in \Lambda$, $\eta \in \mathbb{R}$, is defined in section 4, H_k^0 is a given k–dimensional subspace of the space H.

Remark 6. At each step of the iterative method (5.1), we need to find the number μ^n as a solution of equation (5.2) or the equation $\mu - \gamma_k(\mu, H_k^n) = 0$, $\mu \in \Lambda$. It follows from Theorem 1 that $\mu^n = \lambda_k(H_k^n)$ is the maximal eigenvalue of problem (2.1) with $V = H_k^n$.

Consequently, the number μ^n we may define as the maximal eigenvalue of matrix problem (2.2) of order k (see Remarks 1 and 2).

Lemma 9. *Let $\tau(\mu) = \delta_1^{-1}(\mu)$, $\mu \in \Lambda$. Then H_k^n, $n = 1, 2, \dots$ are k–dimensional subspaces of the space H.*

The proof follows from Lemma 7.

Lemma 10. *The functions $\varphi_n(\mu)$, $\mu \in \Lambda$, $n = 0, 1, \dots$, are continuous nonincreasing functions with positive values. In addition, the following inequalities are valid: $\varphi_n(\mu) \geq \gamma_k(\mu)$, $\mu \in \Lambda$, $n = 0, 1, \dots$*

The proof follows from Lemmas 3 and 4.

Lemma 11. *The functions $\mu - \varphi_n(\mu)$, $\mu \in \Lambda$, $n = 0, 1, \dots$, are continuous and strictly increasing functions with negative and positive values in the neighbourhoods of the points α and β, respectively.*

The proof follows from Lemma 5.

Lemma 12. *Let A and B be linear operators from H to H, the operator A has the inverse operator A^{-1} from H to H and $\|B - A\|\,\|A^{-1}\| < 1$. Then there exists the operator B^{-1} from H to H and the following inequality holds:*

$$\|B^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|B - A\|\,\|A^{-1}\|}.$$

The proof is given, for example, in [19].

Put

$$\rho(\nu) = \frac{1 - \delta_k(1 - \nu/\lambda_{k+s+1})}{1 + \delta_k(1 - \nu/\lambda_{k+s+1})(\nu/\lambda_k - 1)}, \quad \nu \in [\lambda_k, \lambda_{k+s+1}),$$
$$\delta_k = \min_{\mu \in [\lambda_k, \lambda_{k+s+1}]} \delta(\mu), \quad \delta(\mu) = \delta_0(\mu)/\delta_1(\mu), \quad \mu \in \Lambda,$$

for $\lambda_k, \lambda_{k+s+1} \in \Lambda$, $\lambda_k < \lambda_{k+s+1}$. Note that $0 < \delta_k \leq 1$, $0 < \rho(\nu) < 1$ for $\nu \in [\lambda_k, \lambda_{k+s+1})$.

Lemma 13. *The half–open interval $[\lambda_k, \lambda_{k+s+1})$ is contained in the half–open interval $[\gamma_k(\mu), \gamma_{k+s+1}(\mu))$ for any $\mu \in [\lambda_k, \lambda_{k+s+1})$.*

Proof. Taking into account Lemma 4, we get $\gamma_k(\mu) \leq \lambda_k$ and $\gamma_{k+s+1}(\mu) \geq \lambda_{k+s+1}$ for $\mu \in [\lambda_k, \lambda_{k+s+1})$. These inequalities proves the lemma.

Lemma 14. *The following inequality holds: $\rho(\mu, \nu) \leq \rho(\nu)$ for $\mu, \nu \in [\lambda_k, \lambda_{k+s+1})$.*

Proof. By Lemma 13, if $\nu \in [\lambda_k, \lambda_{k+s+1})$, then $\nu \in [\gamma_k(\mu), \gamma_{k+s+1}(\mu))$ for $\mu \in [\lambda_k, \lambda_{k+s+1})$. Now relations $\gamma_k(\mu) \leq \lambda_k$, $\gamma_{k+s+1}(\mu) \geq \lambda_{k+s+1}$, $\mu \in [\lambda_k, \lambda_{k+s+1})$, imply the desired inequality. Thus, the lemma is proved.

Theorem 3. *Let λ_k be an eigenvalue of problem* (1.1) *such that*

$$\lambda_k = \ldots = \lambda_{k+s} < \lambda_{k+s+1},$$

$k \geq 1$, $s \geq 0$, $k+s+1 \leq N$, the sequence μ^n, $n = 0, 1, \ldots$ is calculated by the formulas (5.1), (5.2). *Suppose $\mu^0 < \lambda_{k+s+1}$, $\tau(\mu) = \delta_1^{-1}(\mu)$, $\mu \in \Lambda$. Then $\mu^n \to \lambda_k$ as $n \to \infty$, and the following inequalities are valid*

$$\lambda_{k+s+1} > \mu^0 \geq \mu^1 \geq \ldots \geq \mu^n \geq \ldots \geq \lambda_k.$$

Moreover, the following estimate holds:

$$\mu^{n+1} - \gamma_k(\mu^{n+1}) \leq (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + \rho(\mu^n)(\mu^n - \gamma_k(\mu^n)),$$

where $0 < \rho(\mu) < 1$, $\mu \in [\lambda_k, \lambda_{k+s+1})$, $n = 0, 1, \ldots$

Proof. Let us show that the solutions μ^n, $n = 0, 1, \ldots$ of the equations $\mu - \varphi_n(\mu) = 0$, $\mu \in \Lambda$, $n = 0, 1, \ldots$ satisfy the following inequalities:

$$\lambda_{k+s+1} > \mu^0 \geq \mu^1 \geq \ldots \geq \mu^n \geq \ldots \geq \lambda_k.$$

Assume that the equation $\mu - \varphi_n(\mu) = 0$, $\mu \in \Lambda$, has the solution μ^n such that

$$\lambda_{k+s+1} > \mu^0 \geq \mu^1 \geq \ldots \geq \mu^n \geq \lambda_k, \quad n \geq 0.$$

Hence we obtain

$$\nu^0 = \varphi_n(\mu^n) = \mu^n < \lambda_{k+s+1} = \gamma_{k+s+1}(\lambda_{k+s+1}) \leq \gamma_{k+s+1}(\mu^n).$$

Consequently, by Lemma 8, we have

$$\nu^1 = \varphi_{n+1}(\mu^n) \leq \nu^0 = \varphi_n(\mu^n) = \mu^n.$$

It follows from Lemmas 10 and 11 that the equation $\mu - \varphi_{n+1}(\mu) = 0$, $\mu \in \Lambda$, has the unique solution μ^{n+1} and

$$\lambda_{k+s+1} > \mu^0 \geq \mu^1 \geq \ldots \geq \mu^n \geq \mu^{n+1} \geq \lambda_k.$$

Let us prove that $\mu^n \to \lambda_k$ as $n \to \infty$. Taking into account Lemma 8, 13, 14, we obtain the following relations:

$$\begin{aligned}
\mu^{n+1} - \gamma_k(\mu^{n+1}) &= (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + (\varphi_{n+1}(\mu^n) - \gamma_k(\mu^{n+1})) \leq \\
&\leq (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + (\varphi_{n+1}(\mu^n) - \gamma_k(\mu^n)) = \\
&= (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + (\nu^1 - \gamma_k(\mu^n)) \leq \\
&\leq (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + \rho(\mu^n, \nu^0)(\nu^0 - \gamma_k(\mu^n)) \leq \\
&\leq (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + \rho(\mu^n)(\mu^n - \gamma_k(\mu^n)),
\end{aligned}$$

where $\nu^0 = \varphi_n(\mu^n) = \mu^n$, $\nu^1 = \varphi_{n+1}(\mu^n)$.

Since $\lambda_{k+s+1} > \mu^0 \geq \mu^1 \geq \ldots \geq \mu^n \geq \ldots \geq \lambda_k$, there exists $\mu \in [\lambda_k, \lambda_{k+s+1})$ such that $\mu^n \to \mu$ as $n \to \infty$.

Denote by u^n an element from H_k^n such that

$$R(\mu^n, u^n) = \max_{v \in H_k^n \setminus \{0\}} R(\mu^n, v) = \varphi_n(\mu^n) = \mu^n, \quad \|u^n\|_{B(\mu^n)} = 1,$$

for $n = 0, 1, \ldots$

Let us show that there exists a constant $c > 0$ such that $\|u^n\| \leq c$, $n = 0, 1, \ldots$ Choose a number $n_0 \geq 0$ such that

$$\|B(\mu^n) - B(\mu)\| \, \|B^{-1}(\mu)\| \leq \frac{1}{2}$$

for $n \geq n_0$. We may do this because, by condition b), $\|B(\mu^n) - B(\mu)\| \to 0$ as $n \to \infty$. Then, according to Lemma 12, we obtain

$$\begin{aligned}
\|u^n\|^2 &= (u^n, u^n) = (B^{-1}(\mu^n) B(\mu^n) u^n, u^n) \leq \\
&\leq \|B^{-1}(\mu^n)\| \, \|u^n\|^2_{B(\mu^n)} = \|B^{-1}(\mu^n)\| \leq \\
&\leq \frac{\|B^{-1}(\mu)\|}{1 - \|B(\mu^n) - B(\mu)\| \, \|B^{-1}(\mu)\|} \leq c_1
\end{aligned}$$

for $n \geq n_0$, $c_1 = 2\|B^{-1}(\mu)\|$. Put

$$c_0 = \max_{n=0,1,\ldots,n_0} \|u^n\|.$$

Thus, we obtain that the required constant c is defined by the formula $c = \max\{c_0, c_1\}$.

Since $\|u^n\| \leq c$, $n = 0, 1, \ldots$, there exists an element $w \in H$ and a subsequence u^{n_i+1}, $i = 1, 2, \ldots$, such that $u^{n_i+1} \to w$ as $i \to \infty$.

Let us prove that $\mu^{n_i+1} - \varphi_{n_i+1}(\mu^{n_i}) \to 0$ as $i \to \infty$. We have

$$\begin{aligned}
0 \leq \mu^{n_i+1} - \varphi_{n_i+1}(\mu^{n_i}) &= \max_{v \in H_k^{n_i+1} \setminus \{0\}} R(\mu^{n_i+1}, v) - \max_{v \in H_k^{n_i+1} \setminus \{0\}} R(\mu^{n_i}, v) \leq \\
&\leq R(\mu^{n_i+1}, u^{n_i+1}) - R(\mu^{n_i}, u^{n_i+1}) \to 0
\end{aligned}$$

as $i \to \infty$. Here, we have taken into account that

$$R(\mu^{n_i+1}, u^{n_i+1}) \to R(\mu, w), \quad R(\mu^{n_i}, u^{n_i+1}) \to R(\mu, w),$$

as $i \to \infty$.

Using the inequality

$$\mu^{n_i+1} - \gamma_k(\mu^{n_i+1}) \leq (\mu^{n_i+1} - \varphi_{n_i+1}(\mu^{n_i})) + \rho(\mu^{n_i})(\mu^{n_i} - \gamma_k(\mu^{n_i}))$$

as $i \to \infty$, we get

$$0 \leq \mu - \gamma_k(\mu) \leq \rho(\mu)(\mu - \gamma_k(\mu)),$$

where $0 < \rho(\mu) < 1$, $\mu \in [\lambda_k, \lambda_{k+s+1})$. Hence the number $\mu \in [\lambda_k, \lambda_{k+s+1})$ satisfies the equation $\mu - \gamma_k(\mu) = 0$, i. e. $\mu = \lambda_k$ is an eigenvalue of problem (1.1) and $\mu^n \to \lambda_k$ as $n \to \infty$. This completes the proof of the theorem.

6. Error estimates of the modified subspace iteration method

Assume that there exist positive continuous functions $\alpha_0(\mu, \eta)$ and $\beta_0(\mu, \eta)$, $\mu, \eta \in \Lambda$, such that

$$\|A(\mu) - A(\eta)\| \le \alpha_0(\mu, \eta)|\mu - \eta|, \quad \|B(\mu) - B(\eta)\| \le \beta_0(\mu, \eta)|\mu - \eta|,$$

for $\mu, \eta \in \Lambda$.

Set

$$\alpha_{0,max}(a, b) = \max_{\mu,\eta\in[a,b]} \alpha_0(\mu, \eta), \quad \beta_{0,max}(a, b) = \max_{\mu,\eta\in[a,b]} \beta_0(\mu, \eta),$$

for a fixed segment $[a, b]$ on Λ.

Lemma 15. *Assume that the following inequality holds:*

$$\frac{\alpha_{0,max}(a, b)}{\alpha_{1,min}(a, b)}\,(b - a) \le \frac{1}{2},$$

for a fixed segment $[a, b]$ on Λ, V is a k–dimensional subspace of the space H, $1 \le k \le N$. Then the following estimate is valid:

$$|\gamma_i(\mu, V) - \gamma_i(\eta, V)| \le r_i(a, b, V)\,|\mu - \eta|, \quad \mu, \eta \in [a, b],$$

where

$$r_i(a, b, V) = 2\left(\frac{\alpha_{0,max}(a, b)}{\alpha_{1,min}(a, b)} + \frac{\beta_{0,max}(a, b)}{\beta_{1,min}(a, b)}\right)\gamma_i(a, V), \quad i = 1, 2, \ldots, k.$$

The proof follows from Lemma 3.

Put

$$q(\mu) = \max\{\rho(\lambda_k), \rho(\mu)\}, \quad \mu \in [\lambda_k, \lambda_{k+s+1}),$$
$$\omega_k = \lambda_{k+s+1}\sqrt{1 - \delta_k}/(1 + \sqrt{1 - \delta_k}).$$

Note that $0 < q(\mu) < 1$ for $\mu \in [\lambda_k, \lambda_{k+s+1})$.

Lemma 16. *The following equality is valid:*

$$\max_{\mu\in[\lambda_k,\mu^0]} \rho(\mu) = q(\mu^0)$$

for $\mu^0 \in [\lambda_k, \lambda_{k+s+1})$. If $0 \le \omega_k \le \lambda_k$, then $q(\mu^0) = \rho(\mu^0)$. If $\lambda_k \le \omega_k < \lambda_{k+s+1}$ and $\lambda_k \le \mu^0 \le \omega_k$, then $q(\mu^0) = \rho(\lambda_k)$.

Proof. It is not difficult to make sure (see also [5]) that $\rho'(\omega_k) = 0$, $\rho'(\mu) < 0$ for $\mu \in (0, \omega_k)$, $\rho'(\mu) > 0$ for $\mu \in (\omega_k, \lambda_{k+s+1})$. These relations imply desired results. Thus, the lemma is proved.

Theorem 4. *Let λ_k be an eigenvalue of problem* (1.1) *such that*

$$\lambda_k = \ldots = \lambda_{k+s} < \lambda_{k+s+1},$$

$k \geq 1$, $s \geq 0$, $k+s+1 \leq N$, *the sequence* μ^n, $n = 0, 1, \ldots$ *is calculated by the formulas* (5.1), (5.2). *Assume that* $\mu^0 < \lambda_{k+s+1}$, $\tau(\mu) = \delta_1^{-1}(\mu)$, $\mu \in \Lambda$, *and that numbers* $n_0 \geq 0$ *and* $\varepsilon > 0$ *such that* $\lambda_k \leq \mu^{n+1} \leq \mu^n \leq \lambda_k + \varepsilon < \lambda_{k+s+1}$ *and*

$$\frac{\alpha_{0,max}(\lambda_k, \lambda_k + \varepsilon)}{\alpha_{1,min}(\lambda_k, \lambda_k + \varepsilon)} \varepsilon \leq \frac{1}{2}$$

for $n \geq n_0$. *Then the following estimate is valid:*

$$\mu^{n+1} - \gamma_k(\mu^{n+1}) \leq q_n(\mu^n - \gamma_k(\mu^n)),$$

where $q_n = r_k(\lambda_k, \lambda_k + \varepsilon, H_k^{n+1}) + \rho(\mu^n)$, $n \geq n_0$.

Suppose $r_k(\lambda_k, \lambda_k + \varepsilon, H_k^{n+1}) \leq \sigma$, $n \geq n_0$. *Then*

$$\begin{aligned} \mu^{n+1} - \gamma_k(\mu^{n+1}) &\leq q_0^{n+1}(\mu^0 - \gamma_k(\mu^0)), \\ \mu^{n+1} - \lambda_k &\leq q_0^{n+1}(\mu^0 - \gamma_k(\mu^0)), \end{aligned}$$

for $q_0 = \sigma + q(\mu^0)$, $n \geq n_0$.

Proof. According to Lemma 15, for $n \geq n_0$ we obtain the following relation:

$$\begin{aligned} \mu^{n+1} - \varphi_{n+1}(\mu^n) &= \varphi_{n+1}(\mu^{n+1}) - \varphi_{n+1}(\mu^n) = \\ &= \gamma_k(\mu^{n+1}, H_k^{n+1}) - \gamma_k(\mu^n, H_k^{n+1}) \leq \\ &\leq r_k(\lambda_k, \lambda_k + \varepsilon, H_k^{n+1})(\mu^n - \mu^{n+1}) \leq \\ &\leq r_k(\lambda_k, \lambda_k + \varepsilon, H_k^{n+1})(\mu^n - \gamma_k(\mu^n)), \end{aligned}$$

in which we have taken into account that

$$\gamma_k(\mu^n) \leq \gamma_k(\mu^n, H_k^{n+1}) = \varphi_{n+1}(\mu^n) \leq \varphi_{n+1}(\mu^{n+1}) = \mu^{n+1}.$$

Now, by Theorem 3 and Lemma 16, we obtain desired estimates. Thus, the theorem is proved.

Remark 7. Assume that the operators $A(\mu) = A$, $B(\mu) = B$, $C(\mu) = C$, do not depend on $\mu \in \mathbb{R}$, and that the following relations are valid:

$$\delta_0(Cv, v) \leq (Av, v) \leq \delta_1(Cv, v), \quad v \in H,$$

for given constants δ_0 and δ_1, $0 < \delta_0 \leq \delta_1$. In this case, the iterative method (5.1) and (5.2) has the following form:

$$H_k^{n+1} = S(\mu^n) H_k^n, \quad \mu^n = \lambda_k(H_k^n), \quad n = 0, 1, \ldots,$$

where $S(\mu) = I - \tau C^{-1}(A - \mu B)$, $\mu \in \mathbb{R}$, $\tau = \delta_1^{-1}$, H_k^0 is a given k–dimensional subspace of the space H. Note that the number $\mu^n = \lambda_k(H_k^n)$ is a maximal eigenvalue of problem (2.1) with $V = H_k^n$.

Then the error estimates of Theorem 4 are transformed to the form:

$$\mu^{n+1} - \lambda_k \leq \rho(\mu^n)(\mu^n - \lambda_k),$$

$$\mu^{n+1} - \lambda_k \leq q_0^{n+1}(\mu^0 - \lambda_k),$$

for $n = 0, 1, \ldots$, where $0 < \rho(\mu) < 1$ for $\mu \in [\lambda_k, \lambda_{k+s+1})$, $q(\mu) = \max\{\rho(\lambda_k), \rho(\mu)\}$, $\mu \in [\lambda_k, \lambda_{k+s+1})$, $0 < q_0 = q(\mu^0) < 1$,

$$\rho(\nu) = \frac{1 - \delta(1 - \nu/\lambda_{k+s+1})}{1 + \delta(1 - \nu/\lambda_{k+s+1})(\nu/\lambda_k - 1)}, \quad \delta = \delta_0/\delta_1, \quad \nu \in [\lambda_k, \lambda_{k+s+1}).$$

These error estimates are identical with known results (see, for example, [3]).

Acknowledgement

The work of the author was supported by DFG (German Research Foundation), Sonderforschungsbereich 393.

References

1. B.N. Parlett, *The Symmetric Eigenvalue Problem.* Prentice–Hall, Englewood Cliffs, New Jersey, 1980.

2. B.A. Samokish, The steepest descent method for an eigenelement problem with semibounded operators. *Izv. Vyssh. Uchebn. Zaved., Mat.*, 1958, No. 5, pp. 105–114 (in Russian).

3. E.G. D'yakonov, A.V. Knyazev, The group iterative method for finding low–order eigenvalues. *Vestnik Mosk. Univ., Vychisl. Mat. Kibern.*, Ser. XV. 1982, No. 2, pp. 29–34 (in Russian).

4. E.G. D'yakonov, A.V. Knyazev, On an iterative method for finding smallest eigenvalues. *Preprint* No. 26, OVM, USSR Acad. Sci., Moscow, 1981 (in Russian).

5. E.G. D'yakonov, *Minimization of Computational Work. Asymptotically Optimal Algorithms for Elliptic Problems.* Nauka, Moscow, 1989 (in Russian).

6. A.V. Goolin, A.V. Kregzhde, Difference schemes for certain nonlinear spectral problems. *Preprint* No. 153, Inst. of Applied Math., USSR Acad. Sci., Moscow, 1981 (in Russian).

7. A.V. Kregzhde, On difference schemes for a nonlinear Sturm–Liouville problem. *Differ. Uravn.*, 1981, Vol. 17, No. 7, pp. 1280–1284 (in Russian).

8. A.V. Goolin, S.V. Kartyshov, Numerical study of stability and nonlinear eigenvalue problems. *Surv. Math. Ind.*, 1993, Vol. 3, pp. 29–48.

9. Yu.P. Zhigalko, A.D. Lyashko, S.I. Solov'ev, Oscillations of a cylindrical shell with joned rigid ring elements. *Modelirovanie v Mekhanike*, 1988, Vol. 2, No. 2, pp. 68–85 (in Russian).

10. R.Z. Dautov, A.D. Lyashko, S.I. Solov'ev, Convergence of the Bubnov–Galerkin method with perturbations for symmetric eigenvalue problems with a parameter entering nonlinearly. *Differ. Uravn.*, 1991, Vol. 27, No. 7, pp. 1144–1153 (in Russian).

11. S.I. Solov'ev, Error of the Bubnov–Galerkin method with perturbations for symmetric eigenvalue problems with a parameter entering nonlinearly. *Zh. Vychisl. Mat. Mat. Fiz.*, 1992, Vol. 32, No. 5, pp. 675–691 (in Russian).

12. S.I. Solov'ev, Approximation of symmetric eigenvalue problems with a parameter entering nonlinearly. *Izv. Vyssh. Uchebn. Zaved., Mat.*, 1993, No. 10, pp. 60–68 (in Russian).

13. S.I. Solov'ev, Error estimates of the finite element method for symmetric eigenvalue problems with a parameter entering nonlinearly. *Izv. Vyssh. Uchebn. Zaved., Mat.*, 1994, No. 9, pp. 70–77 (in Russian).

14. R.Z. Dautov, A.D. Lyashko, S.I. Solov'ev, The bisection method for symmetric eigenvalue problems with a parameter entering nonlinearly. *Rus. J. Numer. Anal. Math. Modelling.*, 1994, Vol. 9, No. 5, pp. 417–427.

15. S.I. Solov'ev, The finite element method for symmetric eigenvalue problems with a nonlinear entrance of a spectral parameter. *Zh. Vychisl. Mat. Mat. Fiz.*, 1997, Vol. 37, No. 11, pp. 1311–1318 (in Russian).

16. Yu.P. Zhigalko, S.I. Solov'ev, Approximation of a problem on eigenvibrations of a beam with a harmonic oscillator. In: *Theory of mesh methods for nonlinear boundary value problems.* Proceeding of Russia workshop. Kazan Math. Society, Kazan, 1998, pp. 35–37 (in Russian).

17. V.S. Zheltukhin, On solvability of an nonlinear eigenvalue problem in the theory of high–frequency discharge of low pressure, *Izv. Vyssh. Uchebn. Zaved., Mat.*, 1999, No. 5, pp. 26–31 (in Russian).

18. F.R. Gantmakher, *Theory of Matrices.* Nauka, Moscow, 1988 (in Russian).

19. V.A. Trenogin, *Functional Analysis.* Nauka, Moscow, 1980 (in Russian).

Other titles in the SFB393 series:

99-01 P. Kunkel, V. Mehrmann, W. Rath. Analysis and numerical solution of control problems in descriptor form. January 1999.

99-02 A. Meyer. Hierarchical preconditioners for higher order elements and applications in computational mechanics. January 1999.

99-03 T. Apel. Anisotropic finite elements: local estimates and applications (Habilitationsschrift). January 1999.

99-04 C. Villagonzalo, R. A. Römer, M. Schreiber. Thermoelectric transport properties in disordered systems near the Anderson transition. February 1999.

99-05 D. Michael. Notizen zu einer geometrisch motivierten Plastizitätstheorie. Februar 1999.

99-06 T. Apel, U. Reichel. SPC-PM Po 3D V 3.3, User's Manual. February 1999.

99-07 F. Tröltzsch, A. Unger. Fast solution of optimal control problems in the selective cooling of steel. March 1999.

99-08 W. Rehm, T. Ungerer (Eds.). Ausgewählte Beiträge zum 2. Workshop Cluster-Computing 25./26. März 1999, Universität Karlsruhe. März 1999.

99-09 M. Arav, D. Hershkowitz, V. Mehrmann, H. Schneider. The recursive inverse eigenvalue problem. March 1999.

99-10 T. Apel, S. Nicaise, J. Schöberl. Crouzeix-Raviart type finite elements on anisotropic meshes. May 1999.

99-11 M. Jung. Einige Klassen iterativer Auflösungsverfahren (Habilitationsschrift). Mai 1999.

99-12 V. Mehrmann, H. Xu. Numerical methods in control, from pole assignment via linear quadratic to H_∞ control. June 1999.

99-13 K. Bernert, A. Eppler. Two-stage testing of advanced dynamic subgrid-scale models for Large-Eddy Simulation on parallel computers. June 1999.

99-14 R. A. Römer, M. E. Raikh. The Aharonov-Bohm effect for an exciton. June 1999.

99-15 P. Benner, R. Byers, V. Mehrmann, H. Xu. Numerical computation of deflating subspaces of embedded Hamiltonian pencils. June 1999.

99-16 S. V. Nepomnyaschikh. Domain decomposition for isotropic and anisotropic elliptic problems. July 1999.

99-17 T. Stykel. On a criterion for asymptotic stability of differential-algebraic equations. August 1999.

99-18 U. Grimm, R. A. Römer, M. Schreiber, J. X. Zhong. Universal level-spacing statistics in quasiperiodic tight-binding models. August 1999.

99-19 R. A. Römer, M. Leadbeater, M. Schreiber. Numerical results for two interacting particles in a random environment. August 1999.

99-20 C. Villagonzalo, R. A. Römer, M. Schreiber. Transport Properties near the Anderson Transition. August 1999.

99-21 P. Cain, R. A. Römer, M. Schreiber. Phase diagram of the three-dimensional Anderson model of localization with random hopping. August 1999.

99-22 M. Bollhöfer, V. Mehrmann. A new approach to algebraic multilevel methods based on sparse approximate inverses. August 1999.

99-23 D. S. Watkins. Infinite eigenvalues and the QZ algorithm. September 1999.

99-24 V. Uski, R. A. Römer, B. Mehlig, M. Schreiber. Incipient localization in the Anderson model. August 1999.

99-25 A. Meyer. Projected PCGM for handling hanging in adaptive finite element procedures. September 1999.

99-26 F. Milde, R. A. Römer, M. Schreiber. Energy-level statistics at the metal-insulator transition in anisotropic system. September 1999.

99-27 F. Milde, R. A. Römer, M. Schreiber, V. Uski. Critical properties of the metal-insulator transition in anisotropic systems. October 1999.

99-28 M. Theß. Parallel multilevel preconditioners for thin shell problems. November 1999.

The complete list of current and former preprints is available via
`http://www.tu-chemnitz.de/sfb393/preprints.html`.